PATCHING & ZARISKI DESCENT

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Contents

In this section, we will prove that finite projective modules=finite presented flat=finite locally free modules. All lemmas and theorems can be found in [Aso21].

1. Some modules

Before this section, we review some knowledge about rings and modules. And we will prove that finite projective modules are finite locally free, finite presented flat modules are finite locally free.

Let's review some basic knowedge.

Lemma 1.0.1. *R is a commutative ring, Spec(R) is quasi-compact.*

Lemma 1.0.2. *If R is a local ring, finite projective $R-$module is free.*

Lemma 1.0.3. *Finite projective modules are finite presented*

Let's prove an important lemma.

Lemma 1.0.4. *If R is a commutative ring, $\mathfrak{p}$ is a prime ideal, and P is a finite projective $R-$module, then there exists $s \in R - \mathfrak{p}$ such that P_s is a free R_s-module.*

Proof. P is a finite projective $R-$module, by 1.0.3, we know P is finite presented, then we can know there is a short exact sequence:

$$R^{\oplus m} \to R^{\oplus n} \to P \to 0,$$

so we can use an $m \times n$ matrix M over R to represent P, which means $P \cong R^{\oplus n}/M \cdot R^{\oplus m}$. By 1.0.5, $P_{\mathfrak{p}}$ is a free module, so it's easy to see there is an invertible an $m \times m$ matrix N over $R_{\mathfrak{p}}$ such that MN is the form of

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Because there are only finite elements, we can choose an element $s \in R - \mathfrak{p}$ such that N is an matrix over R_s, which means P_S is a free R_s-module. ∎

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Proposition 1.0.5. *Finite projective modules are finite locally free.*

Proof. Let P be a finite projective $R-$module, we can fix a prime ideal $\mathfrak{p}$ of R, by 1.0.4, we can know there is an element $s_1 \in R - \mathfrak{p}$ such that P_{s_1} is a free $R_{s_1}-$module. Then we can find another prime ideal $\mathfrak{q}$ such that $s_1 \in \mathfrak{q}$ and $s_2 \in R - \mathfrak{q}$, then P_{s_2} is a free $R_{s_2}-$module. Therefore, we can find a sequence $s_1, s_2, \cdots$ such that P_{s_i} is a free $R_{s_i}-$module for any i. And it's easy to see $D(s_i)$ can cover $Spec(R)$, by 1.0.1, we can find $D(s_{i_1}), D(s_{i_2}), \cdots, D(s_{i_n})$ can cover $Spec(R)$. Thus P is finite locally free. $\square$

Let's review a lemma and Nakayama's lemma:

Lemma 1.0.6. *If $0 \to A \to B \to C \to 0$ is a exact sequence of $R-$modules and C is flat, then for any $R-$module N, we have the exact sequence $0 \to N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$.*

Lemma 1.0.7. *If M is a finitely generated module over a commutative ring R, N is a submodule of M, $J = \cap \mathfrak{m}$ is the Jacboson radical of R, and $N + JM = M$, then $M = N$.*

Let's prove that finite presented flat modules are finite locally free.

Proposition 1.0.8. *Finite presented flat modules are finite locally free.*

Proof. Let M be a finite presented flat module, we can consider the basis of $M \otimes_R R_\mathfrak{p}/\mathfrak{p}$, then we can lift them in $M_\mathfrak{p}$, we can get a map $R_\mathfrak{p}^{\oplus n} \to M_\mathfrak{p}$, we consider the image is N, then $N/\mathfrak{p}N = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$, so $N + \mathfrak{p}M_\mathfrak{p} = M_\mathfrak{p}$, by 1.0.7, we can know $M = N$, thus $R_\mathfrak{p}^{\oplus n} \to M_\mathfrak{p}$ is surjective. And M is finite presented, we can find an element $f \in R - \mathfrak{p}$ such that the original map $R_f^{\oplus} \to M_f$ is surjective (we can find finite elements to represent the basis, then we can find the common denominator.). Then we have the exact sequence $0 \to ker \to R_f^{\oplus} \to M_f \to 0$ and ker is finite generated, by 1.0.6, we can find $ker \otimes R_\mathfrak{p}/\mathfrak{p} = 0$, by 1.0.7, we know $ker = 0$. Thus M is finite locally free. $\square$

2. Zariski descent

In this section, we will talk about Zasiki descent. Before talking about this, we need to review some knowledge

Proposition 2.0.1. *If R is a commutative ring and $M \to N \to L$ is a sequence of $R-$modules. The follwing are equivalent:*
(1) $M \to N \to L$ is exact,
(2) $M_\mathfrak{p} \to N_\mathfrak{p} \to L_\mathfrak{p}$ is exact for any prime ideal $\mathfrak{p} \subset R$,
(3) $M_\mathfrak{m} \to N_\mathfrak{m} \to L_\mathfrak{m}$ is exact for any maximal ideal $\mathfrak{m} \subset R$.

Let's give some definitions.

Definition 2.0.2. Let R be a commutative ring, $f_1, f_2, \cdots, f_n$ can generate the whole ring, we define $R' = R_{f_1} \oplus R_{f_2} \oplus \cdots \oplus R_{f_n}$.

Remark 2.0.3. It is easy to see $R'-$module is the direct sum of $R_{f_i}-$module.

Definition 2.0.4. Let R be a commutative ring, there is a ring homomorphism $R \to R'$, we define a category $Mod_R(f_1, f_2, \cdots, f_n)$:
Object: (M', θ), where M' is an $R'-$module, $M'_l = R' \otimes_R M'$, $M'_r = M' \otimes_R R'$ and bimodules homomorphism $\theta : M'_l \cong M'_r$ such that $p_{12}^*\theta \circ p_{23}^*\theta = p_{13}^*\theta$, where p_{12}, p_{13}, p_{23} are extensions of scalars for different factors,

$$p_{12} : X \otimes_R Y \to X \otimes_R Y \otimes_R R', p_{13} : X \otimes_R Y \to X \otimes_R R' \otimes_R Y, p_{23} : X \otimes_R Y \to R' \otimes_R X \otimes_R Y,$$

Morphism: $\phi : M \to N$ compatible with θ_M and θ_N.

Let's prove the important theorem:

Theorem 2.0.5. *Let R be a commutative ring, $f_1, f_2, \cdots, f_n$ can generate the whole ring, the functor $F : Mod_R \to Mod_R(f_1, f_2, \cdots, f_n)$ is defined as $F(M) = (M \otimes_{R_{f_1}} R') \oplus M_{f_2}(M \otimes_{R_{f_2}} R') \oplus \cdots \oplus M_{f_n}(M \otimes_{R_{f_n}} R')$ and the obvious θ. This functor is well-defined and an equivalence. The inverse functor can be given by $Q : Mod_R(f_1, f_2, \cdots, f_n) \to Mod_R$ such that $Q(M', \theta) = ker(id - \theta)$.*

Remark 2.0.6. We should notice these things:
(1) This θ is a little different, it should be the collection of $R_f \otimes M_g \cong M_f \otimes R_g$.
(2) We need to explain $id - \theta$ because θ is a module morphism between M'_l and M'_r, it seems to be wrong. In fact, we have $R' \otimes_R M' \to M' \to M' \otimes_R R'$ to get them. And the kernel should be seen as the element in M'.

Proof. It's easy to see these functors are well-defined by the properties of localization. And we have these maps

$$0 \to M \to \prod_i M_{f_i} \to \prod_{i,j} M_{f_i f_j}$$

the second map should be the difference between M_{f_i} and M_{f_j}. We need to prove the sequence is exact. We have $f_i^{s_i+t_i} f_1^{t_1} m_1 = f_i^{t_i} f_1^{s_1+t_1} m_i$. so $f_i^{s_i+t_i} m_1 f_1^{-s_1} = f_i^{t_i} m_i$ because f_1 is an unit in $R_\mathfrak{p}$, which means it's from $R_\mathfrak{p}$. Because $f_1, f_2, \cdots, f_n$ can generate the whole ring, we can get $1 = (\sum_{i]} a_i f_i)^t$, it's easy to check the first map is injective.By the property of localization, for any element $(\frac{m_i}{f_i^{s_i}})$ of $\prod_i M_{f_i}$ in the kernel of the second map, we can assume $(f_2, \cdots, f_n) = \mathfrak{m}$ is a maximal ideal. Then f_1 is a unit in $R_\mathfrak{m}$, because localization functor is exact and commute with each other, we consider this $M_\mathfrak{m}$ as a $R_\mathfrak{m}-$module, then element $(\frac{m_i}{f_i^{s_i}})$ in the kernel of the second map. And this operation can be derived by any maximal ideal. By local to global principal 2.0.1, we know it is exact. Therefore, $QF(M) = M$.
Then we can assume $M' = \oplus_i M_i$ and $M = ker(id - \theta)$, where M_i is an $R_{f_i}-$module. So we can see the map $M \to M'$ as the collection of $M \to M_i$. By the universal property of localization, we know these maps can be decomposed as $M \to M_{f_i} \to M_i$. So we just need to check these maps $\oplus_i M_{f_i} \to M'$ are bijective. By the decomposition of R' and θ is an bimodule homomorphism, it's easy to see the restriction of θ is $R_{f_i} \otimes M_j \cong M_i \otimes R_{f_j}$, we can see the the restriction of identity must be 0 except $i = j$. so the kernel must be $\oplus_i R_{f_i} \otimes M_i \cong M'$. □

Remark 2.0.7. It seems to be irrelevant to the content of this section, but this gives us motivation to think about whether the local property can imply the global property.

3. Patching and properties of modules

In this section, we will introduce some propertites of modules and finish the proof about finite projective modules=finite locally free modules.

Definition 3.0.1. M is an $R-$module. For any family $\{f_i\}$ in ring R can generate the whole ring R, if M_{f_i} has the property $P \iff M$ has the property P, then we call the property P is local for the Zariski topology.

Let's review some things about the compatibility between the localization and finite presented.

Lemma 3.0.2. *If $S \subset R$ be a multiplicative set of R, and M is a finite presented $R-$module, N is a $R-$module, then $Hom_R(M, N)_S \cong Hom_{R_S}(M_S, N_S)$.*

Then let's prove some examples with this property.

Proposition 3.0.3. *There are some properties which are local for the Zariski topology.*
(1) Finite generation,
(2) Finite presentation,
(3) Coherence,
(4) Finite projectivity.

Proof. Let's prove these things for $R-$module M:
(1) " $\Rightarrow$ " It's easy to see M_{f_i} must be finite generated. " $\Leftarrow$ " If we have some surjective homomorphisms: $R_{f_i}^{\oplus s_i} \to M_{f_i}$ such that $(f_1, \cdots, f_n) = R$, then we can choose suitable generators such that they are in M, then we can get the homomorphism $R^{\oplus \sum_{i=1}^n s_i} \to M$, then we can take the localization for any maximal ideal $\mathfrak{m}$, we can choose an element $f_i \notin \mathfrak{m}$, from the construction we know this is surjective because this is

the sequence after the localization. By local-global 2.0.1, we know $R^{\oplus \sum_{i=1}^{n} s_i} \to M$ is surjective.
(2) " $\Rightarrow$ " Similarly, by using the exact sequence $R^m \to R^n \to M \to 0$, we can take localization to get the first direction i.e. M_{f_i} is finite presented. " $\Leftarrow$ " Conversely, if we have some exact sequences: $R_{f_i}^{\oplus t_i} \to R_{f_i}^{\oplus s_i} \to M_{f_i} \to 0$ such that $(f_1, \cdots, f_n) = R$, then we can choose suitable generators such that they are in M, then we can get the homomorphism $R^{\oplus \sum_{i=1}^{n} t_i} \to R^{\oplus \sum_{i=1}^{n} s_i} \to M \to 0$. Similarly, we can take the localization for any maximal ideal $\mathfrak{m}$, we can choose an element $f_i \notin \mathfrak{m}$, from the construction we know this is exact because this is the sequence after the localization. By local-global 2.0.1, we know $R^{\oplus \sum_{i=1}^{n} t_i} \to R^{\oplus \sum_{i=1}^{n} s_i} \to M \to 0$ is exact.
(3) " $\Rightarrow$ " Localization is exact, which means the localization of coherent module is coherent. " $\Leftarrow$ " For another direction, we can choose $M' \subset M$ as a finite generated submodule, then we can see $M'_{f_i} \subset M_{f_i}$ is finite presented because localization perserves finite generation and M_{f_i} is coherent. Thus we can see M' is finite presented by (2), which means M is coherent.
(4) " $\Rightarrow$ " This is obvious by 1.0.5. " $\leftarrow$ " By 1.0.5, we can assume M_{f_i} is finite free, and for any exact sequence $0 \to A \to B \to C \to 0$ of $R-$modules, we can get the sequence $0 \to Hom_R(M, A) \to Hom_R(M, B) \to Hom_R(M, C)(\to 0)$ (except the $\to 0$, it's exact). Similarly, taking the localization, by 1.0.3 and 3.0.2, we can find this is from $0 \to A_{fi} \to B_{f_i} \to C_{f_i} \to 0$ by applying $Hom_{R_{f_i}}(P_{f_i}, -)$. Because P_{f_i} is finite free, we can see the exactness. And take the further localization, get the the exactness after taking the localization at any maximal ideal $\mathfrak{m}$. Thus P is projective by local-global 2.0.1, by (1), we know P is finite projective. ☐

Now we can prove the final theorem:

Theorem 3.0.4. *If R is a commutative ring, M is a $R-$module, then the following are equivalent:*
(1) M is finite projective,
(2) M is finite presented and flat,
(3) M is finite locally free.

Proof. Let's prove these things:
"(1)$\Rightarrow$ (2)" It's easy by 1.0.3.
"(2)$\Rightarrow$ (3)" By 1.0.8.
"(3)$\Rightarrow$ (1)" By 3.0.3, we can see M_{f_i} is finite free, then finite projective, so M is finite projective. ☐

References

[Aso21] Aravind Asok. Algebraic geometry from an $\mathbb{A}^1$-homotopic viewpoint, May 3, 2021.