HOMOTOPY CLASSIFICATION OF ALGEBRAIC VECTOR BUNDLES

Abstract. In this note, we define naive $\mathbb{A}^1$-homotopy classes of maps between affine schemes as a direct generalization of the notion of homotopy from topology. Just like in the latter, we can then give a homotopy classification of algebraic vector bundles by maps into Grassmannian.

1. Naive $\mathbb{A}^1$-homotopy

Recall that in topology, vector bundles can be classified upto isomorphism as follows.

Theorem 1.1. *Let X be a paracompact space and let $\mathrm{Vect}_n(X)$ be the isomorphism classes of vector bundles over X. Let Gr_n be the complex Grassmannian of n-planes and let $\mathcal{E} \to \mathrm{Gr}_n$ be the universal n-plane bundle over Gr_n. Then there is a natural bijection*

$$[X, \mathrm{Gr}_n] \cong \mathrm{Vect}_n(X)$$

given by $f : X \to \mathrm{Gr}_n$ mapping to $f^*\mathcal{E}$.

In particular, any map $f : X \to \mathrm{Gr}_n$ uniquely determines a vector bundle of rank n upto homotopy on X. Our goal is to generalize this in order to prove a similar result in algebraic geometry.

2. Naive $\mathbb{A}^1$-homotopies

Let $f, g : X \to Y$ be two morphisms of affine schemes $X = \mathrm{Spec}\,(B)$ and $Y = \mathrm{Spec}\,(A)$ over k, corresponding to $\varphi, \psi : A \to B$ respectively. A *naive $\mathbb{A}^1$-homotopy between f and g* is a morphism $H : X \times \mathbb{A}^1 \to Y$ such that $H_0 = f$ and $H_1 = g$. In terms of algebra, naive $\mathbb{A}^1$-homotopy is a homomorphism α

$$\alpha : A \longrightarrow B \otimes_k k[t] = B[t]$$

such that

$$\begin{tikzcd}
 & B \\
A \arrow[ur, "\varphi"] \arrow[r] \arrow[dr, "\psi"'] & B \otimes_k k[t] \cong B[t]\cdot \arrow[u, "t\mapsto 0"'] \arrow[d, "t\mapsto 1"] \\
 & B
\end{tikzcd}$$

One says that f and g are connected by homotopy H.

Lemma 2.1. *The relation of being connected by a naive $\mathbb{A}^1$-homotopy is reflexive and symmetric.*

Proof. Suppose $f, g : X \to Y$ are two maps. Clearly, f is connected to f via the homotopy $(x, t) \mapsto f(x)$. Moreover, it is symmetric as if $H : f \to g$, then $\bar{H}(x, t) = H(x, 1 - t)$ is a naive $\mathbb{A}^1$-homotopy connecting $g \to f$. □

Example 2.2 (Naive $\mathbb{A}^1$-homotopy is not transitive)**.** Consider $Y = \mathrm{Spec}\,(k[x, y]/xy)$, $X = \mathrm{Spec}\,(k)$ and consider three morphisms $f, g, h : X \to Y$ mapping pt. $\mapsto (0, 1), (0, 0), (1, 0)$ respectively. Observe that there are naive homotopies $f \to g$ and $g \to h$; consider $H_{f\to g} : \mathbb{A}^1 \to Y$ given by $t \mapsto (0, 1-t)$ and $H_{g\to h} : \mathbb{A}^1 \to Y$ given by $t \mapsto (t, 0)$.

We claim that there is no map

$$H_{f\to h} : \mathbb{A}^1 \longrightarrow Y$$

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such that $H_{f\to h}(0)=(0,1)$ and $H_{f\to h}(1)=(1,0)$. Indeed, this would correspond to homomorphism

$$k[x,y]/(xy)\longrightarrow k[t],$$

which should map the $\bar{x}\mapsto\alpha(t)$ and $\bar{y}\mapsto\beta(t)$ such that $\alpha(t)\beta(t)=0$, i.e. one of $\alpha(t)$ or $\beta(t)$ must be 0. But since $f(\text{pt.})=(1,0)$ and $h(\text{pt.})=(0,1)$, we deduce that there is no such morphism $H_{f\to h}$.

Hence we define f and g to be *naively* $\mathbb{A}^1$*-homotopic* if they are equivalent in the equivalence relation generated by naive $\mathbb{A}^1$-homotopy. This means that there is a sequence of maps $(f_k)_{0\le k\le n}$ such that $f_0=f$ and $f_n=g$ with $H_k: f_k\to f_{k+1}$ a naive homotopy connecting f_k to f_{k+1}. We denote the equivalence classes of naive $\mathbb{A}^1$-homotopy between X and Y by $[X,Y]_N$. We say two affine schemes X,Y are *naively* $\mathbb{A}^1$*-homotopy equivalent* if there are morphisms $f:X\to Y$ and $g:Y\to X$ such that $f\circ g$ is naively homotopic to id_Y and $g\circ f$ is naively homotopic to id_X. In this case we write $X\simeq_N Y$.

Lemma 2.3. *For any affine scheme X, we have*

$$X\simeq_N X\times\mathbb{A}^n.$$

Proof. Suffices to show that the map $f:X\times\mathbb{A}^n\to X\times\mathbb{A}^n$, $(x,s)\mapsto(x,0)$ is naively homotopic to $\mathrm{id}_{X\times\mathbb{A}^n}$. This is immediate by the naive homotopy $(t,(x,s))\mapsto(x,ts)$. ☐

Let $\mathcal{A}\mathrm{ff}_k$ be the category of affine schemes over k. A functor $F:\mathcal{A}\mathrm{ff}_k^{\mathrm{op}}\to\mathcal{C}$ is an $\mathbb{A}^1$*-invariant* if for the projection map $p:X\times\mathbb{A}^1_k\to X$ for any $X\in\mathcal{A}\mathrm{ff}_k$ induces an isomorphism $F(X)\to F(X\times\mathbb{A}^1_k)$.

Lemma 2.4. *If F is a $\mathcal{C}$-valued $\mathbb{A}^1$-invariant and $f,g:X\to Y$ are naively equivalent, then $F(f)=F(g)$.*

One can define the *naive* $\mathbb{A}^1$*-homotopy category* over a field k as the category $\mathcal{H}_N(k)$ whose objects are affine varieties over k and morphisms are equivalence classes of naive $\mathbb{A}^1$-homotopy equivalent maps.

Lemma 2.5. *If X and Y are affine varieties over k, then the canonical map*

$$[X,Y]_N\longrightarrow[X\times\mathbb{A}^1,Y]_N$$
$$f\mapsto f\circ\pi$$

for $\pi:X\times\mathbb{A}^1\to X$ projection induces a natural bijection.

Proof. Injectivity is clear by inclusion map $X\times\mathbb{A}^1\to\mathbb{A}^1\times(X\times\mathbb{A}^1)$, $(x,s)\mapsto(s,(x,s))$. For surjectivity, observe that if $H:X\times\mathbb{A}^1\to Y$ is a map, then it is sufficient to show that it is homotopic to the map $H_0:X\times\mathbb{A}^1\to Y$, $(x,s)\mapsto H(x,0)$. Indeed, the map $K:\mathbb{A}^1\times(X\times\mathbb{A}^1)\to Y$ mapping $(t,(x,s))\mapsto H(x,st)$ is enough. ☐

3. Grassmannian

3.1. **Scheme structure.** Recall we have the map

$$\mathrm{Gr}_n(N)\hookrightarrow\mathbb{P}\left(\wedge^n V\right)\cong\mathbb{P}^{\binom{N}{n}-1}$$
$$\Lambda\mapsto[v_1\wedge\cdots\wedge v_n]$$

where $v_1,\dots,v_n\in\Lambda$ is a basis. This defines a closed subspace of $\mathbb{P}(\wedge^n V)$ because it is given by

$$\mathrm{Gr}_n(N)=\left\{\eta\in\wedge^n V\mid\dim\mathrm{Ker}\left(V\overset{\wedge\eta}{\to}\wedge^{n+1}V\right)\ge n\right\}$$

3.2. **Maps to** Gr_n**.** Let $f:X\to\mathrm{Gr}_n$ be a morphism to the infinite Grassmannian of n-planes. Let $\mathcal{V}^n\to\mathrm{Gr}_n$ be the canonical n-plane bundle over Gr_n. For a map $f:X\to\mathrm{Gr}_n$, we construct a vector bundle $\mathcal{E}\to X$ of rank n over X. Indeed, by property of colimits, we must have that the map f factors as $X\to\mathrm{Gr}_n(N)\hookrightarrow\mathrm{Gr}_n$ for some large N. Hence, we get a vector bundle $\mathcal{E}\to X$ together with a surjection $\epsilon^N\twoheadrightarrow\mathcal{E}$ where ϵ^N is the trivial bundle of rank N over X. If X is affine, this corresponds to a map $R^N\twoheadrightarrow P$ where P is projective of rank n with N generators, say $e_1,\dots,e_N\in P$.

Conversely, if instead we have a surjection $R^N\twoheadrightarrow P$, we can construct a map $X\to\mathrm{Gr}_{n,N}$ as follows. Find a Zariski cover of $X = \mathrm{Spec}\,(R)$ given by $\{D(f_i)\}_i$ such that P trivializes over $D(f_i)$. Hence we have that the composite $R^n_{f_i} \to R^N_{f_i} \to P_{f_i}$ is an isomorphism. Thus we get a map

$$D(f_i) \longrightarrow \mathbb{A}^{n(N-n)}$$

which we can glue to get a map $X \to \mathrm{Gr}_n$. To conclude, we have a surjective map

$$\mathrm{Hom}\,(X, \mathrm{Gr}_n) \twoheadrightarrow \mathrm{Vect}_n(X).$$

4. Homotopy classification

Theorem 4.1. *Let k be a field and X be a smooth affine k-scheme, then the map*

$$[X, \mathrm{Gr}_n]_N \longrightarrow \mathrm{Vect}_n(X)$$

is an isomorphism.

Proof. Surjectivity is clear from previous. For injectivity, pick two maps $f, g : X \to \mathrm{Gr}_n$ which leads to the same vector bundle of rank n over $X = \mathrm{Spec}\,(R)$, i.e. a projective module P of rank n with f defining r generators $e_1, \dots, e_r \in P$ and g defining s generators $f_1, \dots, f_s \in P$. It suffices to show that there is a homotopy connecting f and g. To this end, it is equivalent to finding a 1-parameter family of maps $X \to \mathrm{Gr}_n$ connecting f to g. By equivalence with projective modules and generators, we must have a one parameter family of projective modules over R interpolating between $(P, (e_1, \dots, e_r))$ and $(P, (f_1, \dots, f_s))$. As $X \to \mathrm{Gr}_n$ can be constructed by constructing $X \to \mathrm{Gr}_n(N)$ for $N = r + s$, therefore we reduce to constructing a 1-parameter family of projective R-modules P_t interpolating between $(e_1, \dots, e_r, 0, \dots, 0)$ and $(0, \dots, 0, f_1, \dots, f_s)$. We can easily construct this by considering the generators $(e_1, \dots, e_r, tf_1, \dots, tf_s)$ and $((1-t)e_1, \dots, (1-t)e_r, f_1, \dots, f_s)$. □