Note on projective modules

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Winter 2026

Last seminar meeting, we went over the definition and basic properties of Zariski descent data. Our ultimate goal is to use these ideas to study projective modules over a ring. In this note, we'll take a look at some specific examples. Nontrivial projective modules are somewhat time to come by.

1 Example One

Let $A = \mathbb{Z}[\sqrt{-5}]$. This is the standard example of a quadratic ring that does not satisfy the unique factorization property. An open cover of A is given by the rings $A_1 = A[1/2]$ and $A_2 = A[1/3]$. Let M_i equal A_i, considered as a module over itself.

Descent data for the modules M_1, M_2 is just an isomorphism $\phi\colon A_{1,2} \otimes_{A_1} M_1 \cong A_{1,2}, \otimes_{A_2} M_2$ where $A_{1,2} = A[1/6]$. Any descent data we construct automatically corresponds to a projective A-module (rather than a more complicated A-module).

Both $A_{1,2} \otimes_{A_1} M_1$ and $A_{1,2}, \otimes_{A_2} M_2$ are free modules (transitivity of the tensor product), so any isomorphism ϕ must be of the form $\phi(r) = f_0 r$ for some fixed $f_0 \in A_{1,2}$.

If we take $f_0 = 1$, then the descent data corresponds to the free module A. The same is true if we take f_0 equal to a unit in A.

A nontrivial example of descent data is given by $f_0 = (1 - \sqrt{-5})/3$. This element is a unit in $A_{1,2}$ since

$$(1 + \sqrt{-5})/2 \cdot (1 - \sqrt{-5})/3 = 1$$

By construction, this descent data correspondents to the submodule M of $M_1 \oplus M_2$ consisting of elements (x, y) satisfying $y - (1 - \sqrt{-5})/3 \cdot x = 0$. Two elements that satisfy this condition are $(1, (1 - \sqrt{-5})/3)$ and $((1 + \sqrt{-5})/2, 1)$.

The elements $(1, (1-\sqrt{-5})/3)$ and $((1+\sqrt{-5})/2, 1)$ are, in fact, generators for M. This module is nothing other than the ideal $I = (2, 1 + \sqrt{-5})$ with an isomorphism being given by $I \to M$, $a \mapsto (a/2, a(1 - \sqrt{-5})/6)$. The ideal is the standard example of a nonprincipal ideal, so M is not isomorphic to A. This our desired example of a projective module that is not free.

In passing, observe that the isomorphism classes of rank 1 projective modules arising from descent data on the modules M_1 and M_2 is the quotient $A_{1,2}^*/G$ where G is the group of elements of the form a/b with $a \in A_1^*$ and $b \in A_2^*$. I think this group is isomorphic to $\mathbb{Z}/2$, and I'd be interesting to see this directly.

2 Example Two

A more geometric example is provided by $A = \mathbb{C}[x, y]/y^2 - x^3 - x^2 - x$. Let $A_1 = A[1/x]$ and $A_2 = A[1/(x^2 + x + 1)]$. These rings define an open cover of $\mathrm{Spec}(A)$ since $(x^2 + x + 1) - x(x + 1) = 1$. Again, let $M_i = A_i$ be the free module. Descent data is just an isomorphism $\phi\colon A_{1,2} \otimes_{A_1} M_1 \to A_{1,2} \otimes_{A_2} M_2$ that is necessarily of the form $\phi(r) = f_0 r$.

Consider the descent data defined by $f_0 = y/(x^2 + x + 1)$ and the associated A-module M. (The element y/x is a unit in $A_{1,2}$ since its inverse is y/x.). The module M is the submodule of $M_1 \oplus M_2$ generated by $(1, y/(x^2 + x + 1))$ and $(y/x, 1)$. The module is isomorphic to the ideal I generated by x and y. This module is not free although this is harder to prove than in the previous example.

Just as in the previous example, the set of rank 1 projective modules arising from descent data on the modules M_1 and M_2 is naturally identified with the cokernel of the map $A_1^* \times A_2^* \to A_{1,2}^*, (a, b) \mapsto a/b$. The element $y/(x^2 + x + 1)$ that defines M has order two in this quotient since

$$\begin{aligned} y^2/(x^2 + x + 1)^2 =& x^3 + x^2 + x/(x^2 + x + 1)^2 \\ =& x/(x^2 + x + 1), \end{aligned}$$

and the last element is the image of $(x, x^2 + x + 1) \in A_1^* \times A_2^*$.

3 Example Three

An example motivated by topology can be constructed over the ring $A = \mathbb{R}[x, y, z]/x^2 + y^2 + z^2 - 1$ as follows. Let $A_1 = A[1/x]$, $A_2 = A[1/y]$, $A_3 = A[1/z]$. Unlike the previous cases, we take $M_i = A_i^{\oplus 2}$ to be free of rank 2. The descent data $\phi_{i,j}\colon M_i \otimes_{A_i} A_{i,j} \to M_j \otimes_{A_j} A_{i,j}$ are defined by

$$\begin{aligned} \phi_{1,2}(r, s) &= (-rx/y, s - rz/y), \\ \phi_{1,3}(r, s) &= (-sx/z, r - sy/z), \\ \phi_{2,3}(r, s) &= (r - sx/z, -sy/z). \end{aligned}$$

These homomorphisms have the property that $\phi_{2,3} \circ \phi_{1,2}$ and $\phi_{2,3}$ coincide as homomorphisms $M_1[1/(xyz)] \to M_3[1/(xyz)]$. We conclude that $\{\phi_{i,j}\}$ defines descent data.

By construction, the module M is the kernel of a homomorphism $M_1 \times M_2 \times M_3 \to M_{1,2} \times M_{1,3} \times M_{2,3}$ constructed using the isomorphisms $\phi_{i,j}$. The module is, in fact, $M = R^{\oplus 3}/R \cdot (x, y, z)$.

This example has a geometric origin. The ring A is the ring of polynomial functions on the 2-sphere $S^2 \subset \mathbb{R}^3$, and M is the module of polynomial cotangent fields. A standard topological result is that the cotangent bundle to S^2 is nontrivial. One can deduce from this the fact that M is not isomorphic to $R^{\oplus 2}$. I don't know a purely algebraic proof of this fact.

Incidentally, M is an example of a module that is stable free but not free. In other words, while M is not free, the direct sum of M and a free module is free. In fact, an isomorphism $R^{\oplus 3} \cong M \oplus R$ is defined by $(a, b, c) \mapsto (q(a, b, c), ax + by + cz)$, where q is the quotient map. The key observation is that this homomorphism sends (x, y, z) to $(0, x^2 + y^2 + z^2) = (0, 1)$.

Is it important that we work over the real numbers? I'm guessing it is, and if replaced $\mathbb{R}$ with $\mathbb{C}$, then the construction produces $R^{\oplus 2}$. Just a guess, though.