GROTHENDIECK TOPOLOGIES AND THE NISNEVICH SITE

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Abstract. In this rather definition-heavy talk, we will motivate and introduce the notion of a Grothendieck topology and sheaves on a site. We will define the classical examples of sites coming from algebraic geometry. The latter half will be dedicated to the Nisnevich site on a category of smooth schemes and highlighting its important properties for homotopy theory.

Contents

The motivation for this lecture is to begin to develop a suitable homotopy category for schemes. As motivation from algebraic topology, recall that a covering space (or locally trivial fiber bundle) over a CW complex M is a space E with an epimorphism $p : E \to M$ such that every $x \in M$ has a neighborhood U such that $p^{-1}(U) \cong \coprod_{i\in I} U$ for a set I. In other terms, $\coprod_i U$ is a pullback

$$\begin{array}{ccc} \coprod_i U & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & \underset{i}{\longrightarrow} & M \end{array}$$

Covering spaces are important in the computation of homotopy groups of spaces - these sorts of fibrational methods and lifting problems are a main tool for homotopy theory. However, this notion does not generalize properly to schemes if we simply take i to be an open embedding of a subscheme. The issue was noticed by Serre and Grothendieck when constructing fiber bundles for algebraic varieties/groups. If the above discussion were to take place in a category of schemes or varieties, it would still make sense to define suitable neighborhoods (open embeddings) over which a preimage would be trivial in a cohomological or homotopical sense. However this is not stable under pullback in that if $x \in U \subset M$ such that $p^{-1}(U)$ is trivial and $U' \to U$ is a covering of U in the usual topological sense, the pullback $E \times_M U' \to U'$ is not trivial. There are further obstructions from Galois correspondence for algebraic groups.

The first section follows the presentation of [MM12].

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1. Covering sieves and topologies

The notion of a Grothendieck topology on a category $\mathcal{C}$ is meant to provide "generalized neighborhoods" where we don't only consider open monomorphisms.

Definition 1.0.1. Let $\mathcal{C}$ be a category with pullbacks. For $c \in \mathcal{C}$, consider a set $S = \{f_i : c_i \to c\}_i$ of arrows in $\mathcal{C}$, and suppose for every $c \in \mathcal{C}$ we have a family $K(c) = \{S, S', S'', \dots\}$ of such indexed sets of arrows into c. Then we call K a *covering* of $\mathcal{C}$ and the set $K(c)$ a covering of c.

The arrows $f_i \in S$ will play the role in our generalized topology of open embeddings in classical topology.

Definition 1.0.2. For a small category $\mathcal{C}$, a family $\{f_i : c_i \to c\}_i$ which is closed under precomposition and isomorphism is called a *sieve* on c. Equivalently, this is a subfunctor $S \subseteq yc$ in $Fun(\mathcal{C}^{op}, Set) = PSh(\mathcal{C})$. Note that if S is a sieve on c and $h : d \to c$ is an arrow in $\mathcal{C}$, then $h^*S = \{g : cod(g) = d, \quad hg \in S\}$ is a sieve on d.

We are now ready to formulate the definition of a Grothendieck topology.

Definition 1.0.3. A *Grothendieck topology* on $\mathcal{C}$ is a function J assigning to each object $c \in \mathcal{C}$ a collection $J(c)$ of sieves on c such that:

(1) $t_c = \{f : cod(f) = c\} = Obj(\mathcal{C}/c) \in J(c)$, the maximal sieve.
(2) $S \in J(c) \implies h^*S \in J(d)$ for all $h : d \to c$ (stability).
(3) If $S \in J(c)$ and R is a sieve on c such that $h^*R \in J(d)$ for all $h : d \to c \in S$, then $R \in J(c)$ (transitivity).

A *site* is a pair $(\mathcal{C}, J)$. We say S *covers* c when $S \in J(c)$ and S *covers* $f : d \to c$ if f^*S covers d. We can give the equivalent definitions in "arrow form" as follows:

(1) ' If S is a sieve on c and $f \in S$ then S covers f.
(2) ' S covers $f : d \to c \implies S$ covers fg for all $g : e \to d$.
(3) ' If S covers $f : d \to c$ and R is a sieve on c covering all arrows of S, then R covers f.

Example 1.0.4. Let X be a topological space and let $Open(X)$ the category (poset) of open sets of X. Then a sieve S on $U \in Open(X)$ is a family of open subsets $V \hookrightarrow U$ such that $V' \subseteq V \in S \implies V' \in S$, and S covers U if $U \subseteq \bigcup_{V \in S} V$. So our definition of a cover recovers that of an ordinary open cover.

Similarly to the situation of an ordinary topology on a set, it is convenient to define the notion of a basis to generate a topology. Indeed in all the examples later we will define a site by a basis.

Definition 1.0.5. A *basis* for a Grothendieck topology is a function K assigning to $c \in \mathcal{C}$ a family of maps into c such that

(1) $\{f : c' \to c \text{ iso}\} \in K(c)$
(2) $\{f_i : c_i \to c\} \in K(c) \iff$ for every $g : d \to c$, the family $\{\pi_2 : c_i \times_c d \to d\} \in K(d)$
(3) $\{f_i : c_i \to c\} \in K(c) \iff$ for every i there exists a family $\{g_{ij} : d_{ij} \to c_i\} \in K(c_i)$ implies $\{f_i \circ g_{ij} : d_{ij} \to c\} \in K(c)$.

Here π_2 is the pullback of f_i along g.

2. Sheaves on a site

This "topologization" of a category $\mathcal{C}$ is essentially characterized by the sheaf condition. A site $(\mathcal{C}, J)$ can also refer to the category of sheaves $Sh(\mathcal{C}, J)$. Recall that for a topological space X, an open subset $U \subset X$, and any open cover $\{U_i\}$ of U, a presheaf $P : Open(X)^{op} \to Set$ is a sheaf if for any family $\{x_i \in P(U_i)\}$ such that when $x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$ implies there exists a unique $x \in P(U)$ such that $x|_{U_i} = x_i$ for every i indexing the open cover. We will define the gluing condition for sheaves in a similar way replacing intersections of open subsets by taking pullbacks along arrows in our covering.

Definition 2.0.1. Let S be a covering of $c \in \mathcal{C}$, and let $f_i : c_i \to c$ and $f_j : c_j \to c$ maps in S. For the pullback square (left) and its image under P (right)

$$\begin{array}{ccc} c_i \times_c c_j & \xrightarrow{h_{ij}} & c_j \\ \downarrow{\scriptstyle v_{ij}} & & \downarrow{\scriptstyle f_j} \\ c_i & \xrightarrow[f_i]{} & c \end{array} \qquad\qquad \begin{array}{ccc} P(c_i \times_c c_j) & \longleftarrow & P(c_j) \\ \uparrow & & \uparrow{\scriptstyle P(f_j)} \\ P(c_i) & \xleftarrow[P(f_i)]{} & P(c) \end{array}$$

we write $xf_i = P(f_i)(x)$ for $x \in P(c)$. Then we say a family $\{x_i \in P(c_i)\}_i$ such that $x_i v_{ij} = x_j h_{ij}$ for all i, j is a *matching family* if there exists a unique $x \in P(c)$ such that $xf_i = x_i$ for all i. In other terms,

$$P(c) \xrightarrow{x \mapsto xf_i} \prod_i P(c_i) \rightrightarrows \prod_{i,j} P(c_i \times_c c_j)$$

is an equalizer of sets, where the parallel maps are restrictions along v_{ij} and h_{ij}.

Definition 2.0.2. Let $\mathcal{C}$ be small, J a Grothendieck topology on $\mathcal{C}$, and $P : \mathcal{C}^{op} \to Set$ a presheaf. For a sieve S on c, a matching family assigns to $f \in S$ an element $x_f \in P(d)$ such that $x_f \cdot g = x_{fg}$ for all $g : e \to d$. (Here $x_f \cdot g := P(g)(x_f)$.) An *amalgamation* of a matching family is $x \in P(c)$ such that $xf = x_f$ for all $f \in S$. We say P is a *sheaf* or a J-sheaf if any matching family has a unique amalgamation.

Remark 2.0.3. P is a J-sheaf if and only if for all covering sieves S of c, every natural transformation $S \to P$ has a unique factorization through $S \hookrightarrow yc$ in that

$$\begin{array}{ccc} S & \longrightarrow & P \\ \big\downarrow & \nearrow & \\ yc & & \end{array}$$

This is the same as a natural bijection

$$\mathrm{Hom}(S, P) \cong \mathrm{Hom}(yc, P) \cong P(c)$$

where the second bijection is Yoneda and the composition is the matching family $f \mapsto x_f$.
Moreover, P is a sheaf if and only if

$$P(c) \xrightarrow{x \mapsto (x_f)_{f \in S}} \prod_{f \in S} P(domf) \rightrightarrows \prod_{f,g \in S, domf = codg} P(domg)$$

is an equalizer of sets where one parallel map is $(x_f) \mapsto (x_{fg})$ and the other is $(x_f) \mapsto (x_f \cdot g)$.

We denote the category of J-sheaves by $Sh(\mathcal{C}, J)$, which is a full subcategory of $PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Set)$. As we should hope, sheaf conditions can be checked on a basis for a Grothendieck topology.

3. Sites as left-exact localizations of presheaf categories

For a short digression, we refer to [Rez10]. Categories of presheaves and sheaves are (the only) examples of topoi. As functor categories into the universal topos Set, these categories are complete, cocomplete, and cartesian closed. By the above remark, it is clear that the inclusion $Sh(\mathcal{C}, J) \hookrightarrow PSh(\mathcal{C})$ has a left adjoint. That is, the inclusion is a fully faithful right adjoint, ie a reflexive subcategory. Indeed we may even *define* a site to be a certain localization of a presheaf category, as Rezk does.

Definition 3.0.1. Let $\mathcal{C}$ be a small category. A *site* $Sh(\mathcal{C}, J)$ is a reflexive subcategory of $PSh(\mathcal{C})$ such that the left adjoint to the inclusion $a : PSh(\mathcal{C}) \to Sh(\mathcal{C}, J)$ preserves limits.

It can be shown that this definition is equivalent to the previous definition of a site.

This won't be particularly important for the following examples, but it is a useful perspective for the wider context of topos in geometry and logic. Moreover, this ends up being the definition most suited to generalizations in higher categories. For an ∞-category $\mathcal{C}$, the Yoneda embedding takes values in the simplicial set $Fun(\mathcal{C}^{op}, \mathcal{S})$ where $\mathcal{S}$ is the ∞-category of spaces (Kan complexes). We then can define an

∞-topos to be a subcategory of a presheaf category where the inclusion is a right adjoint whose left adjoint preserves all homotopy limits.

4. Examples in algebraic geometry

We now want to examine a few examples of sites which are used in algebraic geometry. Throughout this section we will be working with the categories, and in most cases it doesn't much matter which one we choose

- $\mathrm{Aff}_{\mathbb{C}}$ affine varieties over the complex numbers.
- $\mathrm{Var}_{\mathbb{C}}$ varieties over $\mathbb{C}$.
- Aff_k (finite) affine schemes over a ring k.
- Sch_k schemes over a commutative (and better noetherian) ring k.
- Sch_S schemes over a noetherian scheme S.
- Et_S étale schemes over a base scheme S with étale maps between them.
- Sm_S smooth schemes over a smooth noetherian scheme S.

4.1. **The Zariski site.** The category $\mathrm{Aff}_{\mathbb{C}}$ is a subcategory of Top closed under finite limits and Zariski opens. We can take the Zariski open cover topology by taking as a basis

$$\{f_i : V_i \to X\} \in K(X) \iff f_i : V_i \hookrightarrow X \text{ open Zariski embedding and } X = \bigcup_i V_i$$

This generates a site, and can be done mutatis mutandis for any subcategory of Top which is closed under open subspaces and finite limits.
To generalize, let k be a commutative ring, and $k\text{-alg}_{fp}$ the category of finitely presented k-algebras. Under the equivalence

$$\mathrm{Aff}_k \simeq (k\text{-alg}_{fp})^{op}$$

and the lack of Nullstellensatz, we are now thinking of a site on $(k\text{-alg}_{fp})^{op}$ where our objects are not sets of points but algebras, and functions defined on points are replaced by sheaves on $(k\text{-alg}_{fp})^{op}$. So a cover of a k-algebra A is a finite list $a_1, \dots, a_n \in A$ such that $1_A \in (a_1, \dots, a_n)$. For a Grothendieck topology on Aff_k, we take a dual basis by

$$\{A \to A[a_i^{-1}] : i = 1, \dots, n\} \in K(A) \iff 1_A \in (a_1, \dots, a_n)$$

We claim this defines a Grothendieck topology.
Clearly $A \to A$ is a one element family so that $\{A' \xrightarrow{\cong} A\} \in K(A)$. Second, pullbacks in $(k\text{-alg}_{fp})^{op}$ are pushouts in $k\text{-alg}_{fp}$, ie tensor products. So if $1_A \in (a_1, \dots, a_n)$ and $h : A \to B$ is a homomorphism, then $1_B \in (h(a_1), \dots, h(a_n))$ and

$$B \otimes_A A[a_i^{-1}] \cong B[h(a_i)^{-1}]$$

Lastly suppose for every i we have a dual cover $\{A[a_i^{-1}] \to A[a_i^{-1}][c_{ij}^{-1}] : j = 1, \dots, m\}$ where $c_{ij} \in A[a_i^{-1}]$ and $\overline{1_A} \in (c_{i1}, \dots, c_{im})$. Then $c_{ij} = b_{ij}/a_i^{k_{ij}}$. We can rewrite the dual cover as

$$\{A[a_i^{-1}] \to A[a_i^{-1}][b_{ij}^{-1}] \xrightarrow{\sim} A[(a_i b_{ij})^{-1}]\}_{i,j}$$

with $\overline{1_A} \in (b_{i1}, \dots, b_{im})$. so there exists a K_i such that $a_i^{K_i} \in (a_i b_{i1}, \dots, a_i b_{im})$ for each i. Then we take $K > n \max_i\{K_i\}$, so that

$$1 = \left(\sum d_i a_i\right)^K \in (a_1^{K_1}, \dots, a_n^{K_n})$$

showing axiom (3) is satisfied.

This is the Zariski site on the category of affine schemes over k. This generalizes to the category of finite k-schemes by gluing compatibly with the sheaf condition and localization of algebras. All of this structure is stable under pullback, so we also have a Zariski site on the category Sch_S of schemes over a noetherian base scheme S.

4.2. **Étale topology.** Let k be a ring and X a k-scheme. We define an étale covering of X to be a family $\{f_i : X_i \to X\}_i$ such that each f_i is étale and $X = \bigcup_i f_i(X_i)$. Note that since open immersions are étale, any Zariski covering is étale. That is, the étale topology is finer than the Zariski topology.

Remark 4.2.1. Recall that a morphism of schemes $f : Y \to X$ is étale if for every $y \in Y$ there is an affine neighborhood $\mathrm{Spec}(B) = U \ni y$ and $V = \mathrm{Spec}(R) \subset X$ with $f(U) \subset V$ such that the ring map $R \to B$ is étale, meaning it is smooth and the cotangent complex $\Omega_{R/B} = 0$.

4.3. **Nisnevich topology.** When developing a homotopy theory for schemes, the Nisnevich topology is the appropriate topology. Since we usually work with smooth schemes, we will define the site (Sm_S, Nis).

Let U be a smooth (étale) noetherian scheme. We define a family $\{f_i : V_i \to U\}_i$ of morphisms in Sm_S to be a Nisnevich cover if each f_i is étale and if every field valued point $\mathrm{Spec}(F) \to U$ lifts to a field valued point in V_i. That is, for every $x \in U$ there exists $y_i \in V_i$ in the fiber over x such that the induced map of residue fields $\kappa(x) \to \kappa(y_i)$ is an isomorphism. We will talk more about étale morphisms and Kähler differentials in a few weeks.

Remark 4.3.1. The Nisnevich site is finer than the Zariski site but coarser than the étale site.

Remark 4.3.2. At the level of k-algebras, a family of étale homomorphisms of algebras $\{f_i : A \to A_i\}_{i \in I}$ is a Nisnevich covering of A if there exists a finite set $a_1, \dots, a_n \in A$ such that the following hold:

- $(a_1, \dots, a_n) = A$.
- For every $1 \le j \le n$ there exists $i \in I$ and a homomorphism g making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f_i} & A_i \\ \downarrow & \swarrow_{\exists g} & \\ A[a_j^{-1}]/(a_1, \dots, a_{j-1}) & & \end{array}$$

  commute [Lur11].

We are particularly interested in Nisnevich sheaves on a category of smooth schemes over a quasi-compact, quasi-separated, noetherian scheme S. For the usefulness of the Nisnevich site in homotopy theory, the following theorem gives a form of excision called Nisnevich excision.

Theorem 4.3.3. [MV99] *A presheaf of sets $F : Sm_S^{op} \to Set$ is a sheaf for the Nisnevich topology (written $F \in Sh(Sm_S, Nis)$) if and only if F preserves pullback squares of the form*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

with p étale, j an open embedding, and $p^{-1}(X \setminus U) \to X \setminus U$ an isomorphism. That is, the square of sets

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow Fj \\ F(V) & \xrightarrow[Fp]{} & F(U \times_X V) \end{array}$$

is a pullback.

Proof. First, let $\{U \xrightarrow{j} X, V \xrightarrow{p} X\}$ be a Nisnevich covering with j an open embedding and p étale. Since

$$(U \sqcup V) \times_X (U \sqcup V) = (V \times_X V) \sqcup (U \times_X V) \sqcup (V \times_X U) \sqcup (U \times_X U)$$

we get that

$$\{V \xrightarrow{\Delta} V \times_X V, U \times_X V \times_X V \to V \times_X V\}$$

is a Nisnevich covering for $V \times_X V$. Therefore the square of sets

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U \sqcup V) \\ \downarrow & & \downarrow \\ F(U \sqcup V) & \longrightarrow & F((U \sqcup V) \times_X (U \sqcup V)) \end{array}$$

is a pullback.

Conversely, let $F : Sm_S^{op} \to Set$ be a functor taking preserving such pullbacks. Let $W = \{W_i \to X\}_i$ be a Nisnevich covering of X which is assumed noetherian. A sequence of closed subschemes

$$\emptyset = Z_{n+1} \subset Z_n \subset \cdots \subset Z_0 = X$$

is a W-splitting if the map

$$(\sqcup p_i)^{-1}(Z_i \setminus Z_{i+1}) \to Z_i \setminus Z_{i+1}$$

splits for every i. Noetherian schemes always have such a splitting sequence [MV99]. Since p is étale, and $p^{-1}(Z_n) \to Z_n$ has a splitting s, we have $p^{-1}(Z_n) = im(s) \sqcup Y$ where $Y \subset \bigsqcup W_i$ is closed. Set $U = X \setminus Z_n$ and $V = \bigsqcup W_i \setminus Y$, so that we have get a pullback square in Sm_S as in the theorem. Then by the assumption that F preserves such pullbacks, we have that

$$F(X) \to F(U) \times F(V) \rightrightarrows F(U \times_X V)$$

and

$$F(U) \to \prod_i F(W_i \times_X U) \rightrightarrows \prod_{i,j} F(W_i \times_X W_j \times_X U)$$

are equalizers, and F is a Nisnevich sheaf. □

Remark 4.3.4. We want to emphasize that the Nisnevich site is really the correct one for developing a homotopy theory of schemes. We list without proof a number of desirable properties of the Nisnevich site.

(1) The Nisnevich site has Mayer-Vietoris sequences for cohomology.
(2) If $\dim S \leq d$ and F is a Nisnevich sheaf of abelian groups on $(Sm_S)_{Nis}$, then $H^i_{Nis}(S, F) = 0$ for $i > d$ [MV99]. In particular, field-valued points "look like points" in a homotopical sense.
(3) Closed immersions $U \hookrightarrow X$ in $(Sm_S)_{Nis}$ are locally of the form $\mathbb{A}_k^{\dim U} \hookrightarrow \mathbb{A}_k^{\dim X}$.
(4) The Nisnevich site has descent for algebraic K-theory [Lur11].
(5) Cohomology for Nisnevich sheaves can be computed from the Čech complex.

Property (3) does not hold in the Zariski topology. Consider the embedding $\mathrm{Spec}(\mathbb{C}) \hookrightarrow \mathbb{A}^1_{\mathbb{R}}$. And property (2) does not hold in the étale topology. The étale cohomological dimension of $\mathrm{Spec}(\mathbb{R})$ is infinite.

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