EXTENDING BUNDLES AND QUILLEN-SUSLIN THEOREM

Abstract. Any topological vector bundle over a contractible space is trivial. This can be proved easily by homotopy classification of vector bundles. Prototypical example of a contractible space is $\mathbb{R}^n$ or $\mathbb{C}^n$, so one may analogously expect that any algebraic vector bundle over $\mathbb{A}^n_k$ is trivial, for k a field. This is the content of Quillen-Suslin theorem, giving us yet another big hint on the existence of a homotopy theory for schemes. In this note, we give an exposition of the proof of this result, the main idea of which is Quillen's patching - when does a projective module over $R[t]$ is obtained via extension of scalars - combined with understanding the obstruction in extending a vector bundle to $\mathbb{P}^1_R$.

Contents

1. Introduction

We wish to prove the following result.

Theorem 1.1 (Quillen-Suslin)**.** *Let R be a PID. Then any finitely generated projective module over $R[x_1, \ldots, x_n]$ is free.*

We may begin by induction. The base case is simple as we have a structure theorem for modules over PID. So the first difficult case is that of $R[x]$, the affine line over R. Our approach is guided by the following fundamental theorem of Horrocks.

Theorem 1.2 (Horrocks)**.** *Let R be a noetherian local ring. If $\mathcal{E}$ is a vector bundle over $\mathbb{A}^1_R$ which extends to a vector bundle over $\mathbb{P}^1_R$, then $\mathcal{E}$ is trivial.*

This immediately raises two questions:

(1) When does a vector bundle over $\mathbb{A}^1_R$ extends to a vector bundle over $\mathbb{P}^1_R$? Answered by Lemma 2.4.
(2) Is there a way to "globalize" Horrocks theorem? Answered by Quillen's patching (Theorem 3.6).

Vector bundles P over R are simple, but is it true that any vector bundle over $R[t]$ is obtained by extending the one on R? Of-course we may define $P \otimes_R R[t]$, and this is clearly a vector bundle. In particular, we have a map

$$\mathrm{Vect}(R) \xrightarrow{\text{extension}} \mathrm{Vect}(R[t])$$
$$P \longmapsto P \otimes_R R[t]$$

If R is a PID, every module in its image is free. Consequently, we may rephrase the $n = 1$ case of Theorem 1.1 as showing whether this map is an isomorphism or not. This is what we deduce to be true. One way thus to globalize Horrocks would be to prove whether a module is extended if it is extended locally. Hence, answering these two problems naturally leads to proof of Theorem 1.1.

All modules considered in this note are finitely generated.

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2. Extending vector bundles from $\mathbb{A}^1_R$ to $\mathbb{P}^1_R$

We wish to understand the obstruction in extending a vector bundle on $\mathbb{A}^1_R$ to $\mathbb{P}^1_R$. Recall that $\mathbb{P}^1_R$ is obtained by gluing $U = \operatorname{Spec}(R[t])$ and $V = \operatorname{Spec}\big(R[t^{-1}]\big)$ on the intersection $U \cap V = \operatorname{Spec}\big(R[t^{\pm 1}]\big)$ via the maps $R[t] \to R[t^{\pm 1}]$ and $R[t^{-1}] \to R[t^{\pm 1}]$. Of-course, the issue lies at the point ∞. We want to handle this by understanding an $R[t]$-module M around the neighborhood of ∞. Let $s = t^{-1}$ so that $V = \operatorname{Spec}(R[s])$ and the point at infinity of $\mathbb{P}^1_R$ is $s = 0$. Consider functions which do not vanish at ∞, i.e. functions of the form $1 + sf(s)$ for $f(s) \in R[s]$. Thus, consider the localization of $R[s]$ at the multiplicative set $S = 1 + sR[s]$; denote

$$V_\infty = \operatorname{Spec}\big(S^{-1}R[s]\big).$$

One can imagine this as an infinitesimal neighborhood of ∞. Indeed, this is christened by the following result.

Lemma 2.1. *With the notation as above, $U \cap V_\infty = \operatorname{Spec}(R\langle t\rangle)$, where $R\langle t\rangle$ is the localization of $R[t]$ at the multiplicative set of all monic polynomials.*

Let us now begin to extend a vector bundle from $\mathbb{A}^1_R$ to $\mathbb{P}^1_R$. Let P be a projective module over $R[t]$. How does it behave near ∞? Restricting P to $U \cap V_\infty$ gives the module $P_\infty = P \otimes_{R[t]} R\langle t\rangle$.

Remark 2.2. Observe that P_∞ is a projective $R\langle t\rangle$-module since $\otimes$ and $\oplus$ commutes.

Note that if P_∞ is free, then we are done via the following lemma.

Lemma 2.3. *Let X be a scheme covered by two open sets U and V. If $\mathcal{E}$ is a vector bundle of rank r over U such that $\mathcal{E}_{|U\cap V}$ is trivial, then $\mathcal{E}$ extends to a vector bundle over X.*

Proof. We need only show that as sheaves, one can glue $\mathcal{E}$ over U and $\mathcal{O}_V^{\oplus r}$ on V. As there are only two open sets, gluing condition degenerates to existence of an isomorphism

$$\mathcal{E}_{U\cap V} \xrightarrow{\cong} \mathcal{O}_V^{\oplus r}$$

which is exactly our hypothesis. □

As a consequence of this lemma, we get the following result.

Lemma 2.4. *A projective module P over $R[t]$ extends to $\mathbb{P}^1_R$ if $P_\infty = P \otimes_{R[t]} R\langle t\rangle$ is a free $R\langle t\rangle$-module.*

Proof. As P_∞ is free, let $e_i = p_i/f_i(t)$ be a collection of free generators of P_∞, where $f_i(t)$ are monic polynomials in $R[t]$. Let $f = \prod_i f_i$. Observe that f is a monic polynomial. Note that P_f contains e_i (as it contains p_i/f) and is freely generated by them. Take $V = D(f)$ in Lemma 2.3 to complete the proof. □

Remember we are trying to solve the $n = 1$ case of Theorem 1.1. By the above result, we are now reduced to understanding $R\langle t\rangle$-modules, especially, to characterize when they are free. Note that in our case, we need to answer whether all projective $R\langle t\rangle$-modules is free or not (Remark 2.2).

2.1. Prime ideals in $R\langle t\rangle$.

Let us begin by understanding prime ideals in $R\langle t\rangle$. Recall that $R[t] \to R\langle t\rangle$ is a localization map, thus it is a flat $R[t]$-algebra.

Proposition 2.5. *Let R be a noetherian domain.*

1. *If $\dim R = d$, then $\dim R\langle t\rangle = d$.*
2. *If R is a PID, then $R\langle t\rangle$ is a PID.*

Proof. 1. Note that primes of $R\langle t\rangle$ are primes of $R[t]$ not containing any monic polynomials. Note that for a chain

$$0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_d$$

of primes in R, we get a sequence of primes

$$0 = \mathfrak{p}_0[t] \subsetneq \mathfrak{p}_1[t] \subsetneq \mathfrak{p}_2[t] \subsetneq \cdots \subsetneq \mathfrak{p}_d[t]$$

in $R[t]$ of length d. Note that none of the primes above contain a monic; $\dim R\langle t\rangle \geq d$. It suffices to show that any prime $\mathfrak{P}$ of $R[t]$ of height $d+1$ must contain a monic polynomial, i.e. $\dim R\langle t\rangle \leq d$.

Let $\mathfrak{P} \subset R[t]$ be of height $d+1$. Consider $\mathfrak{p} = \mathfrak{P} \cap R$. Then $\mathfrak{p}[t] \subseteq \mathfrak{P}$. As $R[t]/\mathfrak{p}[t] = (R/\mathfrak{p})[t]$, which is atleast 1-dimensional, it follows that $\mathfrak{p}[t] \subsetneq \mathfrak{P}$ (otherwise a contradiction to $\dim R[t] = d+1$). Moreover, this forces $\dim R[t]/\mathfrak{p}[t] = 1$, giving ht $(\mathfrak{p}[t]) = d$ (a chain of length 1 can atmost lie over $\mathfrak{p}$) and $R/\mathfrak{p}$ a field, i.e. $\mathfrak{p}$ is maximal. As ht $(\mathfrak{p}[t]) =$ ht $(\mathfrak{p})$ (Krull's height theorem) and ht $(\mathfrak{p}[t]) = d$, thus $\mathfrak{p}$ is a maximal ideal of R of maximum height.

Pick $f(t) \in \mathfrak{P} - \mathfrak{p}[t]$. Let $f(t) = a_n t^n + \cdots + a_1 t + a_0$. As $a_n \notin \mathfrak{p}$, therefore there exists $b_n \in R$, $d_n \in \mathfrak{p}$ such that $a_n b_n + d_n = 1$. Thus, $b_n f(t) + d_n t^n$ is clearly in $\mathfrak{P}$ and is a monic polynomial, completing the proof.

2. By above, $R\langle t\rangle$ is a dimension 1 noetherian domain. As a PID is a UFD of dimension 1, it suffices to show that $R\langle t\rangle$ is a UFD, but this is immediate since $R[t]$ is a UFD (Gauss' lemma) and $R\langle t\rangle$ is a localization of $R[t]$. ☐

By preceding discussion, we can now deduce the following.

Corollary 2.6. *If R is a PID, then any projective module P over $R[t]$ extends to $\mathbb{P}^1_R$.*

Proof. By Lemma 2.4, P extends if the projective module P_∞ is a free $R\langle t\rangle$-module. By Proposition 2.5, $R\langle t\rangle$ is a PID, thus P_∞ is free. ☐

Our goal is now to understand the behaviour of vector bundles on $\mathbb{P}^1_R$ which comes by extending a projective module over $R[t]$.

3. Extendability & Quillen patching

In our goal of proving Quillen-Suslin theorem in the case of $R[t]$, we have so far shown that any vector bundle over $\mathbb{A}^1_R$ extends to a vector bundle over $\mathbb{P}^1_R$. We now analyze this vector bundle on $\mathbb{P}^1_R$.

Now suppose that we had a global version of Horrocks' theorem; that every vector bundle over $\mathbb{A}^1_R$ which extends to $\mathbb{P}^1_R$ is trivial, for any noetherian R. By Corollary 2.6, it would thus follows that every projective module P over $R[t]$ is trivial if R is a PID. This would hence complete the proof of Theorem 1.1 for the case $n = 1$.

A naive global Horrocks' (locally extends to $\mathbb{P}^1_{A_\mathfrak{m}}$ then globally extends to $\mathbb{P}^1_A$) is not possible

Example 3.1. We will construct an example of noetherian R and a vector bundle over $\mathbb{A}^1_R$ which extends to $\mathbb{P}^1_R$ but is not free. Indeed, let R be $\mathbb{Z}[\sqrt{-5}]$ and M be a projective non-free module over R. By Jesse's example 1, such a module exists. Then the extension $M[t]$ to $R[t]$ is a non-free projective module over $R[t]$ further satisfying the property that it extends to $\mathbb{P}^1_R$. Indeed, $P_\infty = M[t] \otimes_{R[t]} R\langle t\rangle = M\langle t\rangle$, which is not free.

But we can still salvage the following. Recall that localization of a PID at a maximal is a DVR.

Lemma 3.2. *Let R be a DVR. Then any projective module over $R[t]$ is free.*

Proof. Let P be a projective over $R[t]$. Recall that R is a local PID in particular, so by Corollary 2.6, P extends to a vector bundle over $\mathbb{P}^1_R$. By Horrocks' theorem (Theorem 1.2), P is free. ☐

But observe that we can instead do the following. If R is a ring such that all projective modules over $R[t]$ are free, then it has the property that any projective module P over $R[t]$ is obtained by *extending* a projective module M over R to $R[t]$, that is, $P = M \otimes_R R[t]$. This motivates the following.

Definition 3.3. Let R be a noetherian ring and P be an $R[t]$-module. We say P is *extended* if there is M an R-module such that $P = M \otimes_R R[t]$.

The first question in our context that must arise is that if P is a vector bundle obtained by extending a module M from R, is M a vector bundle over R?

Lemma 3.4. *Let R be any noetherian ring. If P is an extended projective $R[t]$-module such that $P = M \otimes_R R[t]$, then M is a projective R-module.*

Proof. Observe that since $P = M[t]$, therefore $M = P/tP$. It is easy to see that

$$P \cong tP \oplus P/tP$$

via the obvious map. As direct summands of projective modules are projective, therefor $P/tP = M$ is projective. □

Remark 3.5. Note that our goal of showing that a projective module P over $R[t]$ being trivial for R a PID can be thus reduced to showing that every projective module P over $R[t]$ is extended. To show this latter statement, we can perhaps get a hint to a possible method from Lemma 3.2. For any maximal ideal $\mathfrak{m} \subset R$ for R PID, $R_\mathfrak{m}$ is a DVR. So if P is a projective module over $R[t]$, we can contemplate the projective $R_\mathfrak{m}[t]$-module $P_\mathfrak{m}$, which we know is free from Lemma 3.2.

In particular, $P_\mathfrak{m}$ is an extended $R_\mathfrak{m}[t]$-module. Suppose we can *patch* this up and are able to conclude that P is an extended $R[t]$-module. Then $P = M \otimes_R R[t]$ for an R-module M, which is a projective R-module by Lemma 3.4. As M is free since R is a PID, therefore by $P = M \otimes_R R[t]$ is a free $R[t]$-module, completing the Quillen-Suslin for $n = 1$.

For the above remark to work, we need a patching argument for extendability. This is the content of Quillen's patching theorem.

Theorem 3.6 (Quillen's patching)**.** *Let R be a noetherian ring and P be a finitely presented module over $R[t]$. If $P_\mathfrak{m}$ is an extended $R_\mathfrak{m}[t]$-module for each maximal ideal $\mathfrak{m} \subset R$, then P is an extended $R[t]$-module.*

Hence we have the following.

Theorem 3.7 (Quillen-Suslin-1)**.** *Let R be a PID and P be a projective module over $R[t]$. Then P is free.*

Proof. Follows from Remark 3.5 and Theorem 3.6. □

Proof of Theorem 1.1. We can now complete the proof of Quillen-Suslin theorem as follows. Let P be a projective module over $R[x_1, \ldots, x_n]$ where R is a PID. We proceed by induction on n. When $n = 0$, then we are done as R is PID. When $n = 1$, then we are done by Theorem 3.7. Now suppose that any projective module over $A = R[x_1, \ldots, x_{n-1}]$ is free for any PID R. We wish to show that any projective module P over $A[t]$ is free, for $t = x_n$. By inducive hypothesis, it is sufficient to show that P is an extended $A[t]$-module. We will use Quillen patching (Theorem 3.6) for that.

Let $\mathfrak{m} \subset A$ be a maximal ideal and consider $P_\mathfrak{m}$ as a projective $A_\mathfrak{m}[t]$-module. Note that $A_\mathfrak{m}$ is a noetherian local ring. Thus by Horrocks' result (Theorem 1.2), it suffices to show that $P_\mathfrak{m}$ extends to a vector bundle on $\mathbb{P}^1_{A_\mathfrak{m}}$. To this end, we will use Lemma 2.4, i.e. we have to show that $P_{\mathfrak{m},\infty} = P_\mathfrak{m} \otimes_{A_\mathfrak{m}[t]} A_\mathfrak{m}\langle t\rangle$ is a free $A_\mathfrak{m}\langle t\rangle$-module. Observe that

$$P_{\mathfrak{m},\infty} = (P \otimes_{A[t]} A\langle t\rangle)_\mathfrak{m},$$

so it suffices to show that $P \otimes_{A[t]} A\langle t\rangle$ is a free $A\langle t\rangle$-module. Observe that we have $A\langle t\rangle = A[t] \otimes_{R[t]} R\langle t\rangle$. Consequently,

$$P \otimes_{A[t]} A\langle t\rangle \cong P \otimes_{A[t]} \left(A[t] \otimes_{R[t]} R\langle t\rangle\right) \cong P \otimes_{R[t]} R\langle t\rangle.$$

Now since $P \otimes_{R[t]} R\langle t\rangle$ is a projective module over $A[t] \otimes_{R[t]} R\langle t\rangle \cong R\langle t\rangle[x_1, \ldots, x_{n-1}]$, which is a polynomial ring over the PID $R\langle t\rangle$ (Proposition 2.5) in $n-1$ variables, therefore by inductive hypothesis, we have that $P \otimes_{R[t]} R\langle t\rangle$ is free. By above isomorphism, we get that $P \otimes_{A[t]} A\langle t\rangle$ is free.

$$\begin{array}{ccccc} R & \hookrightarrow & R[t] & \longrightarrow & R\langle t\rangle \\ \downarrow & & \downarrow & & \downarrow \\ A & \hookrightarrow & A[t] & \longrightarrow & A\langle t\rangle = A[t] \otimes_{R[t]} R\langle t\rangle \end{array} .$$

This completes the proof. □

4. Proof of patching

Proof of Theorem 3.6. Let $Q(P) = \{f \in R \mid P_f \text{ is an extended } R_f[t]\text{-module}\}$. Note that it suffices to show that $Q(P) = R$. Indeed, in this case, for $f, g \in R$ such that $f + g = 1$, we get a cover of $X = \mathbb{R}^1_R$ by two open sets $U = D(f), V = D(g)$. Rs the modules $P_f = M^f \otimes_{R_f} R_f[t]$ and $P_g = M^g \otimes_{R_g} R_g[t]$ for M^f, M^g being projective R_f, R_g-modules, therefore by Zariski descent, one can glue M^f, M^g to a projective module M over R and then $M \otimes_R R[t] \cong P$.

We will show $Q(P) = R$ by showing $Q(P)$ is a unit ideal. Let us first assume that $Q(P)$ is an ideal. We show that $Q(P)$ is unit. Note that it suffices to show that $Q(P) \cap (R - \mathfrak{m}) \neq \emptyset$ for each maximal ideal $\mathfrak{m} \subset R$, i.e. $Q(P)$ is an ideal which has no maximal ideal containing it. So fix $\mathfrak{m} \subset R$ a maximal ideal. Indeed, consider $P' = R[t] \otimes_R P/tP$, where P/tP is an R-module. Observe that by properties of localizations, we have an isomorphism of $R_\mathfrak{m}[t]$-modules

$$\varphi_\mathfrak{m} : P_\mathfrak{m} \xrightarrow{\cong} P'_\mathfrak{m}.$$

As both of these are finitely presented $R_\mathfrak{m}[t]$-modules, therefore by clearing the denominators, there exists $f \in R - \mathfrak{m}$ for which we get an isomorphism of finitely presented R_f-modules

$$\varphi : P_f \longrightarrow P'_f$$

such that φ localized at $\mathfrak{m}$ is $\varphi_\mathfrak{m}$. Hence by definition, $f \in Q(P) - \mathfrak{m}$, completing the proof that $Q(P)$ is a unit ideal.

It suffices to show that $Q(P)$ is an ideal. Let $f, g \in Q(P)$. We wish to show that $f + g \in Q(P)$. By replacing R by localization of R at $f + g$, we reduce to assuming that f, g are comaximal in R such that P_f and P_g are extended by M_f and M_g - which are obtained by localizing $M = P/tP$ at f and g, respectively - and we reduce to showing that P is extended by M, i.e. $P \cong M[t]$. We have isomorphisms

$$\varphi : P_f \longrightarrow M_f \otimes_{R_f} R_f[t] = M_f[t]$$
$$\psi : P_g \longrightarrow M_g \otimes_{R_g} R_g[t] = M_g[t],$$

in order to patch which, we need to show that on $D(fg) = D(f) \cap D(g)$, the composite

$$\begin{array}{ccc} & P_{fg} & \\ {\scriptstyle \varphi_g} \swarrow & & \searrow {\scriptstyle \psi_f} \\ M_{fg}[t] & \xrightarrow[\psi_f \circ \varphi_g^{-1}]{} & M_{fg}[t] \end{array}$$

is identity. Quillen showed that by suitable composition of φ and ψ with certain endomorphisms of $M_f[t]$ and $M_g[t]$, we can assume that $\psi_f \circ \varphi_g^{-1} = \text{id}$, completing the proof. A reference to this result is given in Corollary V.1.2-3, [Lam10]. □

References

[Lam10] T.Y. Lam. *Serre's Problem on Projective Modules.* Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2010.