BASS-QUILLEN CONJECTURE

THOMAS ARNSTEIN

Abstract. We return to the study of $\mathbb{A}^1$-invariance vector bundles after seeing a non-example in week 6. The Bass-Quillen conjecture suggests a local regularity hypothesis to obtain a desired statement on invariance. We'll then look at the contribution of Lindel for the case of a smooth algebra over a field.

Contents

1. Roitman's theorem

Recall from lecture 6, Example 5.0.4 that there exist vector bundles on $\mathbb{P}^1 \times_k \mathbb{A}^1$ which are not extended from a vector bundle on $\mathbb{P}^1$. In particular we have a rank 2 vector bundle $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ such that $\mathcal{F}|_{\mathbb{P}^1 \times \{0\}}$ is trivial and $\mathcal{F}|_{\mathbb{P}^1 \times \{1\}} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$. The Bass-Quillen conjecture suggests that, at least locally, some obstruction could be overcome with a regularity condition.

Conjecture 1.1. For a regular commutative ring R and for all $n \geq 1$, there is a bijection

$$Vect_r(\mathrm{Spec}(R)) \cong Vect_r(\mathrm{Spec}(R[x_1, \dots, x_n]))$$

By regular ring, we mean a ring such that the localization at every prime ideal is a regular local ring. Let's recall some characterizations of a regular local ring:

Definition 1.2. A noetherian local ring $(R, \mathfrak{m})$ of Krull dimension d is a regular local ring if any of the following equivalent statements hold.

(1) d is equal to the minimal number of generators of the maximal ideal $\mathfrak{m}$.
(2) $d = \dim_k \mathfrak{m}/\mathfrak{m}^2$ the dimension of the Zariski tangent space at $\mathfrak{m}$.
(3) Every R-module has a projective resolution of length $\leq d$.
(4) The completion $\widehat{R} \cong R_\mathfrak{m}[[x_1, \dots, x_d]]$.

Example 1.3. An example of a non-regular ring is $R = \mathbb{Q}[x, y]/(y^2 - x^3)$ at the origin $\mathfrak{m} = (x, y)$. The Krull dimension $\dim_\mathbb{Q}(R_\mathfrak{m}) = 1$ but $\mathfrak{m}$ cannot be written with fewer than two generators. That is, (x, y) is a regular system of parameters for R at $\mathfrak{m}$.

We recall (a mild generalization of) Quillen's patching theorem to work locally. To fix terminology, for a ring homomorphism $f : R \to S$ we say that an S-module M is extended from R if there exists an R-module N such that $N \otimes_R S \cong M$. Geometrically, this is to say that the vector bundle M on $\mathrm{Spec}(S)$ is obtained by pulling back the vector bundle N on $\mathrm{Spec}(R)$.

Lemma 1.4. *Let R be commutative and let M be a finitely presented $R[t_1, \dots, t_n]$-module. If $M_\mathfrak{m}$ is extended from an $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m} \subset R$, then M is extended from an R-module.*

Date: March 6, 2026.

Remark 1.5. Recall that Quillen-Suslin says that every finitely generated projective $R[x_1, \dots, x_n]$-module is free when R is a PID. In the language of Lemma 1.4, this is the statement that if R is a PID, then every $R[x_1, \dots, x_n]$-module is extended from R.

We have a theorem of Roitman to address the converse at localizations at general multiplicative subsets.

Theorem 1.6. *Let R be commutative and $S \subset R$ multiplicative set. Let $n \geq 1$. If every finitely generated projective $R[x_1, \dots, x_n]$-module is extended from R, then every finitely generated projective $R[S^{-1}][x_1, \dots, x_n]$-module is extended from $R[S^{-1}]$.*

Proof. By induction, it suffices to demonstrate the case $n = 1$. Suppose every finitely generated projective $R[x]$-module is extended from R and let P be a finitely generated projective $R[S^{-1}][x]$-module. Without loss of generality we can say P is a finitely generated $R_\mathfrak{p}[x]$-module for some prime ideal $\mathfrak{p} \subset R$. This step relies on Quillen's patching plus the observation that for any maximal $\mathfrak{m} \subset R[S^{-1}]$, there exists a prime $\mathfrak{p} \subset R$ such that $(R[S^{-1}])_\mathfrak{m} \cong R_\mathfrak{p}$ and we may pass to local rings at a prime without loss of generality. To show that P is free, consider the $R_\mathfrak{p}[x]$-linear map

$$f(x) : R_\mathfrak{p}[x]^{\oplus n} \to R_\mathfrak{p}[x]^{\oplus n}$$

with $\ker(f(x)) \oplus P \cong R_\mathfrak{p}[x]^{\oplus n}$. We may view $f(x)$ as a composition

$$R_\mathfrak{p}[x]^{\oplus n} \xrightarrow{\pi} P \xrightarrow{s} R_\mathfrak{p}[x]^{\oplus n}$$

where s is a section of π (existing by the projectivity assumption on P). It is also clear that $f(x)$ is idempotent as $(\pi \circ s) \circ (\pi \circ s) = \pi \circ id_P \circ s = \pi \circ s$. Denoting $\operatorname{rank}(P) = k$, it suffices to show that $f(x)$ is conjugate to the matrix $\operatorname{diag}(1, \dots, 1, 0, \dots, 0)$ with k non-zero entries, now working with $f(x) \in M_n(R_\mathfrak{p}[x])$. In particular, for every $\alpha \in R_\mathfrak{p}$, we have $f(\alpha) \in M_n(R_\mathfrak{p})$ is idempotent. Since $R_\mathfrak{p}$ is local, every idempotent in $M_n(R_\mathfrak{p})$ is conjugate to a standard idempotent $\operatorname{diag}(1, \dots, 1, 0, \dots, 0)$. To check that $f(x)$ is conjugate to a standard idempotent in the matrix ring over $R_\mathfrak{p}[x]$, we note that there exists a $t \in R \setminus \mathfrak{p}$ such that $e(tx) \in im(M_n(R[x]) \to M_n(R_\mathfrak{p}[x]))$ where the map is induced by the localization $R \to R_\mathfrak{p}$. So there is a matrix $e_0 \in M_n(R[x])$ such that $(e_0)_\mathfrak{p} = e(tx)$ and $e_0(0)$ a standard idempotent. Then there exists $u \in R \setminus \mathfrak{p}$ such that $u(e_0^2 - e_0) = 0$ hence $e_0^2(ux) = e_0(ux)$ (since x is not a zero-divisor). Then we can conjugate by $p(x) \in GL_n(R[x])$ to obtain

$$p^{-1}(x)e_0(ux)p(x) = e_0(0)$$

where we localize the coefficients of $p(x)$. Therefore, in $M_n(R_\mathfrak{p}[x])$, we have

$$p^{-1}(x)e(tux)p(x) = e(0) = p^{-1}(x/tu)e(x)p(x/tu)$$

So $f(x)$ is conjugate to a standard idempotent, thus P is extended from $R_\mathfrak{p}$. □

2. Bass-Quillen conjecture for smooth algebras over a field

Recall that a Nisnevich cover for a scheme X is generated by maps $U_i \to X$ which are étale and lift to isomorphisms on residue fields. Similary, we define a *Nisnevich neighborhood of* $x \in X$ to be an étale map $f : U \to X$ and a $u \in U$ such that $f(u) = x$ and $\kappa(x) \xrightarrow{\simeq} \kappa(u)$. Moreover, for a scheme X of finite type and a point $x \in X$ with residue field κ and any separable extension k/κ, there exists an étale neighborhood U of x with $u \in U$ with residue field k realizing this extension. In particular there exists a Nisnevich neighborhood of $x \in X$ whenever X is finite type. We have the stronger statement due to [Lin81].

Theorem 2.1. *Let k be a field and A a finite dimensional regular k-algebra with* $\dim A = d \geq 1$. *Let $\mathfrak{m} \subset A$ a maximal ideal such that the residue field $\kappa = A/\mathfrak{m}$ is a simple separable extension of k. Then there exists a polynomial ring $B = k[x_1, \dots, x_d]$ and a maximal ideal $\mathfrak{n} \subset B$ such that $A_\mathfrak{m}$ is a Nisnevich neighborhood of $B_\mathfrak{n}$ with $\mathfrak{n} = \mathfrak{m} \cap B$.*

The theorem can be restated as saying there is a local étale map $A_\mathfrak{m} \to B_\mathfrak{n}$ such that $A/\mathfrak{m} \cong B/\mathfrak{n}$ and that this residue field is a simple separable extension of the ground field k. In other words, a regular local ring containing a field k with residue a separable extension of k is a Nisnevich neighborhood of a localization of a polynomial ring at a maximal ideal. Note that the fractions fields of A and B both have transcendence degree d over k. And the fraction field of A is an extension of the fraction field of B, so this is an algebraic extension.

Remark 2.2. Generally a regular local ring containing a field k need not have its residue field at the maximal ideal be a separable extension. We could guarantee this with the additional hypothesis that we work with regular varieties over a perfect field, since the residue field will be a finite algebraic extension, and every algebraic extension over a perfect field is separable. With these further assumptions we obtain the following [Aso21].

Corollary 2.3. *Let k be a perfect field and A be a localization of regular finite-type algebra over k of dimension d. Let κ denote the residue field of A at $\mathfrak{m}$. If $\mathfrak{n}$ is the maximal ideal $\mathfrak{n} = \mathfrak{m} \cap B$ in the polynomial ring $B = \kappa[x_1, \ldots, x_d] \subset A$, then the induced map $B_\mathfrak{n} \to A_\mathfrak{m}$ is an étale local ring map.*

Moreover, we can replace étale neighborhood with étale cover in the above cases. If

$$\begin{array}{ccc} S & \longrightarrow & S_f \\ \downarrow & & \downarrow \\ R & \longrightarrow & R_f \end{array}$$

is an étale cover of S, then for any $n \geq 0$, using étale descent,

$$\begin{array}{ccc} S[x_1, \ldots, x_n] & \longrightarrow & S_f[x_1, \ldots, x_n] \\ \downarrow & & \downarrow \\ R[x_1, \ldots, x_n] & \longrightarrow & R_f[x_1, \ldots, x_n] \end{array}$$

is an étale cover of $S[x_1, \ldots, x_n]$. We can therefore view projective $S[x_1, \ldots, x_n]$-modules as a pair of projective $R[x_1, \ldots, x_n]$- and $S_f[x_1, \ldots, x_n]$-modules which agree upon extension of scalars to $R_f[x_1, \ldots, x_n]$. In the situation of the previous corollary, we could take $f \in \mathfrak{n}$ to obtain an étale cover.

The strongest worked out case of the Bass-Quillen conjecture is the following, which drops the above assumption that k is perfect. Through fairly elementary means, we reduce to the case that k is perfect.

Theorem 2.4. *Let R be a regular k-algebra of finite type, where k is a field. Then for every $r, n \geq 0$, we have*

$$Vect_r(R) \to Vect_r(R[t_1, \ldots, t_n])$$

is a bijection.

Proof. We reduce to the case that k is a perfect field so that the previous results prove the claim. We write $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. If P is a finitely generated projective R-module, then P is in the image of an idempotent endomorphism of $R^{\oplus N}$ for some N. Denote $k' \subset k$ the subfield of k generated by the coefficients of the f_is and $k_0 \subset k$ the subfield generated by the multiplicative identity 1. Note k_0 is surely a perfect field (we have $k_0 \cong \mathbb{Q}$ if $\mathrm{char}(k) = 0$ and $k_0 \cong \mathbb{F}_p$ if $\mathrm{char}(k) = p$). Setting

$$R' = k'[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

and denote P' to extension of P. Note that the map $R' \to R' \otimes_{k'} k \cong R$ is faithfully flat since $k' \to k$ is faithfully flat and $- \otimes_{k'} k = - \otimes_{R'} R$. Therefore R' is also regular and k' is a finite extension of the prime subfield $k_0 \subset k$. This means that R' is essentially of finite type over k_0, which is a perfect field. We have reduced the claim to the case when k is perfect, completing the proof of the theorem. □

References

[Aso21] Aravind Asok. Algebraic geometry from an $\mathbb{A}^1$-homotopic viewpoint, May 3, 2021.

[Lin81] Hartmut Lindel. On the bass-quillen conjecture concerning projective modules over polynomial rings. *Inventiones mathematicae*, 65:319–324, 1981.