SERRE'S SPLITTING THEOREM

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1. Serre's splitting theorem

In this section, we will prove Serre's splitting theorem. Let's give an important theorem without proof to show some motivations.

Theorem 1.0.1. *Let M be a manifold of dimension d. If $p : E \to M$ is a topological vector bundle of rank $r > d$, then E is isomorphic to $E \oplus F$, where E is vector bundle of rank $\leq d$ and F is a free module.*

Proof. We can consider the surjection $V \to E \to 0$, there is a kernel F, consider the subspace P_x corresponding to the projective space of F_x in $\mathbb{P}(V)$. Then $\cup P_x$ is a subspace Y of $\mathbb{P}(V)$. We can know $dim(Y) = d + dim(P_x) = d + dim(V) - r - 1$. So we can find a subspace of V such that it doesn't intersect with F at any closed point. And we can see there is a nowhere vanishing section, so we can see E *is* isomorphic to $E \oplus F$, where E is vector bundle of rank $\leq d$ and F is a free module. □

So we want to prove similar things in algebraic geometry.

Before proving this, we need to give some lemmas.

Lemma 1.0.2. *P is a finite projective $R-$module. If $s_1, \cdots, s_n \in P$, then*

$$\{x \in Spec(R) | x \text{ is closed and } \bar{s_1}, \cdots, \bar{s_n} \text{ are linearly dependent in } R/\mathfrak{m}_x\}$$

is the set of closed points of a closed subscheme of $Spec(R)$.

Proof. Because P is finite projective, we can assume that P is a free module of finite rank. So we can choose a basis $\{a_1, \cdots, a_m\}$ of R, we can get the expression $s_i = \sum_{i,j} b_{ij} a_j$, then we can get the $\bar{s_1}, \cdots, \bar{s_n}$ are linearly dependent in $R/\mathfrak{m}_x$ if only if $det(B) = det((b_{ij}))$ is 0 in $R/\mathfrak{m}_x$, which means the former set should be the set of closed points of $V(det(B))$. □

Lemma 1.0.3. *If P is a finite projective $R-$module and $x_1, \cdots, x_n$ are distinct closed points of $Spec(R)$ and $v_i \in P/\mathfrak{m}_x, 1 \leq i \leq n$, then there exists a section $s \in P$ such that $s(x_i) := \bar{s} = v_i$ in $P/\mathfrak{m}_x$ for any i.*

Proof. We have known the closed points $x_1, \cdots, x_n$ are pairwise distinct, the corresponding maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$ are pairwise comaximal, by CRT, we have the surjection $R \twoheadrightarrow \oplus_{i=1}^n R/\mathfrak{m}_i$. Then we can find the preimage r_i of t_i in $\oplus_{i=1}^n R/\mathfrak{m}_i$, where $(t_i)_j = 0$ if $i \neq j$ and $(t_i)_i = 1$. And the preimage in P of v_i is q_i, then we can define $s := \sum_{i=1}^n r_i q_i$. Therefore $s(x_i) = \sum_{j=1}^n r_j(x_i) q_j(x_i) = r_i(x_i) q_i(x_i) = v_i$. □

Lemma 1.0.4. *P is a finite projective $R-$module and $s \in P$, we can define a map $R \mapsto P, r \mapsto rs$. Then R is a direct summand of P through this map if and only if $\bar{s} \neq 0$ in $P/\mathfrak{m}_x$ for every closed point $x \in Spec(R)$*

Proof. Let us prove this.

"$\Rightarrow$": R is a one-dimensional subspace of P, we can get s must be the basis of $P/\mathfrak{m}_x$. Then $\bar{s} = 0$ in $P/\mathfrak{m}_x$ means that $R_{\mathfrak{m}_x}/\mathfrak{m}_x R_{\mathfrak{m}_x} \to P_{\mathfrak{m}_x}/\mathfrak{m}_x P_{\mathfrak{m}_x}$ is 0, by **??**, then we can know that $R_{\mathfrak{m}_x} \to P_{\mathfrak{m}_x}$ is 0, because the localization functor is exact, a contradiction!

"$\Leftarrow$": We can think of M as the image in P. Then we can get the map $R \twoheadrightarrow M \hookrightarrow P$, so it is easy to see $R_{\mathfrak{m}_x}/\mathfrak{m}_x R_{\mathfrak{m}_x} \to M_{\mathfrak{m}_x}/\mathfrak{m}_x M_{\mathfrak{m}_x}$ is bijective for any closed point x. By **??**, we can use **??** for kernel and image. So $R_{\mathfrak{x}} \cong M_{\mathfrak{m}_x}$. By **??**, we know $R \cong M$. □

Let's give a different version of Serre's splitting theorem.

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Theorem 1.0.5. *Let R is a Noetherian ring of Krull dimension d, and $Z \subset Spec(R)$ is a closed subset, $x_1, \cdots, x_n \in Z$ are distinct closed points and $v_i \in P(x_i)$. Assume*
(1) *P is a finite projective $R-$module and $s_1, \cdots, s_r$ are elements of P that are linearly independent at every closed point $x \in Spec(R) - Z$,*
(2) *There is an integer $k \geq 0$ such that $r + k \leq rk_x(P)$ for every closed point $x \in Z$.*
Then, there is an element $s \in P$ and a closed subset $X \subset Spec(R)$ such that
1. $s(x_i) = v_i$ for any $1 \leq i \leq n$,
2. $s_1, \cdots, s_r, s$ are linearly independent at every closed point $x \notin Z \cup X$,
3. The codimension of X in $Spec(R)$ is $codim(X) \geq k$.

Proof. By 1.0.3, there is an element $s \in P$ such that $s(x_i) = v_i$ for any $1 \leq i \leq n$. By 1.0.2, we can choose this closed subscheme X such that $s_1, \cdots, s_r, s$ are linearly dependent. Then $codim(X) \geq r+k-r = k$ since we can consider the closed point x in X, we have $rk_x(P) \geq r + k$ and $s_1, \cdots, s_r, s$ are linearly dependent, which means we still can choose at least $r + k - r = k$ sections which are linearly independent and this implies that we can find the height of this closed point must $\geq k$. ☐

Theorem 1.0.6. *If R is a Noetherian ring of Krull dimension d, then any finite projective module P of rank $r > d$ can be written as the direct sum $Q \oplus F$, where Q is finite projective of rank $\leq d$ and F is free.*

Proof. We can consider the section $s \in P$. We can know the set of closed points vanishing for section s is X, and $codim(X) \geq d$ by 1.0.2. Then X must be the union of finite points since R is Noetherian. We can choose these closed points and $\bar{1} \in P(y)$ for these closed points y. By 1.0.5, we can find another element t such that s, t are linearly independent at every closed points $y \notin X \cup T$ and T is closed and has finite closed points. So we can just choose $s + t$, this must be a nowhere vanishing section. By 1.0.4, we know $P \cong Q_1 \oplus R$. We can deduce this by Q_i, then $P \cong Q_1 \oplus R \cong Q_2 \oplus R^{\oplus 2} \cong \cdots \cong Q \oplus F$. ☐