Singular Spaces

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1 Stratified spaces

1.0.1 (Stratified spaces)**.** We would like to understand the topology of singular spaces. The fundamental definition is that of a stratified pseudomanifold. An **n-dimensional stratified topological space** X is a space with a closed filtration $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_1 \supseteq X_0$ where we call each connected component of the difference $X_j \setminus X_{j-1}$ the **j^{th}-strata** such that for each point $x \in X_j \setminus X_{j-1}$, there exists an open neighborhood $x \in N_x \subseteq X$ and an $n-j-1$-dimensional compact stratified space L called the **link** with a homeomorphism $\phi : N_x \to \mathbb{R}^j \times C(L)$ where $C(L) = L \times [0,1)/L \times \{0\}$ and ϕ must take $N_x \cap X_{j+i+1}$ into $\mathbb{R}^j \times C(L_i)$ for each $n-j-1 \geq i \geq 0$ and $N_x \cap X_j$ maps into $\mathbb{R}^j \times \{\text{vertex of } C(L)\}$. In particular each difference $X_j \setminus X_{j-1}$ is a j-dimensional manifold. Thus each j-strata is a j-dimensional connected topological manifold. We call this latter condition for points in j^{th}-strata to be the *local conical triviality condition for j^{th}-strata.*

The way to think about such spaces is that we are trying to "filter" all the singular points of the space by their dimension in which they occur.

1.0.2. A **topological pseudomanifold of dimension** n is a paracompact, Hausdorff stratified space X such that $X_{n-1} = X_{n-2}$ (no codimension 1 singularity) and $X \setminus X_{n-1}$ is dense in X (the **non-singular stratum** is dense). The **singular locus of** X is X_{n-2}.

1.0.3. Some trivial examples of pseudomanifolds are $S^n \vee S^n$ (dimension n), pinched torus, etc. A non-example is $S^1 \vee S^1$ as it has a codimension 1 singularity, T^2 with a disc covering its central hole by the same reason is not a pseudomanifold.

2 Simplicial & singular intersection homology

2.1 Definitions

We begin with simplicial intersction homology.

2.1.1 (Recollections on simplicial homology)**.** Recall that a *simplicial complex* is a collection N of simplices in a fixed Euclidean space $\mathbb{R}^n$ such that for each $\sigma \in N$, each of the $n+1$-faces of σ is again in N and any two simplices can only intersect in another simplex; no non-trivial intersection between simplices allowed. The *support* of simplicial complex N in $\mathbb{R}^n$ is just the union of all simplices of N in $\mathbb{R}^n$. It is denoted by $|N|$. A *triangulation* of a space X is a tuple (N, φ) where N is a simplicial complex and $\varphi : |N| \to X$ a homeomorphism.

Fix a field F and an oriented simplicial complex N (an ordering on vertices of σ for each $\sigma \in N$). Denote $N^{(k)} = \{\sigma \in N \mid \sigma \text{ is a } k\text{-simplex}\}$. Let $C_k(N)$ be the free F-vector space generated by $N^{(k)}$ as a basis. An element $\xi \in C_k(N)$ is called a *simplicial k-chain.* Define $\partial : C_k(N) \to C_{k-1}(N)$ on basis by sending $\sigma \mapsto \sum_{i=0}^{k}(-1)^i \partial_i \sigma$ where i is even iff the chosen orientation on $\partial_i \sigma \in N$ is same as the one obtained by $v_0, \dots \widehat{v_i}, \dots, v_k$. This determines a chain complex $(C_k(N), \partial)$, called the *simplicial chain complex.* The homology of this complex is defined to be the simplicial homology of N, denoted $H_i(N)$. If (X, T, N) is a traingulated space, then we define $C_k^T(X) := C_k(N)$ and $H_i^T(X) := H_i(N)$. For a simplicial k-chain $\xi \in C_k(N)$, its *support* is the closed subspace of X given by $|\xi| = \bigcup_{\xi_\sigma \neq 0} T(\sigma)$.

The standard example is of S^1, which is homeomorphic to triangle Δ^2 via $T : |\Delta^2| \to S^1$.

One then easily computes that $H_1^T(S^1) \cong F$.

2.1.2 (Recollections on singular homology)**.** Let X be a space and fix a field F. Let $S_i(X)$ be the free F-vector space generated by the set of all i-simplices $\{f : \Delta^i \to X \mid f \text{ is continuous}\}$. An element of $S_i(X)$ is called *singular i-chain*. Consider the map $\partial : S_i(X) \to S_{i-1}(X)$ which on an i-simplex σ is given by $\sigma \mapsto \sum_{j=0}^{i}(-1)^j \partial_j \sigma$ where $\partial_j \sigma$ is the σ restricted to the face opposite to j^{th}-vertex. It follows that $\partial^2 = 0$. Thus, we have a chain complex $(S_i(X), \partial)$, called the singular chain complex.

2.1.3 (Singular homology with closed support/Borel-Moore homology)**.** Let X be a space and F be a field. A formal infinite linear combination

$$\xi = \sum_{\sigma} \xi_\sigma \sigma$$

where σ runs over continuous maps $\Delta^i \to X$ is said to be a locally finite singular i-chain if for all points $x \in X$, there exists an open set $x \in U_x \subseteq X$ such that the set $\{\xi_\sigma \mid \xi_\sigma \neq 0 \text{ and } \sigma^{-1}(U_x) \neq \emptyset\}$ is finite. Denote the F-vector space generated by set of all locally finite singular i-chains by $S_i((X))$. We thus get a chain complex $(S_i((X)), \partial)$, whose homology is called the *homology with closed supports* or the *Borel-Moore homology*.

2.1.4 (Triangulated pseudomanifolds)**.** An n-dimensional pseudomanifold X is triangulated by a homeomorphism $T : |N| \to X$ where N is a simplicial complex if T respects the filtration of X, that is, for each $1 \leq j \leq n$, each X_j is a union of simplices of N (under T).

2.1.5 (Perversities)**.** A *perversity* $\bar{p} : \mathbb{N}_{\geq 2} \to \mathbb{N} \cup \{0\}$ is a function such that $\bar{p}(2) = 0$ and $\bar{p}(k+1) = \bar{p}(k)$ or $\bar{p}(k) + 1$ for all $k \geq 2$. Consequently, it follows that $\bar{p}(k) \in \{0, \ldots, k-2\}$ for all $k \geq 2$.

Top perversity is $\bar{p}(k) = k-2$, *zero perversity* is the zero function. For a perversity $\bar{p}$, its *complementary perversity* is $\bar{q}$ such that $\bar{q}+\bar{p}$ is the top perversity, that is, $\bar{q}(k) = k - \bar{p}(k) - 2$.

An important perversity is the *lower-middle perversity* defined by $\bar{p}(k) = \left\lfloor \frac{k-2}{2} \right\rfloor$.

2.1.6 (Simplicial intersection homology)**.** Let (X, T, N) be a triangulated n-dimensional pseudomanifold with $T : |N| \to X$ a homeomorphism. Fix a perversity $\bar{p}$. A simplicial i-chain $\xi \in C_i^T(X)$ is *$\bar{p}$-allowable* if for all $k \geq 2$, we have

$$\dim_{\mathbb{R}} |\xi| \cap X_{n-k} \leq i - k + \bar{p}(k).$$

A quick computation shows that if ξ is a $\bar{p}$-allowable 1-chain, then it cannot intersect the singular locus! If ξ is a $\bar{p}$-allowable 2-chain on the other hand, then it can only intersect the singular locus at a discrete set of points!

Let $I^{\bar{p}}C_i^T(X) \subseteq C_i^T(X)$ denote the subspace of all those simpicial i-chains ξ which are $\bar{p}$-allowable and $\partial\xi$ is also $\bar{p}$-allowable $i-1$-chain. Then $I^{\bar{p}}C_i^T(X)$ is called *simplicial intersection i-chains*. Under the usual boundary maps, this forms a chain-complex, whose homology we denote by $I^{\bar{p}}H_i^T(X)$. The refinement of triangulation yields a directed system of simplicial intersection i-chain, taking limit, we obtain $I^{\bar{p}}C_i(X)$ together with boundary maps ∂, which makes $I^{\bar{p}}C_i(X)$ into a chain complex. The homology of this chain complex is called the simplicial intersection homology, denoted $I^{\bar{p}}H_i(X)$.

2.1.7 (Singular intersection homology)**.** Let X be an n-dimensional pseudomanifold and $\bar{p}$ be a perversity. Recall that the j-skeleton of Δ^i is the set of all j-simplices of Δ^i, that is, the j-dimensional subsimplex of Δ^i. A simplex $\sigma : \Delta^i \to X$ is said to be $\bar{p}$-*allowable* if for all $k \geq 2$, we have

$$\sigma^{-1}\left(X_{n-k} \setminus X_{n-k-1}\right) \subseteq n-k+\bar{p}(k)\text{-skeleton of } \Delta^i.$$

One can interpret this as saying that a simplex σ is $\bar{p}$-allowable if its intersection with the codimension k-strata happens only in $n-k+\bar{p}(k)$-skeleton of the simplex. A singular i-chain $\sigma \in S_i(X)$ is p-allowable if each simplex f_i of $\sigma = \sum_{i=1}^{n} c_i f_i$ is $\bar{p}$-*allowable.*

Let $I^{\bar{p}}S_i(X) \subseteq S_i(X)$ be the subspace of all $\bar{p}$-allowable i-chains σ such that $\partial\sigma$ is also $\bar{p}$-allowable. Under the usual boundary map of complex $(S_i(X), \partial)$, we get a chain complex $(I^{\bar{p}}S_i(X), \partial)$. The homology groups of this chain complex is defined to be the *singular perversity $\bar{p}$-intersection homology of X*. The following theorem allows us to denote this as $I^{\bar{p}}H_i(X)$ as well.

2.2 Basic tools

Theorem 2.2.1. *If X is a PL-pseudomanifold, then both simplicial and singular intersection homology for a fixed perversity $\bar{p}$ are isomorphic.*

2.2.2 (Relative intersection homology groups)**.** We first define the relative intersection homology groups. Let X be an n-dimensional pseudomanifold and $\bar{p}$ be a perversity. For an open set $U \subseteq X$. Note that the following is a map of F-vector spaces:

$$\begin{aligned} I^{\bar{p}}S_i(U) &\hookrightarrow I^{\bar{p}}S_i(X) \\ \sigma &\longmapsto \iota \circ \sigma \end{aligned}$$

where $\iota : U \hookrightarrow X$. Note that this defines an inclusion of chain complexes $I^{\bar{p}}S_\bullet(U) \hookrightarrow I^{\bar{p}}S_\bullet(X)$. We then define the relative chain complex $(I^{\bar{p}}S_i(X,U), \partial)$ as follows:

$$I^{\bar{p}}S_i(X,U) = \frac{I^{\bar{p}}S_i(X)}{I^{\bar{p}}S_i(U)}$$

and the differential is given by the map induced on the quotient by universla property of the quotients (using the fact that $I^{\bar{p}}S_\bullet(U) \hookrightarrow I^{\bar{p}}S_\bullet(X)$ is an inclusion of chain complexes):

$$I^{\bar{p}}S_i(X,U) \xrightarrow{\bar{\partial}} I^{\bar{p}}S_{i-1}(X,U).$$

Hence we define the i^{th}-*relative intersection homology groups* of the pair (X,U) by $I^{\bar{p}}H_i(X,U)$.

2.2.3 (Some tools for intersection homology)**.** We now gather some important calculational tools akin to usual homology theory. Fix an n-dimensional pseudomanifold X and a perversity $\bar{p}$.

1. *Homology long exact sequence*: Let $U \hookrightarrow X$ be an open set. Then we have a long exact sequence

$$\begin{array}{ccccc} \cdots \to I^{\bar{p}}H_i(U) & \longrightarrow & I^{\bar{p}}H_i(X) & \longrightarrow & I^{\bar{p}}H_i(X,U) \to \\ \to I^{\bar{p}}H_{i-1}(U) & \longrightarrow & I^{\bar{p}}H_{i-1}(X) & \longrightarrow & I^{\bar{p}}H_{i-1}(X,U) \to \cdots \end{array}.$$

2. *Excision*: Let $U \hookrightarrow X$ be an open subset and $A \subseteq U$ be a closed subset of U such that $X - A$ is a pseudomanifold. Then, the inclusion $(X - A, U - A) \hookrightarrow (X, U)$ induces an isomorphism

$$I^{\bar{p}}H_i(X - A, U - A) \cong I^{\bar{p}}H_i(X, U).$$

3. *Mayer-Vietoris*: Let $X = U \cup V$ where $U, V \subseteq X$ are open sets. Then there is a long exact sequence

$$\begin{array}{ccccc} \cdots \to I^{\bar{p}}H_i(U \cap V) & \longrightarrow & I^{\bar{p}}H_i(U) \oplus I^{\bar{p}}H_i(V) & \longrightarrow & I^{\bar{p}}H_i(X) \to \\ \to I^{\bar{p}}H_{i-1}(U \cap V) & \longrightarrow & I^{\bar{p}}H_{i-1}(U) \oplus I^{\bar{p}}H_{i-1}(V) & \longrightarrow & I^{\bar{p}}H_{i-1}(X) \to \cdots \end{array}$$

4. *Künneth formula*: We have that $X \times (0, 1)$ is a pseudomanifold of dimension $n + 1$ and

$$I^{\bar{p}}H_i(X \times (0, 1)) \cong I^{\bar{p}}H_i(X).$$

The following showcases an important calculation using the above tools.

Proposition 2.2.4 (Intersection homology of a cone)**.** *Let X be a compact pseudomanifold of dimension $n \geq 1$. Then for any perversity $\bar{p}$, we have*

$$I^{\bar{p}}H_i(C(X)) = \begin{cases} I^{\bar{p}}H_i(X) & \text{if } i < n - \bar{p}(n+1) \\ 0 & \text{else.} \end{cases}$$

and

$$I^{\bar{p}}H_i(C(X), C(X) - \{*\}) = \begin{cases} 0 & \text{if } i \leq n - \bar{p}(n+1) \\ I^{\bar{p}}H_{i-1}(X) & \text{else.} \end{cases}$$

Proof. For the first statement, observe that for any $\sigma \in I^{\bar{p}}S_i(CX)$ such that $i \leq n - \bar{p}(n+1)$, σ cannot intersect the vertex of the cone CX because σ is $\bar{p}$-allowable, vertex lives in codimension $n+1$ and $i-(n+1)+\bar{p}(n+1) \leq -1$. We thus deduce that $I^{\bar{p}}S_i(CX) = I^{\bar{p}}S_i(CX - \{*\})$ if $i \leq n - \bar{p}(n+1)$. Hence, for $i < n - \bar{p}(n+1)$, by Künneth formula, we have

$$I^{\bar{p}}H_i(CX) = I^{\bar{p}}H_i(CX - \{*\}) \cong I^{\bar{p}}H_i(X \times (0,1)) \cong I^{\bar{p}}H_i(X).$$

Now fix $i \geq n - \bar{p}(n+1)$. We will show that every cycle in $I^{\bar{p}}S_i(CX)$ is a boundary, which will complete the proof. To this end, we first pick any $\sigma : \Delta^i \to CX$ such that $\partial\sigma = 0$, then it is a boundary of some $i-1$-simplex. Indeed, we claim the following; there is a map $c : I^{\bar{p}}S_i(X) \longrightarrow I^{\bar{p}}S_{i+1}(CX)$ which establishes a chain homotopy between $\kappa : I^{\bar{p}}S_\bullet(X) \to I^{\bar{p}}S_\bullet(CX)$ and $0 : I^{\bar{p}}S_\bullet(X) \to I^{\bar{p}}S_\bullet(CX)$, where κ is obtained by the inclusion of $X \hookrightarrow CX$. We define c as follows on the basis elements and then extend linearly:

$$\begin{aligned} c : I^{\bar{p}}S_i(X) &\longrightarrow I^{\bar{p}}S_{i+1}(CX) \\ \sigma : \Delta^i \to X &\longmapsto c\sigma : \Delta^{i+1} \to CX \end{aligned}$$

where $c\sigma([s,t]) = t\sigma(s)$ where $[s,t] \in \Delta^{i+1}$ is a representation where $s \in \Delta^i$ and $t \in [0,1]$ and $c\sigma$ is oriented by labelling the cone point as 0 and the rest starting from 1 to i. That is, $c\sigma$ is the cone of σ. We now wish to show that $c\sigma$ is an $\bar{p}$-allowable $i+1$-singular chain of CX. Indeed, pick any stratum $S_k = X_{n-k} - X_{n-k-1}$ for $2 \leq k \leq n$. Note that since CX is $n+1$-dimensional pseudomanifold with filtration $C(X_n) \supseteq C(X_{n-1}) \supseteq C(X_0) \supseteq C(X_{-1}) = \{*\} \supseteq \emptyset$, therefore the codimensiona k-strata of $C(X)$ are $T_k = (X_{n-k} - X_{n-k-1}) \times (0,1)$ for $0 \leq k \leq n$ and $T_{n+1} = \{*\}$. We wish to show that $(c\sigma)^{-1}(T_k) \subseteq (i+1-k+\bar{p}(k))$-skeleton of Δ^{i+1}. This can be checked by conditioning on each $0 \leq k \leq n+1$, whether $\sigma^{-1}(S_k)$ is empty or not. If not, then $\sigma^{-1}(S_k) \subseteq i-k+\bar{p}(k)$-skeleton of Δ^i, from which it follows by construction of $c\sigma$ that $(c\sigma)^{-1}(T_k) \subseteq i+1-k+\bar{p}(k)$-skeleton of Δ^{i+1}. If empty, then $(c\sigma)^{-1}(T_k) =$ or $(c\sigma)^{-1}(T_{n+1}) \subseteq$ 0-skeleton of Δ^{i+1}, as $(i+1)-(n+1)+\bar{p}(n+1) \geq 0$.

One can see that $\partial c + c\partial = \kappa - 0 = \kappa$. This completes the proof of first statement. For the latter, a simple use of Mayer-Vietoris concludes the proof. □

2.3 Local coefficients

Recall that a local system is just another name for a locally constant sheaf on a space X. We will define singular intersection homology groups with coefficient in a local system $\mathcal{L}$. We first recall some technicalities of local systems. Let us first show that any local system over $[0,1]$ is constant, as it will be useful to define monodromy.

Lemma 2.3.1. *Every local system $\mathcal{L}$ on $[0,1]$ is constant.*

Proof. Let $\mathcal{L}$ be locally system on $[0,1]$. We may assume $I_i = (t_i, t_{i+1})$ is a finite cover of $[0,1]$ such that $\mathcal{L}$ restricted on each I_i is constant. As each I_i by construction intersects I_{i+1}, it follows that the restriction of $\mathcal{L}$ to each I_i is a constant sheaf with the constant abelian group being the same. Let $t \in \mathcal{L}_x = A$ for any $x \in I_{i_0}$ for some i_0. Then t can be glued to each i to give a constant global section $t \in \Gamma(X, \mathcal{L})$, which is unique by sheaf condition. □

Lemma 2.3.2. *If X is path-connected and $\mathcal{L}$ is a local system over X, then all stalks of $\mathcal{L}$ are same.*

Proof. Pick two points $x \neq y \in X$ and a path $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We wish to show that $\mathcal{L}_x \cong \mathcal{L}_y$. Indeed, consider the sheaf $\gamma^*\mathcal{L}$. As inverse image of a locally constant sheaf is again locally constant, it follows at once from Lemma 2.3.1 that $\gamma^*\mathcal{L}$ is a constant sheaf in say abelian group A. Pick any $t \in I$ and observe that $A \cong (\gamma^*\mathcal{L})_t \cong \mathcal{L}_{\gamma(t)}$. It follows that stalks of $\mathcal{L}$ along the path γ are constant, as needed. □

2.3.3 (Local systems-Monodromy)**.** Let (X, x_0) be a path-connected, locally path-connected and semi-locally simply connected space. We wish to show the following equivalence between local systems with local groups A and $\mathrm{Aut}(A)$-representations of $\pi_1(X)$:

$$\mathrm{LocSys}_A(X) \cong \mathrm{Hom}_{\mathcal{G}rp}\left(\pi_1(X, x_0), \mathrm{Aut}(A)\right).$$

Let $\mathcal{L}$ be a local system of abelian groups over X. By Lemma 2.3.2, it follows that $\mathcal{L}$ is a local system with fiber (stalk) a fixed abelian group A. We will construct a representation of $\pi_1(X, x_0)$ in the group $\mathrm{Aut}(A)$. Indeed, consider the map

$$\begin{aligned} \varphi : \pi_1(X, x_0) &\longrightarrow \mathrm{Aut}(A) \\ [\gamma] &\longmapsto \gamma^\times : (\gamma^*\mathcal{L})_0 \cong A \cong (\gamma^*\mathcal{L})_1. \end{aligned}$$

We omit the proof that this is well-defined. Conversely, pick any map $\varphi : \pi_1(X, x_0) \to \mathrm{Aut}(A)$. We wish to construct a local system $\mathcal{L}$ over X. Let $p : \tilde{X} \to X$ be the universal cover over X, which exists by our hypotheses over X. Recall from covering space theory that $\pi_1(X, x_0) \cong G(\tilde{X}/X)$, the latter being the Deck-group of $(\tilde{X}, p, X)$. Now consider the constant sheaf $\underline{A}$ on $\tilde{X}$. Let $U \subseteq X$ be an evenly covered neighborhood of X. Then, $p^{-1}(U) = \coprod_{\alpha \in \pi_1(X)} V_\alpha$. Now consider the following sheaf $\mathcal{L}$ which on an open set $U \subseteq X$ gives the following set of sections:

$$\mathcal{L}(U) := \{s \in \underline{A}(p^{-1}(U)) \mid s \circ \theta = \varphi(\theta)s \ \forall \theta \in G(\tilde{X}/X) = \pi_1(X, x_0)\}$$

where $s \in \underline{A}(p^{-1}(U))$ is a section of p as in $s : p^{-1}(U) \to \underline{A} = \coprod_{a \in A} \tilde{X}$ and it is in $\mathcal{L}(U)$ if and only if for any deck transformation $\theta \in G(\tilde{X}/X) = \pi_1(X, x_0)$, we must get that the following commutes[1][2]

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\theta} & p^{-1}(U) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \underline{A} & \xrightarrow[\varphi(\theta)]{\cong} & \underline{A} \end{array} .$$

We wish to show that $\mathcal{L}$ is a locally constant sheaf and its associated monodromy coincides with φ. The local triviality follows from covering property. Monodromy coinciding again follows by simple unravelling of underlying definitions.

The action of $\pi_1(X, x_0)$ on A obtained by a local system $\mathcal{L}$ is called the **monodromy**

[1] θ by restriction gives a homeomorphism of $p^{-1(U)}$ by definition.

[2] further, any automorphism of A, say $\kappa : A \to A$ gives a homeomorphism of $\underline{A} = \coprod_{a \in A} \tilde{X}$ by permuting the index A by κ.

action of $\mathcal{L}$ on A. Observe that under the above bijection, the trivial action $0 : \pi_1(X, x_0) \to \mathrm{Aut}(A)$ which maps $[\gamma]$ to the identity automorphism of A corresponds to the local system $\mathcal{L}$ on X whose local sections are $G(\tilde{X}/X)$-invariant sections over X.

We thus obtain the following result which gives a characterization of local systems.

Proposition 2.3.4. *Let X be a path-connected, locally path-connected and semi-locally simply connected space. The following are equivalent:*

1. *$\mathcal{L}$ is a local system on X of finite dimensional vector spaces,*
2. *$\{\mathcal{L}_x\}_{x \in X}$ is a collection of finite dimensional vector spaces such that for any path $\gamma : I \to X$ we get a linear isomorphism*

$$\gamma^* : \mathcal{L}_{\varphi(0)} \to \mathcal{L}_{\varphi(1)}$$

such that the following two conditions are satisfied:

(a) *for any two paths γ, η homotopic rel end points, the maps γ^* and η^* are same,*
(b) *if $\gamma * \eta$ is the concatenation of two paths, then $(\gamma * \eta)^* = \gamma^* \circ \eta^*$.*

Proof. (1. $\Rightarrow$ 2.) Pick any local system $\mathcal{L}$. By hypothesis on X, we immediately have a collection of isomorphic vector spaces $A \cong \mathcal{L}_x$ for each $x \in X$. Pick any path $\gamma : I \to X$. We get a map

$$\gamma^* : \mathcal{L}_{\varphi(0)} \to \mathcal{L}_{\gamma(1)}$$

by the usual process of taking the inverse image of $\mathcal{L}$ under γ and calculating stalks (see proof of Lemma 2.3.2). Consider the corresponding monodromy (see 2.3.3)

$$\pi_1(X, x_0) \to \mathrm{Aut}(A).$$

The two conditions of item 2 now follows from the conditions of monodromy.
(2. $\Rightarrow$ 1.) From the given data, we wish to construct a locally constant sheaf. By 2.3.3, it suffices to obtain an action $\varphi : \pi_1(X, x_0) \to \mathrm{Aut}(A)$. Indeed, pick any $[\gamma] \in \pi_1(X, x_0)$. Define $\varphi([\gamma])$ to be the automorphism associated to the loop γ, the γ^*, as provided by the hypothesis. Its well-definedness follows from the first condition of item 2. That it defines a group homomorphism follows from second condition of item 2. □

Corollary 2.3.5. *Let X be a locally path-connected, simply-connected space. Then any local system $\mathcal{L}$ over X is constant.*

Proof. If X is simply-connected, then the deck group of its universal cover is singleton. Hence $\mathrm{Hom}_{\mathcal{G}rp}(\pi_1(X, x_0), \mathrm{Aut}(A))$ consists of only one map, the trivial map. It follows by 2.3.3 that the local system associated to this is the constant sheaf associated to A, $\underline{A}$ over X (which is its own universal cover). □

We now first define homology with local coefficients on a space X.

2.3.6 (Homology with local coefficients)**.** Let X be a path-connected, locally path-connected and semi-locally simply-connected space and let $\mathcal{L}$ be a local system over X. We will construct **homology groups with local coefficients** $\mathcal{L}$, denoted $H_i(X, \mathcal{L})$, as follows.

Let $A \cong \mathcal{L}_x$ for all $x \in X$. Pick any i-simplex $\sigma : \Delta_i \to X$. Taking inverse image of $\mathcal{L}$ under σ, we obtain a local system $\sigma^*\mathcal{L}$ over Δ_i. By Corollary 2.3.5, it follows that $\sigma^*\mathcal{L}$ is the constant sheaf over A which we denote by $\underline{A}_\sigma$ i.e. $\sigma^*\mathcal{L} \cong \underline{A}_\sigma$. To each $\sigma : \Delta_i \to X$, we attach a copy of A by considering the global sections of $\underline{A}_\sigma$ which is just A. Thus, we construct the group of i-chains with coefficients in $\mathcal{L}$ as follows:

$$S_i(X, \mathcal{L}) = \left\{ \sum_{\sigma} a_\sigma \sigma \mid \sigma : \Delta^i \to X,\ a_\sigma \in A_\sigma \ \&\ a_\sigma \neq 0 \text{ only for finitely many } \sigma \right\}.$$

We further define the boundary map

$$d : S_i(X, \mathcal{L}) \longrightarrow S_{i-1}(X, \mathcal{L})$$

by first defining an isomorphism $\rho^\sigma_\tau : A_\sigma \to A_\tau$ where $\tau = \sigma \circ d_j$ is the j^{th}-face of σ. Indeed, observe that for any point $p \in \Delta_i$, we can define the following isomorphism:

$$\rho^\sigma_p : A = A_\sigma = \Gamma(\Delta_i, \underline{A}_\sigma) \to \mathcal{L}_{\sigma(p)} \cong A.$$

Using this, we can then define the restriction map ρ^σ_τ as in the following diagram where $p \in \Delta^{i-1}$:

$$\begin{array}{ccc} A_\sigma & \xrightarrow{\rho^\sigma_\tau} & A_\tau \\ {\scriptstyle \rho^\sigma_p}\downarrow & \nearrow_{(\rho^\tau_p)^{-1}} & \\ \mathcal{L}_{\sigma(p)} & & \end{array} .$$

This map ρ^σ_τ is independent of any choice of point $p \in \Delta^{i-1}$ by path-connectedness of Δ_{i-1} and the isomorphisms between the stalks by a path as given by Proposition 2.3.4, item 2. Using this map ρ^σ_τ, we obtain the following differential defined on a simple i-chain $\sigma : \Delta^i \to X$:

$$\begin{aligned} d : S_i(X, \mathcal{L}) &\longrightarrow S_{i-1}(X, \mathcal{L}) \\ a_\sigma \sigma &\longmapsto \sum_{j=0}^{i} (-1)^j \rho^\sigma_{\partial_j \sigma}(a_\sigma) \partial_j \sigma. \end{aligned}$$

This makes $(S_\bullet(X, \mathcal{L}), d)$ into a chain complex, whose homology is defined to be homology groups with local coefficients, $H_i(X, \mathcal{L})$.

The following is a simple lemma, showcasing what we have done so far is indeed a generalization of coefficients in the usual homology.

Lemma 2.3.7. *Let X be a path-connected, locally path-connected and semi-locally simply-connected space. The usual homology groups of X with coefficient $\mathbb{Z}$ are in isomorphism with the homology with coefficient in the constant local system $\underline{\mathbb{Z}}$. That is, for all $i \in \mathbb{Z}$, we have*

$$H_i(X, \mathbb{Z}) \cong H_i(X, \underline{\mathbb{Z}}).$$

Proof. We need only show that the map ρ_τ^σ in 2.3.6 are identity. As $\rho_\tau^\sigma : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism of abelian groups, therefore it has to be id. $\square$

Next, we define *intersection* homology with local coefficients.

2.3.8 (Intersection homology with local coefficients)**.** Let X be an n-pseudomanifold with a fixed stratification $\mathbb{S}$. Recall that $X - X_{n-2}$ is the non-singular locus and it is a manifold of dimension n. To define intersection homology with local coefficients, it will suffice to consider a local system on the manifold $X - X_{n-2}$. Indeed, if $\mathcal{L}$ is a local system defined on $X - X_{n-2}$ and $\bar{p}$ is a perversity, then we can still define $I^{\bar{p}}S_i(X, \mathcal{L})$ even though $\mathcal{L}$ is only defined on the non-singular locus by the following procedure. Define

$$I^{\bar{p}}S_i(X, \mathcal{L}) = \left\{ \sum_\sigma a_\sigma \sigma \mid \sigma : \Delta_i \to X \ \& \ d\sigma \text{ are } \bar{p} - \text{allowable}, \ a_\sigma \in A_\sigma \text{ is} \neq 0 \text{ for finitely many } \sigma \right\}.$$

This is well-defined as if σ is $\bar{p}$-allowable and $a_\sigma \neq 0$ in $A_\sigma = \Gamma(\Delta_i, \sigma^* \mathcal{L})$, therefore it has to intersect the non-singular locus $X - X_{n-2}$. Similarly for any face τ of σ. Hence, we get **intersectiom homology groups with coefficients in a local system** $\mathcal{L}$ over the non-singular stratum $X - X_{n-2}$, denoted by $I^{\bar{p}}H_i(X, \mathcal{L})$.

We define the dual of a local system now.

2.3.9 (Dual local systems)**.** Let X be a space and $\mathcal{L}$ be a local system over X. We can then define a new local system

$$\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \underline{K}).$$

Then $\mathcal{L}_x^\vee = \text{Hom}\,(\mathcal{L}_x, K)$.

3 Sheaf theoretic intersection homology

Let X be an n-dimensional pseudomanifold. We construct a complex of sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i}$ whose local sections at open U are the locally-finite intersection i-singular chains over U.

Construction 3.0.1 (The sheaves $\mathcal{S}_X^{-i}$ and $\mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i}$)**.** Let X be a paracompact Haudorff space and F be a field. We globalize the construction of homology with closed supports by constructing a complex of sheaves $\mathcal{S}_X^{-i}$.

We first define the presheaf over X whose sections over the open set $U \subseteq X$ is given by

$$\mathcal{S}_X^{-i}(U) = S_i((U)),$$

the set of locally-finite singular i-chains. The main difficulty is in defining the restriction maps. Let $V \hookrightarrow U$ be an inclusion of open sets of X. We define the restriction map

$$\rho : S_i((U)) \longrightarrow S_i((V))$$

as follows. As the map to be constructed must be F-linear, hence it suffices to define ρ only on a single singular i-simplex $\sigma : \Delta^i \to X$. Indeed, from σ, define the set J_σ by the following process. If $\mathrm{Im}\,(\sigma) \subseteq V$, then set $J_\sigma = \{\sigma\}$. If $\mathrm{Im}\,(\sigma) \not\subseteq V$, then subdivide σ and put those τ in the subdivision whose $\mathrm{Im}\,(\tau) \subseteq V$ in J_σ. Further subdivide those $\mathrm{Im}\,(\tau) \not\subseteq V$ and repeat the process.

At the end of this process, we have a set of i-simplices in V, denoted by J_σ. We thus define

$$\rho(\sigma) = \sum_{\tau \in J_\sigma} \tau.$$

This is a locally-finite singular i-chain in V.

We now wish to show that $\mathcal{S}_X^{-i}$ is a sheaf. Indeed, for any open set $U \subseteq X$, an open cover $\{U_j\}_{j\in J}$ of U and $\xi_j \in \mathcal{S}_X^{-i}(U_j) = S_j((U_i))$ which agrees on intersection, we wish to glue the matching family (ξ_j) to a locally finite i-chain $\xi \in S_i((U))$. Indeed, define ξ as the sum $\sum_{j\in J} \xi_j$. This is a locally-finite i-chain in U as for each $x \in U$ we have that $x \in U_j$, and thus there is an open set $x \in U_x \subseteq U_j$ which intersects atmost finitely many simplices in ξ_j with non-zero coefficient by local compactness of X. Observe that $\xi|_{U_j} = \xi_j$ as by definition of restriction.

Now we define a map of sheaves

$$\partial : \mathcal{S}_X^{-i} \longrightarrow \mathcal{S}_X^{-i+1}$$

which is defined on an open set $U \subseteq X$ by

$$\partial_U : S_i((U)) \longrightarrow S_{i-1}((U))$$

in the usual manner. The fact that this commutes with restrictions follows from checking it on a simplex, where it is immediate.

It follows that we have a complex of sheaves $(\mathcal{S}_X^\bullet, \partial)$ which is bounded above.

In exactly the same mannerism, we construct the presheaf

$$\mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i} : U \mapsto I^{\bar{p}}S_i((U))$$

which becomes a subsheaf of $\mathcal{S}_X^{-i}$ such that the map ∂ restricts to define a differential

$$\partial : \mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i} \longrightarrow \mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i+1}.$$

We thus have a subcomplex of $\mathcal{S}_X^\bullet$ given by $(\mathcal{I}^{\bar{p}}\mathcal{S}_X^\bullet, \partial)$, called the *intersection complex.*

The main point of sheafifying the above construction is the following theorem, which makes computing intersection homology amenable to tools from sheaves.

Theorem 3.0.2. *Let X be an n-pseudomanifold. Then,*

1. *the sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}_{\bar{S},\mathcal{L}}^{-i}$ are soft,*
2. *the hypercohomology of $\mathcal{I}^{\bar{p}}\mathcal{S}_{\bar{S},\mathcal{L}}^\bullet$ is same as intersection homology with coefficients in $\mathcal{L}$.*

We next study singular intersection complex with coefficients in a local system.

3.1 Intersection complex with local coefficients

We will now construct a complex of sheaves with local coefficients on the non-singular stratum.

Construction 3.1.1 (Intersection complex with local coefficients, $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{X,\mathcal{L}}$)**.** Let X be an n-pseudomanifold and $\mathcal{L}$ be a local system on $X - X_{n-2}$, the non-singular stratum. Fix a perversity $\bar{p}$. We will construct sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{X,\mathcal{L}}$ for each $i \in \mathbb{Z}$, called the **intersection sheaves with local coefficients**.

Fix an $i \in \mathbb{Z}$. Consider the following presheaf

$$U \mapsto I^{\bar{p}}S_i((U, \mathcal{L}))$$

where $I^{\bar{p}}S_i((U, \mathcal{L}))$ is the vector space of all locally-finite intersection i-chains with coefficients in $\mathcal{L}$ where the restriction map is defined exactly in the same manner as in Construction 3.0.1. For the same reasons as for $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_X$, we get that this is a sheaf, which we denote by $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{X,\mathcal{L}}$. Moreover, the differential again lifts to a map of sheaves, giving us a cochain complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{X,\mathcal{L}}$, called the intersection complex with local coefficient $\mathcal{L}$.

We'll later see that intersection complex with local coefficients actually forms a prototypical example of a perverse sheaf.

Let $S \subseteq X$ be any stratum in X and a local system $\mathcal{L}$. From this data, we will construct a new complex over X, which will thus give us many examples of perverse sheaves over X.

3.1.2 (From a pair $(S, \mathcal{L})$ to $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$)**.** Let $i_S : S \hookrightarrow X$ be a stratum of (complex) codimension k and $\mathcal{L}$ be a local system over S. We will construct a complex of sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ over X which we call the **extended intersection complex with local coefficient** (the name will make sense in a minute).

As $i_S : S \hookrightarrow X$ is a stratum, then $\bar{S}$ is a pseudomanifold of dimension $n-k$ and thus $\mathcal{L}$ is a local system defined on the non-singular locus of $\bar{S}$ (which is S). By Construction 3.1.1, we get the intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ on $\bar{S}$ with coefficient in $\mathcal{L}$. Now consider the inclusion of the closed set $i : \bar{S} \hookrightarrow X$. Consider the extension by zeroes of each sheaf of the complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ to obtain a complex of sheaves over X, which, to reduce linguistic baggage, we again write as $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$. This complex we call the extended intersection complex with coefficient in $\mathcal{L}$.

We will later see that this is a quintessential example of a perverse sheaf.

Remark 3.1.3. As was the case before, the following are true for extended intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$:

1. the sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{\bar{S},\mathcal{L}}$ are soft,
2. the hypercohomology of $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ is same as intersection homology with coefficients in $\mathcal{L}$.

3.2 Characterization of $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_X$ in $\mathcal{D}^b(X)$

There are certain axioms which completely classify the intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_X$ upto isomorphism in $\mathcal{D}^b(X)$.

3.2.1 (Axioms for $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_X$ in $\mathcal{D}^b_{\mathbb{S}}(X)$)**.** Fix an n-pseudomanifold X and a perversity $\bar{p}$. Further, fix a stratification $\mathbb{S}$ of X. Denote for each $2 \leq k \leq n+1$ the following two susbsets of X:

$$U_k = X - X_{n-k}$$
$$S_k = X_{n-k} - X_{n-k-1}$$

and denote the inclusions as follows:

$$S_k \overset{j_k}{\hookrightarrow} U_{k+1} \overset{i_k}{\hookleftarrow} U_k.$$

We now lay down a set of axioms which will uniquely characterize $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_X$ upto isomorphism in $\mathcal{D}^b(X)$. Let $\mathcal{F}^{\bullet} \in \mathcal{D}^b_{\mathbb{S}}(X)$. We call the following axioms $[\text{AX1}]_{\bar{p},\mathbb{S}}$:

1. [Normalization] : We have a quasi-isomorphism on the non-singular stratum $\mathcal{F}^{\bullet}|_{X-X_{n-2}} \simeq \mathbb{R}_{X-X_{n-2}}[n]$ where $\mathbb{R}_{X-X_{n-2}}$ is a local system on $X - X_{n-2}$.
2. [Lower bound on cohomology] : $\mathcal{H}^i(\mathcal{F}^{\bullet}) = 0$ for all $i < -n$.
3. [Vanishing condition] : $\mathcal{H}^i(\mathcal{F}^{\bullet}|_{U_{k+1}}) = 0$ for all $i > \bar{p}(k) - n$ and $k \geq 2$.
4. [Attaching condition] : The map

$$\mathcal{H}^i(j_k^* \, \mathcal{F}^{\bullet}|_{U_{k+1}}) \longrightarrow \mathcal{H}^i(j_k^* R i_{k*} i_k^* \, \mathcal{F}^{\bullet}|_{U_{k+1}})$$

   is an isomorphism for $i \leq \bar{p}(k) - n$ and $k \geq 2$.

Furthermore, these axioms are equivalent to the following axioms which we call $[\text{AX2}]_{\bar{p},\mathbb{S}}$:

1. [Lower bound on stalk cohomology] : For any $x \in X$, we have $H^i(j_x^*\mathcal{F}^{\bullet}) = 0$ for all $i < -n$.
2. [Non-singular stalk cohomology] : For any $x \in X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^{\bullet})$ is a constant local system on $X - X_{n-2}$ such that for all $x \in X - X_{n-2}$,

$$H^i(j_x^*\mathcal{F}^{\bullet}) = \begin{cases} \mathbb{R} & \text{if } i = -n \\ 0 & \text{else.} \end{cases}$$

   Furthermore, over $X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^{\bullet})|_{X-X_{n-2}} \cong \underline{\mathbb{R}}$.
3. [Stalk cohomology in positive stratum] : for any $x \in X_{n-k} - X_{n-k-1}$ for $k > 0$, we have

$$H^i(j_x^*\mathcal{F}^{\bullet}) = 0 \text{ for } i > \bar{p}(k) - n.$$

4. [Costalk cohomology in positive stratum] : for any $x \in X_{n-k} - X_{n-k-1}$ for $k > 0$, we have

$$H^i(j_x^!\mathcal{F}^{\bullet}) = 0 \text{ for } i < -\bar{q}(k)$$

   where $\bar{q}$ is the complementary perversity of $\bar{p}$, i.e. $\bar{p}(k) + \bar{q}(k) = k - 2$.

The benefit of $[\text{AX2}]_{\bar{p},\mathbb{S}}$ over the $[\text{AX1}]_{\bar{p},\mathbb{S}}$ is that we are only talking about local conditions for a sheaf to satisfy; all conditions in $[\text{AX2}]_{\bar{p},\mathbb{S}}$ are about stalks and costalks.

For local coefficients, we have the following.

3.2.2 (Axioms for $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ in $\mathcal{D}^b_{\mathbb{S}}(X)$)**.** Let $\Sigma = X - X_{n-2}$ be the non-singular stratum and consider a local system $\mathcal{L}$ on Σ. By 3.1.2, we get the intersection complex with local coefficients $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ where we don't need to extend by zeros as $\bar{\Sigma} = X$. Following the notations as in 3.2.1, we again get the same axioms for $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ as for the usual intersection complex, but the only difference is in the normalization axiom where we demand $\mathcal{F}^\bullet|_{X-X_{n-2}} \simeq \mathcal{L}[n]$. We call these axioms $[\mathrm{AX1}]_{\bar{p},\mathbb{S},\mathcal{L}}$. Similarly, we can form $[\mathrm{AX2}]_{\bar{p},\mathbb{S},\mathcal{L}}$ by replacing the axiom of non-singular stalk cohomology by

$$H^i(j_x^*\mathcal{F}^\bullet) = \begin{cases} \mathcal{L}_x & \text{if } i = -n \\ 0 & \text{else.} \end{cases}$$

Furthermore, over $X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^\bullet)|_{X-X_{n-2}} \cong \mathcal{L}$.

3.2.3 ($[\mathrm{AX2}]_{\bar{p},\mathbb{S}}$ via Verdier dual)**.** Following the notations of 3.2.1, we can reframe the axioms of $[\mathrm{AX2}]_{\bar{p},\mathbb{S}}$ using the Verdier dual functor D_X as follows and we call them $[\mathrm{AX2V}]_{\bar{p},\mathbb{S}}$:

1. [Lower bound on stalk cohomology] : For any $x \in X$, we have $H^i(j_x^*\mathcal{F}^\bullet) = 0$ for all $i < -n$.
2. [Non-singular stalk cohomology] : For any $x \in X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^\bullet)$ is a constant local system on $X - X_{n-2}$ such that for all $x \in X - X_{n-2}$,

$$H^i(j_x^*\mathcal{F}^\bullet) = \begin{cases} \mathbb{R} & \text{if } i = -n \\ 0 & \text{else.} \end{cases}$$

   Furthermore, over $X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^\bullet)|_{X-X_{n-2}} \cong \underline{\mathbb{R}}$.

3⋆ [Costalk cohomology in positive stratum of twisted dual] : for any $x \in X_{n-k} - X_{n-k-1}$ for $k > 0$, we have

$$H^i(j_x^!((D_X\mathcal{F}^\bullet)[n])) = 0 \text{ for } i < -\bar{p}(k).$$

4⋆ [Stalk cohomology in positive stratum of twisted dual] : for any $x \in X_{n-k} - X_{n-k-1}$ for $k > 0$, we have

$$H^i(j_x^*((D_X\mathcal{F}^\bullet)[n])) = 0 \text{ for } i > \bar{q}(k) - n.$$

   where $\bar{q}$ is the complementary perversity of $\bar{p}$, i.e. $\bar{p}(k) + \bar{q}(k) = k - 2$.

The proof of this is immediate from Theorem A.4.3, 5 (the version for inverse images).

Theorem 3.2.4. *The complex $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_X$ satisfies $[AX2]_{\bar{p},\mathbb{S}}$. The complex $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ satisfies $[AX1]_{\bar{p},\mathbb{S},\mathcal{L}}$.*

4 Perverse sheaves

Recall notions surrounding constructibility from Appendix B. We fix a complex algebraic or analytic varieties of $\mathbb{R}$-dimension $2n$ (so $\dim_{\mathbb{C}} X = n$) with a fixed Whitney stratification $\mathbb{S}$. Hence there is no odd-dimensional strata. We further fix our perversity as the lower middle perversity $\bar{m}$.

We will now construct a subcategory of $\mathcal{D}^b_{\mathbb{S}}(X)$ which will satisfy various properties which are ideal for further development of intersection complex.

4.0.1 (j^{th}-support and cosupport of a complex)**.** Let $\mathcal{F}^\bullet$ be a complex of sheaves over X and $j \in \mathbb{Z}$. For any $x \in X$, denote the inclusion $i_x : \{x\} \hookrightarrow X$. Then, the j^{th}-*support* of $\mathcal{F}^\bullet$ is defined by

$$\operatorname{supp}^j(\mathcal{F}^\bullet) := \overline{\{x \in X \mid H^j(i_x^* \mathcal{F}^\bullet) \neq 0\}}$$

and the j^{th}-*cosupport* of $\mathcal{F}^\bullet$ is defined by

$$\operatorname{cosupp}^j(\mathcal{F}^\bullet) := \overline{\{x \in X \mid H^j(i_x^! \mathcal{F}^\bullet) \neq 0\}}.$$

4.0.2 (Perverse sheaves)**.** A cohomologically $\mathbb{S}$-constructible complex $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathbb{S}}(X)$ is said to be a **perverse sheaf** if the following two conditions are satisfied:

1. $\dim_{\mathbb{C}} \operatorname{supp}^{-j}(\mathcal{F}^\bullet) \leq j$ for all $j \in \mathbb{Z}$,
2. $\dim_{\mathbb{C}} \operatorname{cosupp}^{j}(\mathcal{F}^\bullet) \leq j$ for all $j \in \mathbb{Z}$.

We denote the subcategory of perverse sheaves in $\mathcal{D}^b_{\mathbb{S}}(X)$ as $\mathcal{P}erv_{\mathbb{S}}(X)$.

There is an alternate characterization of this definition which is very helpful to keep in mind.

Theorem 4.0.3. *Let $\mathcal{F}^\bullet$ be a complex of sheaves in $\mathcal{D}^b_{\mathbb{S}}(X)$. Then the following are equivalent:*

1. *$\mathcal{F}^\bullet$ is a perverse sheaf,*
2. *[Beilinson-Bernstein-Deligne] for any non-empty stratum S (so that it is a complex manifold) with inclusion $i_S : S \hookrightarrow X$, we have*

$$\mathcal{H}^j(i_S^* \mathcal{F}^\bullet) = 0 \; \forall j > -\dim_{\mathbb{C}} S$$
$$\mathcal{H}^j(i_S^! \mathcal{F}^\bullet) = 0 \; \forall j < -\dim_{\mathbb{C}} S.$$

3. *[Kashiwara-Schapira] for any non-empty stratum $i_S : S \hookrightarrow X$ and any point $x \in S$ with inclusion $i_x : \{x\} \hookrightarrow X$, we have*

$$H^j(i_x^* \mathcal{F}^\bullet) = 0 \; \forall j > -\dim_{\mathbb{C}} S$$
$$H^j(i_x^! \mathcal{F}^\bullet) = 0 \; \forall j < \dim_{\mathbb{C}} S.$$

4. *[Kirwaan-Woolf] the shifted complex $\mathcal{F}^\bullet[\dim_{\mathbb{C}} X]$ satisfies that for any stratum $S \subseteq X$ and any $x \in S$ with inclusion $i_x : \{x\} \hookrightarrow X$, we have*

$$H^j(i_x^* \mathcal{F}^\bullet[\dim_{\mathbb{C}} X]) = 0 \; \forall j > -\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} S$$
$$H^j(i_x^! \mathcal{F}^\bullet[\dim_{\mathbb{C}} X]) = 0 \; \forall j < \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} X.$$

5. *[Kirwaan-Woolf] for any stratum $i_S : S \hookrightarrow X$ and any point $x \in X$ with inclusion $i_x : \{x\} \hookrightarrow S$, we have*

$$H^j(i_x^! D_X \mathcal{F}^\bullet) = 0 \; \forall j < \dim_{\mathbb{C}} S$$
$$H^j(i_x^* D_X \mathcal{F}^\bullet) = 0 \; \forall j > -\dim_{\mathbb{C}} S$$

Proof. Proposition 10.2.4 and Corollary 10.2.5 of Kashiwara and Schapira.

(3. $\iff$ 5.) Observe that by Theorem A.4.3, we have

$$\begin{aligned} H^j(i_x^! D_X \mathcal{F}^\bullet) = 0 &\iff H^j(D_\star i_x^* \mathcal{F}^\bullet) = 0 \\ &\iff H^j(i_x^* \mathcal{F}^{-\bullet})^\vee = 0 \\ &\iff H^{-j}(i_x^* \mathcal{F}^\bullet) = 0. \end{aligned}$$

Using this and Theorem A.4.3, we also deduce

$$\begin{aligned} H^j(i_x^* D_X \mathcal{F}^\bullet) = 0 &\iff H^{-j}(i_x^! D_X^2 \mathcal{F}^\bullet) = 0 \\ &\iff H^{-j}(i_x^! \mathcal{F}^\bullet) = 0, \end{aligned}$$

as required. □

Let us now give some examples of perverse sheaves.

4.0.4 (Perverse sheaves over a point)**.** Let $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathbb{S}}(\{x\})$, which we may thus think as a complex of $\mathbb{C}$-vector spaces. Then, we claim that the following are equivalent:

1. $\mathcal{F}^\bullet$ is perverse,
2. $\mathcal{H}^j(\mathcal{F}^\bullet) = 0$ for all $j \neq 0$.

Hence we may call a complex of vector spaces perverse if the only non-zero cohomology is in degree 0.

Indeed, this follows immediately from the equivalence of first and second item of Theorem 4.0.3 with $i_S = \mathrm{id}$.

4.0.5 (Perverse sheaves over manifolds)**.** Let X be a complex non-singular variety of $\dim_{\mathbb{C}} X = n$ and let $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathbb{S}}(X)$ be a cohomologically $\mathbb{S}$-constructible complex over X where $\mathbb{S}$ is the trivial stratification of X as there are no-singularities. We claim that the following are equivalent:

1. $\mathcal{F}^\bullet$ is perverse,
2. $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{H}^{-n}(\mathcal{F}^\bullet)[n]$.

Indeed, we may use the second item of Theorem 4.0.3 by putting $S = X$ and $i_X = \mathrm{id}$ to yield

$$\mathcal{H}^j(\mathcal{F}^\bullet) = 0 \; \forall j \neq -n.$$

Now, shifting the constant complex $\mathcal{H}^j(\mathcal{F}^\bullet)$ by $-n$, we immediately get that both complexes have same cohomology. Now as this is a complex of sheaves of vector spaces, so there exists a quasi-isomorphism as required.

4.0.6 (Local systems and perverse sheaves over manifolds)**.** Let X be a complex non-singular variety of $\dim_{\mathbb{C}} X = n$. Let $\mathcal{L}$ be a local system over X. Then we claim that $\mathcal{L}[n]$ is a perverse sheaf over X with the trivial stratification $\mathbb{S}$.

Indeed, observe that we are treating $\mathcal{L}$ as a complex concentrated in degree 0, thus $\mathcal{L}[n]$ represents complex which has $\mathcal{L}$ at $-n$-position and 0 elsewhere. As X has trivial stratification, therefore by 4.0.5, $\mathcal{L}[n]$ is perverse if and only if $\mathcal{L}[n]$ has same cohomology sheaves as $\mathcal{H}^{-n}(\mathcal{L}[n])[n]$ and the latter is just $\mathcal{L}[n]$ again, as required.

It is not true that if $\mathcal{L}$ is a local system on X where X is an n-complex singular variety that $\mathcal{L}[n]$ is a perverse sheaf.

We will later see that still in the case of local complete intersection X (see Appendix **??**) and any local system $\mathcal{L}$ over X, the complex $\mathcal{L}[n]$ would be perverse!

4.0.7 (Cohomology sheaves of a perverse sheaf)**.** Let $\mathcal{F}^\bullet$ be a perverse sheaf in $\mathcal{D}^b_{\mathbb{S}}(X)$. We then claim that

$$\mathcal{H}^j(\mathcal{F}^\bullet) = 0 \;\forall j \notin [-\dim_{\mathbb{C}} X, 0].$$

Fix $j > 0$. We need only show that that for any $x \in X$, the stalk $\mathcal{H}^j(\mathcal{F}^\bullet)_x = 0$. Indeed, by definition of perverse sheaves, we have

$$\dim_{\mathbb{C}} \operatorname{supp}^j(\mathcal{F}^\bullet) \le -j < 0$$

and since $\operatorname{supp}^j(\mathcal{F}^\bullet) = \overline{\{x \in X \mid \mathcal{H}^j(i_x^*\mathcal{F}^\bullet) \neq 0\}}$ and $\mathcal{H}^j(i_x^*\mathcal{F}^\bullet) \cong i_x^*\mathcal{H}^j(\mathcal{F}^\bullet)$, we see that $\mathcal{H}^j(\mathcal{F}^\bullet)_x = 0$ for all $x \in X$ and $j > 0$. The other side can also be seen easily.

4.0.8 (Perverse sheaves over a cone)**.** In a paper of Beilinson in which he given an alternate construction of nearby and vanishing cycles construction, it is shown that if X is the usual open cone on S^1, then the category of perverse sheaves over X is equivalent to category of diagrams of complex vector spaces $f : V \leftrightarrows W : g$ such that $\mathrm{id} - gf$ and $\mathrm{id} - gf$ are invertible operators.

We now show that the intersection complexes $\mathcal{IS}^\bullet_X$ and $\mathcal{IC}^\bullet_X$ are both perverse (with lower middle perversity, which is omitted from notation).

Theorem 4.0.9. *The shifted intersection complexes* $\mathcal{IS}^\bullet_X[-\dim_C X]$ *and* $\mathcal{IC}^\bullet_X[-\dim_C X]$ *are both*

1. *cohomologically* $\mathbb{S}$*-constructible (i.e. in* $\mathcal{D}^b_{\mathbb{S}}(X)$*)*
2. *perverse (i.e. in* $\mathcal{P}erv_{\mathbb{S}}(X)$*).*

Proof. We have shown that both these complexes are isomorphic in $\mathcal{D}^b(X)$ as both admit an isomorphism to Deligne's sheaf (**TODO**). We hence only prove this for $\mathcal{IC}^\bullet_X$. Recall that we showed earlier that $\mathcal{IC}^\bullet_X$ is cohomologically $\mathbb{S}$-constructible (**TODO**). As shifting only shifts the cohomology, therefore $\mathcal{IC}^\bullet_X[-\dim_{\mathbb{C}} X]$ is also cohomologically $\mathbb{S}$-constructible.

Now we wish to show that $\mathcal{IC}^\bullet_X[-\dim_{\mathbb{C}} X]$ is perverse. To this end, we will show that for any stratum $S \subseteq X$, the item 4 of Theorem 4.0.3 is satisfied. First, pick stratum S of positive codimension and any point $x \in S$. We wish to the conditions in item 4 of Theorem 4.0.3 for $\mathcal{IC}^\bullet_X[-\dim_{\mathbb{C}} X][\dim_{\mathbb{C}} X] = \mathcal{IC}^\bullet_X$. Thus, we wish to show that

$$H^j(i_x^*\mathcal{IC}^\bullet_X) = 0 \;\forall j > -\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} S$$
$$H^j(i_x^!\mathcal{IC}^\bullet_X) = 0 \;\forall j < \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} X.$$

But both of these are immediate from Theorem 3.2.4 for lower middle perversity. This completes the proof. □

A main source of examples of perverse sheaves for us will be extended intersection complexes (see 3.1.2).

Theorem 4.0.10. *Let $S \subseteq X$ be a stratum of X and $\mathcal{L}$ be a local system over S. Consider the associated extended intersection complex with lower middle perversity $\mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}$ over X. Then, the shifted extended intersection complex $\mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$ is*

1. *cohomologically $\mathbb{S}$-constructible (i.e. in $\mathcal{D}^b_{\mathbb{S}}(X)$)*
2. *perverse (i.e. in $\mathcal{P}erv_{\mathbb{S}}(X)$).*

Proof. The cohomological constructibility of $\mathcal{IS}^\bullet_{X,\mathcal{L}}$ we omit. To show that $\mathcal{IS}^\bullet_{X,\mathcal{L}}$ is perverse, we follow the same proof as Theorem 4.0.9, where the item 4 Theorem 4.0.3 is true even for intersection complex with local coefficient. □

4.1 Properties of category of perverse sheaves

We now state the main properties of $\mathcal{P}erv_{\mathbb{S}}(X)$. The first being that it is abelian with triangles only coming from short exact sequences.

Theorem 4.1.1 ($\mathcal{P}erv_{\mathbb{S}}(X)$ is abelian)**.** *The category of perverse sheaves $\mathcal{P}erv_{\mathbb{S}}(X)$ is abelian with*

$$0 \to \mathcal{F}^\bullet \overset{u}{\to} \mathcal{G}^\bullet \overset{v}{\to} \mathcal{C}^\bullet \to 0$$

is an exact sequence in $\mathcal{K}om(X)$ if and only if there is a map $\mathcal{C}^\bullet \longrightarrow \mathcal{F}^\bullet[1]$ such that

$$\mathcal{F}^\bullet \overset{u}{\longrightarrow} \mathcal{G}^\bullet \overset{v}{\longrightarrow} \mathcal{C}^\bullet \longrightarrow \mathcal{F}^\bullet[1]$$

is a standard triangle in $\mathcal{P}erv_{\mathbb{S}}(X)$.

The proof that $\mathcal{P}erv_{\mathbb{S}}(X)$ is abelian follows from setting up a triangulated structure on the category $\mathcal{D}_{\mathbb{S}}(X)$, finding a t-structure on $\mathcal{D}_{\mathbb{S}}(X)$ and then showing that the core of that t-structure is exactly the subcategory $\mathcal{P}erv_{\mathbb{S}}(X)$. The result will then follow from the general result that core of any t-structure is an abelia subcategory.

If we are given a short exact sequence in $\mathcal{K}om(X)$, then by Theorem A.3.6, we know that we get a distinguished triangle. The non-trivial statement here is that every distinguished triangle comes only from a short exact sequence of perverse sheaves.

The next important theorem is that on $\mathcal{P}erv_{\mathbb{S}}(X)$ the Verdier duality (see §A.4) functor D_X restricts to $D_X : \mathcal{P}erv_{\mathbb{S}}(X)^{op} \to \mathcal{P}erv_{\mathbb{S}}(X)$.

Theorem 4.1.2 (Verdier duality)**.** *The Verdier duality functor $D_X : \mathcal{D}^b(X)^{op} \to \mathcal{D}^b(X)$ restricts to a functor $D_X : \mathcal{P}erv_{\mathbb{S}}(X) \to \mathcal{P}erv_{\mathbb{S}}(X)$. Furthermore, Verdier duality on $\mathcal{P}erv_{\mathbb{S}}(X)$ is an exact functor.*

Proof. We will prove the first part of the claim here. This is clear from Theorem A.4.3 and 4.0.3. The other part follows from theory of t-structures. □

4.1.3 (Verdier dual of extended intersection complexes)**.** Let $S \hookrightarrow X$ be a stratum and $\mathcal{L}$ be a local system over S. We know that we get an extended intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ over X and we have shown that this is also perverse (Theorem 4.0.10). We now show that the Verdier dual of $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$ is just $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}^{\vee}}[-\dim_{\mathbb{C}} S]$, that is,

$$D_X \mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S] \cong \mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}^{\vee}}[-\dim_{\mathbb{C}} S],$$

for a stratum $S \hookrightarrow X$ in X. It follows from Theorem A.4.3. This infact verifies that Verdier dual of a perverse sheaves of the form $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$ are indeed perverse again.

We next show that the category of perverse sheaves is both Noetherian and Artinian. Recall that an object a in an abelian category is said to be *simple* if there is no non-trivial short-exact sequence $0 \to p \to a \to q \to 0$. For example, prime cyclic groups are exactly the simple objects of $\mathcal{A}b$. We first state an equivalent formulation for a category to be both Artinian and Noetherian. Recall that an abelian category is Artinian (Noetherian) if each object is Artinian (Noetherian).

Proposition 4.1.4. *Let $\mathcal{A}$ be an abelian category and $A \in \mathcal{A}$. Then the following are equivalent:*

1. *A is Noetherian and Artinian.*
2. *There exists a filtration $0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n = A$ by subobjects such that A_i/A_{i-1} is simple for $i = 1, \dots, n$.*

Proof. StacksProject 0FCJ. □

Theorem 4.1.5 ($\mathcal{P}erv_{\mathbb{S}}(X)$ is Noetherian and Artinian)**.** *Consider the category of perverse sheaves $\mathcal{P}erv_{\mathbb{S}}(X)$ over X. The following are true:*

1. *Category $\mathcal{P}erv_{\mathbb{S}}(X)$ is Noetherian and Artinian; every perverse sheaf satisfies acc and dcc for its subobjects.*
2. *For any perverse sheaf $\mathcal{F}^{\bullet}$, there exists finitely many perverse sheaves $\mathcal{E}^{\bullet}_i$, $0 \leq i \leq k$ as in the following*

$$0 \hookrightarrow \mathcal{E}^{\bullet}_0 \hookrightarrow \mathcal{E}^{\bullet}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{E}^{\bullet}_k = \mathcal{F}^{\bullet}$$

   such that they form a composition series, that is, $\mathcal{E}^{\bullet}_i/\mathcal{E}^{\bullet}_{i-1}$ is a simple perverse sheaf.
3. *If $\mathcal{F}^{\bullet}$ is a simple perverse sheaf, then there is a quasi-isomorphism*

$$\mathcal{F}^{\bullet} \simeq \mathcal{I}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$$

   where $S \hookrightarrow X$ is a stratum of X and $\mathcal{L}$ is an irreducible local system on S.

To prove the third statement would require realizing the extended intersection complex with coefficients in a local system as a functor, whose formal properties will yield it. Therefore we postpone it for the next section.

Proof. (*Sketch*) 1. Pick any perverse sheaf $\mathcal{F}^{\bullet}$ over X and consider a strictly decreasing filtration $\mathcal{F}^{\bullet} = \mathcal{E}^{\bullet}_1 \supset \mathcal{E}^{\bullet}_2 \supset \dots$. We claim that there exists an $N \in \mathbb{N}$ such that $\mathcal{E}^{\bullet}_N$ is supported in a strictly smaller dimensional subset than X. As X is finite dimensional, it will then follow that $\mathcal{E}_i^\bullet$ eventually goes to 0 after some large i. This shows that $\mathcal{P}erv_{\mathbb{S}}(X)$ is Artinian. Verdier duality reverses inclusions and is exact, therefore we get Noetherian for free.

2. Follows from Proposition 4.1.4.

□

4.2 Intermediate extensions

Let X be a $\mathbb{C}$-analytic or algebraic variety with a fixed Whitney stratification $\mathbb{S}$ with $\dim_{\mathbb{C}} X = n$.

4.2.1 (Intermediate extension)**.** Let $j : U \hookrightarrow X$ be an open subvariety and let $\mathcal{G}^\bullet \in \mathcal{P}erv_{\mathbb{S}}(U)$. Then there exists a unique extension $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathbb{S}}(X)$ of $\mathcal{G}^\bullet$ (that is, $j^*\mathcal{F}^\bullet = \mathcal{G}^\bullet$) such that for any stratum $i_V : V \hookrightarrow X - U = Z$, we have

$$\mathcal{H}^k(i_V^*\mathcal{F}^\bullet) = 0 \; \forall k \geq -\dim_{\mathbb{C}} V$$
$$\mathcal{H}^k(i_V^!\mathcal{F}^\bullet) = 0 \; \forall k \leq -\dim_{\mathbb{C}} V.$$

Moreover, this is functorial in nature and we denote $\mathcal{F}^\bullet = j_{!*}(\mathcal{G}^\bullet)$, i.e. $j_{!*} : \mathcal{P}erv_{\mathbb{S}}(U) \to \mathcal{P}erv_{\mathbb{S}}(X)$.

The following are important properties of intermediate extensions which shows that they give us the extended intersection complexes.

Theorem 4.2.2. *We have the following statements for intermediate extensions.*

1. *[Maxim] If $j : U \hookrightarrow X$ is the non-singular locus of X and $\mathcal{L}$ a local system on U, then*

$$j_{!*}(\mathcal{L}[\dim_{\mathbb{C}} X]) \simeq \mathcal{IS}^\bullet_{X,\mathcal{L}}[-\dim_{\mathbb{C}} X].$$

2. *[Maxim] Let $j : U \hookrightarrow X$ be an open subvariety. If $\mathcal{G}^\bullet \in \mathcal{P}erv_{\mathbb{S}}(X)$ is simple, then $j_{!*}(\mathcal{G}^\bullet) \in \mathcal{P}erv_{\mathbb{S}}(X)$ is simple.*
3. *[BBD] Let $j : U \hookrightarrow X$ be the non-singular locus and $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathbb{S}}(X)$. Then,*

$$j_{!*}(j^*\mathcal{F}^\bullet) \simeq \mathcal{F}^\bullet.$$

4. *[BBD] If $j : U \hookrightarrow X$ is an open subvariety, $i : Z = X - U \hookrightarrow X$ and $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathbb{S}}(X)$ then we have a short exact sequence*

$$0 \to j_{!*}(\mathcal{G}_1^\bullet) \to \mathcal{F}^\bullet \to i_*(\mathcal{G}_2^\bullet) \to 0$$

   where $\mathcal{G}_1^\bullet \in \mathcal{P}erv_{\mathbb{S}}(U)$ and $\mathcal{G}_2^\bullet \in \mathcal{P}erv_{\mathbb{S}}(Z)$.

Using these, we first prove Theorem 4.1.5, 3.

Proof of Theorem 4.1.5, 3. Let $\mathcal{F}^\bullet \in \mathcal{P}erv_{\mathbb{S}}(X)$ be simple. Begin with an arbitrary stratum $T \subseteq X - X_{n-2}$ in the non-singular locus. We have the following decomposition

$$U = X - \bar{T} \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} \bar{T} = Z.$$

Now, by Theorem 4.2.2, 4 and simplicity, we get that $\mathcal{F}^\bullet = j_{!*}(\mathcal{G}_1^\bullet)$ for some $\mathcal{G}_1^\bullet \in \mathcal{P}erv_{\mathbb{S}}(U)$ or $\mathcal{F}^\bullet = i_*\mathcal{G}_2^\bullet$ for some $\mathcal{G}_2^\bullet \in \mathcal{P}erv_{\mathbb{S}}(Z)$. Consider the former case. Replacing X by U and repeating the above process inducitvely, we see by finite dimensionality of X that we can reduce to the latter case, that $\mathcal{F}^\bullet = i_*\mathcal{G}_2^\bullet$ for some $\mathcal{G}_2^\bullet \in \mathcal{P}erv_{\mathbb{S}}(Z)$.

Now, in this case, note that we may replace X by Z and $\mathcal{F}^\bullet$ by $\mathcal{G}_2^\bullet$. Thus we may assume that X is an irreducible pseudomanifold and $\mathcal{F}^\bullet$ is supported on X. Now, let $\Sigma = X - X_{n-2}$ be the non-singular stratum (only one because of irreducibility). Then, let $j : \Sigma \hookrightarrow X$ be the open inclusion. As $j^*\mathcal{F}^\bullet$ is a perverse sheaf on Σ, so it is quasi-isomorphic to $\mathcal{L}[\dim_{\mathbb{C}} X]$ where $\mathcal{L}$ is irreducible since $j^*\mathcal{F}^\bullet$ is simple. Now, we see by Theorem 4.2.2, 3 that

$$j_{!*}(j^*\mathcal{F}^\bullet) \simeq \mathcal{F}^\bullet$$

and by Theorem 4.2.2, 1

$$j_{!*}(j^*\mathcal{F}^\bullet) \simeq j_{!*}(\mathcal{L}[\dim_{\mathbb{C}} X]) \simeq \mathcal{IS}^\bullet_{X,\mathcal{L}}[-\dim_{\mathbb{C}} X].$$

Thus unravelling, we see that the original $\mathcal{F}^\bullet$ on X is quasi-isomorphic to $i_*\mathcal{IS}^\bullet_{X,\mathcal{L}}[-\dim_{\mathbb{C}} X]$, where $i : Z = \bar{T} \hookrightarrow X$, as needed. □

4.3 Examples in complex varieties

We show two instances of perverse sheaves that appears in AG.

Proposition 4.3.1. *Let X, Y be complex quasi-projective varieties with Whitney stratifications $\mathbb{S}$ and $\mathbb{T}$ respectively. Let $f : X \to Y$ be a finite map. Then $Rf_* : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ descends to a functor $\mathcal{P}erv_{\mathbb{S}}(X) \to \mathcal{P}erv_{\mathbb{T}}(Y)$.*

Proof. (*Very brief sketch*) Show that it takes cohomologically constructible complexes onto itself is a long process and is done in §3.8-3.10 of Achar's book.

Next, to see it preserves perverse sheaves, it suffices to show it induces a "t-exact" functor $Rf_*\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$. By Theorem 4.1.1, it further suffices to show that $f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ is exact. Thus we need to show that all higher i^{th}-right derived functors of f_* are zero. This is also a long and complicated procedure, done in Achar's book. □

Next, we study local complete intersections and show that any twisted constant sheaf over a local complete intersection is a perverse sheaf.

Theorem 4.3.2. *Let X be an analytic variety of pure dimension n (every component is of dimension n) in $\mathbb{C}^m$. If X is local complete intersection and K a field, then $\underline{K}[n]$ is a perverse sheaf over X*[3].

Proof. (*Brief sketch*) We may write $X = X_n \amalg X_0$, where X_n is the non-singular stratum and X_0 the discrete collection of isolated complete intersection singularities of X. Observe that $\mathcal{H}^j(\underline{K}[n])_x$ is K for $j = -n$ and 0 else. Now, we claim the following:

$$\mathcal{H}^j(i_0^*\underline{K}[n]) = 0 \text{ if } j > 0$$
$$\mathcal{H}^j(i_n^*\underline{K}[n]) = 0 \text{ if } j > -n.$$

[3]This is really worthwhile because of remarks made in 4.0.6.

The first follows from the previous calculation for if it is non-zero, then for one of the stalks will be non-zero. The latter follows from oberving that $i_n^* \underline{K}[n]$ is a local system on X_n, which is a manifold, so it is perverse, and thus we win by 4.0.7. So this makes $\underline{K}[n]$ satisfy the support condition of the definition. For the cosupport condition, we need some local calculations as we did last time and the properties of Verdier duality we saw above, to reduce to calculating reduced cohomology of the link L_x of $x \in X_0$. Then one concludes by using a result that link of such points in an n-dimensional local complete intersection are $n-2$-connected. □

A Derived category of a space

Let X be a space and $\mathcal{S}h(X)$ be the abelian category of abelian sheaves over X. We want to construct the derived category of the abelian category $\mathcal{S}h(X)$ so that we can have a very clean setup for homological algebra. However, defining this category requires a considerable amount of preparation, which we first undertake.

A.1 Preliminary-I : Injective resolutions, soft sheaves and hypercohomology

A.1.1 (Acyclic sheaves for global sections)**.** Let X be a paracompact space. We find a class of sheaves in $\mathcal{S}h(X)$ which is acyclic for global sections left-exact functor $\Gamma : \mathcal{S}h(X) \to \mathcal{A}b$. Indeed, define a sheaf $\mathcal{F} \in \mathcal{S}h(X)$ to be **soft** if the restriction map $\Gamma(X, \mathcal{F}) \to \mathcal{F}(C)$ is surjective for all closed sets, where $\mathcal{F}(C) = \varinjlim_{U \supseteq C} \mathcal{F}(U)$. We also call $\mathcal{F}$ **c-soft** if $\Gamma(X, \mathcal{F}) \to \mathcal{F}(K)$ for all compact sets $K \subseteq X$ is surjective. The following theorem tells us what we want.

Theorem A.1.2. *Let X be a paracompact space. Then,*

1. *Any c-soft sheaf is soft.*
2. *Any soft sheaf $\mathcal{F}$ is Γ-acyclic. That is, for each $i \geq 1$, we have*

$$H^i(X, \mathcal{F}) = 0.$$

Hence, using soft-resolutions, we may compute cohomology in $\mathcal{S}h(X)$.

We will soon define derived functor hypercohomology. But before that, we need preparations for resolutions of complexes.

A.1.3 (Direct image with proper support)**.** Let $f : X \to Y$ be a map of spaces and let $\mathcal{F} \in \mathcal{S}h(X)$. Define $f_!\mathcal{F}$ to be the sheaf over Y obtained by the sheafification of the presheaf

$$V \mapsto \{s \in \mathcal{F}(f^{-1}(V)) \mid f|_{|s|} : |s| \to Y \text{ is proper}\}.$$

This is called the *direct image with proper supports.* Recall $|s| = \overline{\{x \in X \mid s_x \neq 0\}}$ is the support of a section $s \in \mathcal{F}(U)$.

We then observe that if $p_X : X \to \{*\}$ is the terminal map and $\mathcal{F} \in \mathcal{S}h(X)$, then $p_{X!}\mathcal{F} = \Gamma_c(X, \mathcal{F})$, the global sections with compact support.

A.1.4 (Injective resolution of complex of sheaves)**.** Let $\mathcal{F}^\bullet$ be a complex of sheaves. An injective resolution of $\mathcal{F}^\bullet$ is a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. Recall $f_\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism if on the i^{th}-cohomology sheaves we have an isomorphism $\mathcal{H}^i(f_\bullet) : \mathcal{H}^i(\mathcal{F}^\bullet) \to \mathcal{H}^i(\mathcal{G}^\bullet)$.

Note that these cohomology sheaves are not the ones which will actually compute our groups; this will be only useful if we will have an additive left exact functor $\mathcal{A} \to \mathcal{S}h(X)$ and then wish to compute its derived functor cohomology. Such a situation would rarely arise. However, we will use these in our discussion of perverse sheaves.

For any complex of sheaves, we can get two more complexes; for each point $x \in X$ we may compute the stalks and get the stalk complex and we may use global sections functor to get a complex of abelian groups. The latter would be later used to construct hypercohomology of a complex of sheaves.

A.1.5 (Stalk cohomology)**.** Let $\mathcal{F}^\bullet$ be a complex of sheaves. By taking stalk at $x \in X$ to get $\mathcal{F}_x^\bullet$, we obtain a complex of abelian groups called the *stalk complex at* $x \in X$. The following lemma tells us that we can compute its cohomology without leaving the category $\mathcal{S}h(X)$. We will later see this in the setting of derived functors.

Lemma A.1.6 (Cohomology object preservation)**.** *Exact functor on an abelian category takes cohomology objects to cohomology objects.*

A.1.7 (Hypercohomology)**.** Let $\mathcal{F}^\bullet$ be a bounded complex of sheaves and $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}^\bullet$. Then, the hypercohomology of $\mathcal{F}^\bullet$ is defined to be the global sections cohomology of $\mathcal{I}^\bullet$:

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) := H^i(\Gamma(X, \mathcal{I}^\bullet)).$$

The following list of results justify the well-definedness of this construction.

Theorem A.1.8 (Hypercohomology preparation theorem)**.** *Let $\mathcal{F}^\bullet$ be a bounded complex of sheaves. Then,*

1. *Complex $\mathcal{F}^\bullet$ has an injective resolution.*
2. *Any two injective resolutions of $\mathcal{F}^\bullet$ are homotopy equivalent.*
3. *Injective resolutions of quasi-isomorphic complexes are homotopy equivalent.*

Thus for two injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $\mathcal{F}^\bullet \to \mathcal{J}^\bullet$, we get a homotopy $\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ to which we apply $\Gamma(X, -)$ to get a homotopy $\Gamma(X, \mathcal{I}^\bullet) \to \Gamma(X, \mathcal{J}^\bullet)$. As homotopic complexes induce same maps in cohomology, it follows that hypercohomology is well-defined.

For purposes of calculation, we have an important tool which calculates the hypercohomology of a complex of soft sheaves by calculating its global sections cohomology.

Theorem A.1.9 (Soft complex theorem)**.** *Let $\mathcal{F}^\bullet$ be a complex of soft sheaves. Let $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ be an injective resolution. Then, the global sections cohomology of $\mathcal{F}^\bullet$ is isomorphic to hypercohomology of $\mathcal{F}^\bullet$. That is for all $i \geq 0$*

$$H^i(\Gamma(X, \mathcal{F}^\bullet)) \cong \mathbb{H}^i(X, \mathcal{F}^\bullet).$$

In the main text, we will see that the intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}_X^\bullet$ is a complex of soft sheaves, therefore computing its hypercohomology is equivalent to computing its global sections cohomology.

A.2 Preliminary-II : Operations and functors on complexes

Let X be a space and $\mathcal{S}h(X)$ be the category of sheaves over X. Further denote the category of cochain complexes and maps of complexes by $\mathcal{K}om(X)$.

A.2.1 (Shift functor)**.** For each $n \in \mathbb{Z}$, we have a functor

$$\begin{aligned} [n] : \mathcal{K}om(X) &\longrightarrow \mathcal{K}om(X) \\ \mathcal{F}^\bullet &\longmapsto \mathcal{F}^\bullet[n] \end{aligned}$$

where $(\mathcal{F}^\bullet[n])^i = \mathcal{F}^{i+n}$ and the boundary map

$$d^i[n] : (\mathcal{F}^\bullet[n])^i = \mathcal{F}^{i+n} \longrightarrow (\mathcal{F}^\bullet[n])^{i+1} = \mathcal{F}^{i+n+1}$$

is defined to be $d^i[n] = (-1)^n d^{i+n}$. For a map $f_\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$, we define $f[n]_i = f_{i+n}$.

A.2.2 (Truncation functor)**.** For each $n \in \mathbb{Z}$, we define two functors:

$$\tau_{\leq n}, \tau_{\geq n} : \mathcal{K}om(X) \to \mathcal{K}om(X)$$

where $\tau_{\leq n}\mathcal{F}^\bullet = \cdots \to \mathcal{F}^{n-1} \to \operatorname{Ker}(d^n) \to 0 \ldots$ and $\tau_{\geq n}\mathcal{F}^\bullet = \cdots \to 0 \to \operatorname{CoKer}\left(d^{n-1}\right) \to \mathcal{F}^{n+1} \to \ldots.$

The main point of truncation functors is that they truncate the cohomology sheaves as well.

Proposition A.2.3 (Cohomological truncation)**.** *Let $\mathcal{F}^\bullet \in \mathcal{K}om(X)$ and fix $n \in \mathbb{Z}$. Then,*

$$\mathcal{H}^i(\tau_{\leq n}\mathcal{F}^\bullet) = \begin{cases} \mathcal{H}^i(\mathcal{F}^\bullet) & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

and

$$\mathcal{H}^i(\tau_{\geq n}\mathcal{F}^\bullet) = \begin{cases} \mathcal{H}^i(\mathcal{F}^\bullet) & \text{if } i \geq n \\ 0 & \text{if } i < n. \end{cases}$$

We now define a list of operations on complexes that will come in handy in the derived setting.

A.2.4 (Operations on complexes)**.** Let $f : X \to Y$ be a continuous map of spaces.

1. *Hom complex* : Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{K}om(X)$. Then, the hom complex $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ given by the complex of sheaves $\mathcal{H}om^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \prod_{p\in\mathbb{Z}} \mathcal{H}om(\mathcal{F}^p, \mathcal{G}^{n+p+1})$ with the differential

$$d^n : \mathcal{H}om^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \longrightarrow \mathcal{H}om^{n+1}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

which on an open set $U \subseteq X$ gives the following map

$$\begin{aligned} d^n_U : \prod_{p\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{S}h(X)}\left(\mathcal{F}^p|_U, \mathcal{G}^{n+p}|_U\right) &\longrightarrow \prod_{p\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{S}h(X)}\left(\mathcal{F}^p|_U, \mathcal{G}^{n+p+1}|_U\right) \\ (f_p)_{p\in\mathbb{Z}} &\longmapsto d \circ f_p + (-1)^{n+1} f_{p+1} \circ d \end{aligned}$$

obtained from the following diagram (which may NOT be commutative, as we don't have that (f_p) forms a cochain map!)

$$\begin{array}{ccc} \mathcal{F}^p|_U & \xrightarrow{f_p} & \mathcal{G}^{n+p}|_U \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \mathcal{F}^{p+1}|_U & \xrightarrow[f_{p+1}]{} & \mathcal{G}^{n+p+1}|_U \end{array} .$$

2. *Direct and inverse image complexes* : We have two functors corresponding to map $f : X \to Y$. The first is the direct image functor:

$$\begin{aligned} f_* : \mathcal{K}om(X) &\longrightarrow \mathcal{K}om(Y) \\ \mathcal{F}^\bullet &\longmapsto (f_*\mathcal{F}^\bullet)^i := f_*\mathcal{F}^i \\ d : \mathcal{F}^i \to \mathcal{F}^{i+1} &\longmapsto f_*d : f_*\mathcal{F}^i \to f_*\mathcal{F}^{i+1}. \end{aligned}$$

   The other is the inverse image functor:

$$\begin{aligned} f^* : \mathcal{K}om(Y) &\longrightarrow \mathcal{K}om(X) \\ \mathcal{G}^\bullet &\longmapsto (f^*\mathcal{G}^\bullet)^i := f^*\mathcal{G}^i \\ d : \mathcal{G}^i \to \mathcal{G}^{i+1} &\longmapsto f^*d : f^*\mathcal{G}^i \to f^*\mathcal{G}^{i+1}. \end{aligned}$$

3. *Tensor product of complexes* : Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ be two bounded complexes in $\mathcal{K}om(X)$. Then, the tensor product complex $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is obtained as follows:

$$(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^i := \bigoplus_{p+q=i} \mathcal{F}^p \otimes \mathcal{G}^q$$

   where to define the differential, we first reduce to defining its restriction on $\mathcal{F}^p \otimes \mathcal{G}^q \to \bigoplus_{k+l=i+1} \mathcal{F}^k \otimes \mathcal{G}^l$. To this end, we again reduce to defining a map of presheaves

$$(\mathcal{F}^p \otimes \mathcal{G}^q)^- \longrightarrow \bigoplus_{k+l=i+1} \mathcal{F}^k \otimes \mathcal{G}^l$$

   where $(\mathcal{F}^p \otimes \mathcal{G}^q)^-$ is the presheaf $U \mapsto \mathcal{F}^p(U) \otimes \mathcal{G}^q(U)$. To this end, pick an open set $U \subseteq X$ and define the following map on simple tensors:

$$\begin{aligned} \mathcal{F}^p(U) \otimes \mathcal{G}^q(U) &\longrightarrow \bigoplus_{k+l=i+1} (\mathcal{F}^k \otimes \mathcal{G}^l)(U) \\ x^p \otimes y^q &\longmapsto dx^p \otimes y^q + (-1)^p x^p \otimes dy^q, \end{aligned}$$

   just like the usual tensor product of complexes of modules. Most of the time in our purposes, we would be satisfied by the knowledge of the differential of $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ on the simple tensors of the components, hence this description would suffice.

A.3 Derived category

Our fundamental goal is to construct a category of complexes where quasi-isomorphisms are isomorphisms. We also want a good hypercohomology theory to appear out of this category; short exact sequences of complexes in this category must yield long exact sequence in hypercohomology. Experience says that such categories also have more richer properties than one initially expects, making them ideal for conceptually understanding homological calculations.

We begin by first constructing a category where homotopy equivalences are inverted.

A.3.1 (Homotopy category and its defect)**.** Define $h\mathcal{K}om(X)$ to be the homotopy category of $\mathcal{K}om(X)$ where objects are same as $\mathcal{K}om(X)$ but maps are homotopy classes of maps of $\mathcal{K}om(X)$. This category satisfies some of our needs:

1. If $f^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a map of complexes which is a homotopy equivalence, then $[f^\bullet]$ determines a map in $h\mathcal{K}om(X)$ which is an isomorphism. Consequently, any two injective resolutions are isomorphic in $h\mathcal{K}om(X)$.
2. $h\mathcal{K}om(X)$ is an additive category.

However, a big drawback is that $h\mathcal{K}om(X)$ is NOT an abelian category! Consequently, we cannot make sense of short-exact sequences, let alone cohomology. However we can salvage this by considering a replacement of exact sequences in $h\mathcal{K}om(X)$, which we discuss next.

A.3.2 (Mapping cones, standard triangles and triangles)**.** Let $h\mathcal{K}om(X)$ be the homotopy complex category over X. A standard triangle in $h\mathcal{K}om(X)$ is given by the *mapping cone* of a map $\varphi^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathrm{Cone}^\bullet(\varphi)$, which is defined as the following complex:

$$\mathrm{Cone}^i(\varphi) = \mathcal{F}^{i+1} \oplus \mathcal{G}^i$$

with differential given by

$$d^i : \mathrm{Cone}^i(\varphi) = \mathcal{F}^{i+1} \oplus \mathcal{G}^i \longrightarrow \mathrm{Cone}^{i+1}(\varphi) = \mathcal{F}^{i+2} \oplus \mathcal{G}^{i+1}$$

given by the matrix

$$d^i = \begin{bmatrix} -d^{i+1} & 0 \\ \varphi^{i+1} & d^i \end{bmatrix}.$$

The motivation behind this definition follows from analyzing singular chain complex over the mapping cone of a map between two spaces.

A *standard triangle* in $\mathcal{K}om(X)$ is a map of the following type:

$$\mathcal{F}^\bullet \xrightarrow{\varphi} \mathcal{G}^\bullet \xrightarrow{[0 \quad 1]} \mathrm{Cone}^\bullet(\varphi) \xrightarrow{[1 \quad 0]} \mathcal{F}^\bullet[1].$$

A *triangle* in $\mathcal{K}om(X)$ is given by any map of the following type:

$$\mathcal{F}^\bullet \xrightarrow{\varphi} \mathcal{G}^\bullet \xrightarrow{\phi} \mathcal{C}^\bullet \xrightarrow{[1]} \mathcal{F}^\bullet[1].$$

A *distinguished triangle* is a triangle which is quasi-isomorphic to a standard triangle.

The following proposition gives another good property of $h\mathcal{K}om(X)$.

Proposition A.3.3. *Any triangle in $h\mathcal{K}om(X)$ is isomorphic to a standard triangle.*

The following shows that one of our main goals that we wanted out of $h\mathcal{K}om(X)$ is achieved:

Lemma A.3.4. *Let the following be a triangle in $h\mathcal{K}om(X)$:*

$$\mathcal{F}^\bullet \xrightarrow{\varphi} \mathcal{G}^\bullet \xrightarrow{\phi} \mathcal{C}^\bullet \xrightarrow{[1]} \mathcal{F}^\bullet[1].$$

Then,

1. *The above triangle induces a long exact sequence in cohomology sheaves as in*

$$\cdots \to \mathcal{H}^i(\mathcal{F}^\bullet) \to \mathcal{H}^i(\mathcal{G}^\bullet) \to \mathcal{H}^i(\mathcal{C}^\bullet) \to \mathcal{H}^{i+1}(\mathcal{F}^\bullet) \to \dots.$$

2. *If the above triangle is a triangle of bounded complexes, we get a long exact sequence in hypercohomology as in*

$$\cdots \to \mathbb{H}^i(X, \mathcal{F}^\bullet) \to \mathbb{H}^i(X, \mathcal{G}^\bullet) \to \mathbb{H}^i(X, \mathcal{C}^\bullet) \to \mathbb{H}^{i+1}(X, \mathcal{F}^\bullet) \to \dots.$$

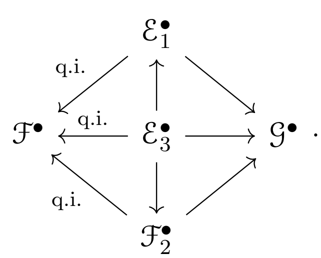
With all these properties of $h\mathcal{K}om(X)$, we still have a big glaring issue at hand; how do we associate a short exact sequence of complexes to a triangle in $h\mathcal{K}om(X)$? It can be shown that not all triangles in $h\mathcal{K}om(X)$ comes from a short exact sequence of complexes. For this, we observe the following statement for a category where objects are complexes and quasi-isomorphisms are isomorphisms:

Every short exact sequence of complexes gives a standard triangle in such a category.

Indeed, let $0 \to \mathcal{F}^\bullet \xrightarrow{u} \mathcal{G}^\bullet \xrightarrow{v} \mathcal{C}^\bullet \to 0$ be a short exact sequence in $\mathcal{K}om(X)$. We will construct a quasi-isomorphism $f^\bullet : \mathrm{Cone}^\bullet(u) \to \mathcal{C}^\bullet$. Indeed, consider the map $f^i : \mathrm{Cone}^i(u) = \mathcal{F}^{i+1} \oplus \mathcal{G}^i \to \mathcal{C}^i$ given by $[0\ v^i]$. To see this is a quasi-isomorphism, we need to check isomorphism on cohomology sheaves, which one can see by Lemma A.3.4.

Hence we now define a category where objects are complexes and arrows are such that quasi-isomorphisms are isomorphisms.

A.3.5 (Derived category)**.** Define $\mathcal{D}^b(X)$ to be the **bounded derived category of complexes over** X whose objects are complexes and arrows are as follows. An element of $\mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is an equivalence class of maps of complexes $\mathcal{F}^\bullet \xleftarrow{\text{q.i.}} \mathcal{E}^\bullet \longrightarrow \mathcal{G}^\bullet$ which is called a roof, where two roofs $\mathcal{F}^\bullet \xleftarrow{\text{q.i.}} \mathcal{E}_1^\bullet \longrightarrow \mathcal{G}^\bullet$ and $\mathcal{F}^\bullet \xleftarrow{\text{q.i.}} \mathcal{E}_2^\bullet \longrightarrow \mathcal{G}^\bullet$ are equivalent if there is a third roof $\mathcal{F}^\bullet \xleftarrow{\text{q.i.}} \mathcal{E}_3^\bullet \longrightarrow \mathcal{G}^\bullet$ which fits in the following commutative diagram:



We now state some collection of important facts about derived category of sheaves over a space.

Theorem A.3.6 (Basic properties of $D^b(X)$)**.** *Let X be a space and $\mathcal{D}^b(X)$ be the bounded derived category of sheaves over X. Then, the following are true:*

1. *There is a functor $D : \mathcal{K}om(X) \to \mathcal{D}^b(X)$ taking $\varphi^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ to $\mathcal{F}^\bullet \xleftarrow{\text{id}} \mathcal{F}^\bullet \xrightarrow{\varphi^\bullet} \mathcal{G}^\bullet$.*
2. *Every quasi-isomorphism of complexes induces an isomorphism in the derived category under the functor D.*
3. *Every short exact sequence in $\mathcal{K}om(X)$ has an associated standard triangle in $\mathcal{D}^b(X)$.*

4. *Every standard triangle in $\mathcal{D}^b(X)$ induces a long exact sequence of complexes of cohomology sheaves.*

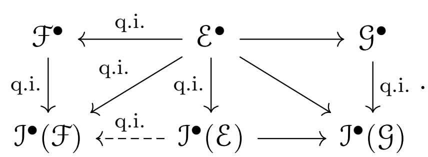
The following lemma is an important one for well-definedness of derived functors.

Lemma A.3.7 (Derived functor preparation lemma)**.** *Let X be a space and $F : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ be a left exact functor. If $\varphi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ is a quasi-isomorphism of injective complexes, then $F\varphi^\bullet : F\mathcal{I}^\bullet \to F\mathcal{J}^\bullet$ is a quasi-isomorphism, where $F\mathcal{I}^\bullet$ and $F\varphi^\bullet$ is point-wise application of functor F.*

A.3.8 (Derived functors)**.** Let $F : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ be a left exact functor. Then consider a functorial assignment of injective resolutions for each complex; consider the following functor:

$$\begin{aligned} \mathcal{I}^\bullet(-) : \mathcal{D}^b(X) &\longrightarrow \mathcal{D}^b(X) \\ \mathcal{F}^\bullet &\longmapsto \mathcal{I}^\bullet(\mathcal{F}) \end{aligned}$$

where $\mathcal{I}^\bullet(\mathcal{F})$ is an injective resolution of complex $\mathcal{F}^\bullet$. To define on morphisms of $D^b(X)$, we first observe the following diagram obtained by taking a map in $\mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ (as in the top row of the diagram below) and using the fact that any two injective resolutions of a same complex are homotopy equivalent, and thus quasi-isomorphic (Theorem A.1.8):

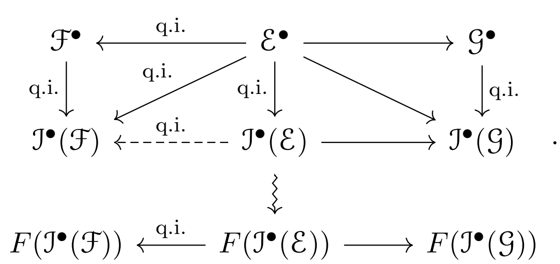


The bottom row of the above diagram is the required map on arrows.

Using this and Lemma A.3.7, we define the **right derived functor** of F as the following functor on derived categories:

$$\begin{aligned} RF : \mathcal{D}^b(X) &\longrightarrow \mathcal{D}^b(Y) \\ \mathcal{F}^\bullet &\longmapsto F(\mathcal{I}^\bullet(\mathcal{F})) \end{aligned}$$

where on arrows, we define the map as



Moreover, the i^{th}**-right derived functor of** F is defined to be the following composite:

$$\mathcal{D}^b(X) \xrightarrow{RF} \mathcal{D}^b(Y) \xrightarrow{\mathcal{H}^i(-)} \mathcal{S}h(Y)$$

where

$$\begin{aligned}\mathcal{H}^i(-) : \mathcal{D}^b(Y) &\longrightarrow \mathcal{S}h(Y)\\ \mathcal{G}^\bullet &\longmapsto \mathcal{H}^i(\mathcal{G}^\bullet)\end{aligned}$$

where on arrows, we map as

$$\begin{array}{ccc} \mathcal{G}^\bullet & & \mathcal{H}^i(\mathcal{G}^\bullet) \\ \uparrow \text{q.i.} & & \uparrow \cong \\ \mathcal{E}^\bullet & \longmapsto & \mathcal{H}^i(\mathcal{E}^\bullet) \\ \downarrow & & \downarrow \\ \mathcal{K}^\bullet & & \mathcal{H}^i(\mathcal{K}^\bullet) \end{array} .$$

We now give plenty examples of derived functors in this generality.

Example A.3.9 (Stalk, costalk and global sections cohomology)**.** Let X be a space and $x \in X$ be a point. Consider the inclusion $j_x : \{x\} \hookrightarrow X$. Consider also the projection $p_X : X \to \{x\}$. These gives the following functors on the sheaf categories:

$$j_x^* : \mathcal{S}h(X) \longrightarrow \mathcal{A}b \tag{A.3.10.1}$$

which is the *stalk functor,* which is exact as it is an inverse image map,

$$p_{X*} : \mathcal{S}h(X) \longrightarrow \mathcal{A}b \tag{A.3.10.2}$$

which is the *global sections functor*, which is left exact as it is direct image map.

We construct another functor

$$\Gamma_x : \mathcal{S}h(X) \longrightarrow \mathcal{A}b \tag{A.3.10.3}$$

which is called the *costalk functor*. Indeed, define

$$\Gamma_x(\mathcal{F}) := \varprojlim_{U \supseteq x} \Gamma_c(U, \mathcal{F})$$

where $\Gamma_c(U, \mathcal{F}) = \{s \in \mathcal{F}(U) \mid |s| \subseteq V \text{ is compact}\}$ and for $x \in V \subseteq U$, we define $\Gamma_c(V, \mathcal{F}) \to \Gamma_c(U, \mathcal{F})$ by taking $s \mapsto s_!$ where $s_!$ is the section obtained by extending s by zeros outside the support (which we can do by considering a finite open cover of $|s|$ in V and then obtaining a cover of U by considering that open cover together with $U \setminus |s|$). This defines the required map, so we can take limit. This gives the required functor of (A.3.10.3).

We can right derive these three functors to obtain the following three functors[4]:

$$Rj_x^*, Rp_{X*}, R\Gamma_x : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(x)$$

[4]we choose to write $\mathcal{D}^b(\mathcal{A}b)$ as the bounded derived category over a point to streamline all three functors.

where if $\mathcal{F}^\bullet \xrightarrow{\text{q.i.}} \mathcal{I}^\bullet(\mathcal{F})$ is an injective resolution, then we have

$$j_x^*(\mathcal{F}^\bullet) = Rj_x^*(\mathcal{F}^\bullet) = j_x^*\mathcal{I}^\bullet(\mathcal{F})$$

is the stalk complex of the injective resolution[5],

$$Rp_{X*}(\mathcal{F}^\bullet) = p_{X*}\mathcal{I}^\bullet(\mathcal{F})$$

is the global sections complex of the injective resolution and

$$j_x^!(\mathcal{F}^\bullet) := R\Gamma_x(\mathcal{F}^\bullet) = \Gamma_x\mathcal{I}^\bullet(\mathcal{F})$$

is the costalk complex of the injective resolution.

Furthermore, the i^{th}-right derived functor of j_x^*, p_{X*} and Γ_x which respectively assigns cohomology of the above three complexes of abelian groups are called **stalk cohomology**, **global sections cohomology** and **costalk cohomology** respectively.

Note that global sections cohomology is just the hypercohomology of $\mathcal{F}^\bullet$. In particular, i^{th}-hypercohomology is the i^{th}-right derived functor of global sections functor p_{X*}:

$$\begin{aligned} \mathbb{H}^i(X,-) : \mathcal{D}^b(X) &\longrightarrow \mathcal{A}b \\ \mathcal{F}^\bullet &\longmapsto H^i(p_{X*}\mathcal{I}^\bullet(\mathcal{F})). \end{aligned}$$

These three will be heavily used in the main text as they store vital information about the complex $\mathcal{F}^\bullet$.

There is a notion of **compactly supported hypercohomology** as well. For this, consider $p_X : X \to \{x\}$ and consider the direct image with proper support (see A.1.3)

$$p_{X!} : \mathcal{S}h(X) \longrightarrow \mathcal{A}b.$$

Again, this is left-exact, we may thus right derive it to obtain a functor

$$\begin{aligned} Rp_{X!} : \mathcal{D}^b(X) &\longrightarrow \mathcal{D}^b(x) \\ \mathcal{F}^\bullet &\longmapsto p_{X!}\mathcal{I}^\bullet(\mathcal{F}) = \Gamma(X, \mathcal{I}^\bullet(\mathcal{F})). \end{aligned}$$

The i^{th}-right derived functor of $p_{X!}$ is called i^{th}-hypercohomology with compact support, denoted $\mathbb{H}_c^i(X, \mathcal{F}^\bullet)$.

A.4 Verdier duality

Let X be an n-pseudomanifold. We now construct a functor $D_X : \mathcal{D}^b(X)^{op} \to \mathcal{D}^b(X)$ which will generalize the notion of the dual of a vector space/abelian groups/modules; i.e. it generalizes the contravariant functor $\mathrm{Hom}_{\mathcal{M}od(R)}(-, R)$. It follows after some reasoning that the mapping $\mathcal{E} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{D}_X)$ for an appropriate "dualizing sheaf" satisfies all the usual properties of the "expected dual of $\mathcal{E}$" if we assume that $\mathcal{D}_X$ is not just a sheaf, but a complex of sheaves, as we will see that it is this which satisfies the required properties we expect from a "dual object" (i.e. things like having a natural map into the double dual and generalizing the one for vector spaces, etc.)

[5] the equality $j_x^*(\mathcal{F}^\bullet) = Rj_x^*(\mathcal{F}^\bullet)$ is true by exactness of inverse image where j_x^* is treated as a functor on $\mathcal{D}^b(X)$ which applies j_x^* pointwise.

A.4.1 (Contravariant sheaf hom functor)**.** Let X be a space and $\mathcal{E}^\bullet \in \mathcal{K}om(X)$ be any complex. Consider the functor

$$\begin{aligned} \mathcal{H}om(-, \mathcal{E}^\bullet) : \mathcal{K}om(X) &\longrightarrow \mathcal{K}om(X) \\ \mathcal{F}^\bullet &\longmapsto \mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \end{aligned}$$

where the complex $\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$ is defined as follows:

$$(\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet))^i = \mathcal{H}om(\mathcal{F}^i, \mathcal{E}^i)$$

for all $i \in \mathbb{Z}$. For a map of complexes $\varphi_\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$, we get

$$\varphi_\bullet^* : \mathcal{H}om(\mathcal{G}^\bullet, \mathcal{E}^\bullet) \longrightarrow \mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$$

which on degree $i \in \mathbb{Z}$ is

$$\begin{aligned} \varphi_i^* : \mathcal{H}om(\mathcal{G}^i, \mathcal{E}^i) &\longrightarrow \mathcal{H}om(\mathcal{F}^i, \mathcal{E}^i) \\ f : \mathcal{G}^i \to \mathcal{E}^i &\longmapsto f \circ \varphi_i. \end{aligned}$$

It can be seen that $\mathcal{H}om(-, \mathcal{E}^\bullet)$ functor is left exact as it taking sections is left exact and taking direct limits is an exact operation. Hence we get a contravariant functor at the derived level by right-deriving the $\mathcal{H}om(-, \mathcal{E}^\bullet)$:

$$R\mathcal{H}om(-, \mathcal{E}^\bullet) : \mathcal{D}^b(X)^{op} \longrightarrow \mathcal{D}^b(X).$$

A.4.2 (Verdier dual)**.** Consider X to be an n-pseudomanifold and consider the singular complex $\mathcal{S}_X^\bullet$. We define the Verdier dual functor to be the following right derived contrvariant hom of singular complex:

$$D_X(-) := R\mathcal{H}om(-, \mathcal{S}_X^\bullet) : \mathcal{D}^b(X)^{op} \longrightarrow \mathcal{D}^b(X).$$

We now see that the Verdier duality functor D_X is indeed the "right" duality functor as it satisfies the usual properties we expect from duals.

Theorem A.4.3. *Let X be an n-pseudomanifold. The functor D_X satisfies the following properties:*

1. *D_X takes distinguished triangles to distinguished triangles where*

$$D_X(\mathcal{F}^\bullet[1]) = D_X(\mathcal{F}^\bullet)[-1]$$

That is, if

$$\mathcal{F}^\bullet \xrightarrow{\varphi} \mathcal{G}^\bullet \xrightarrow{\phi} \mathcal{C}^\bullet \xrightarrow{[1]} \mathcal{F}^\bullet[1]$$

is a distinguished triangle, then

$$D_X\mathcal{F}^\bullet \xleftarrow{\varphi^*} D_X\mathcal{G}^\bullet \xleftarrow{\phi^*} D_X\mathcal{C}^\bullet \xleftarrow{[1]^*} D_X(\mathcal{F}^\bullet)[-1]$$

is a distinguished triangle.

2. *If $X = \{\star\}$, then $D^b(X)$ has objects as bounded chain complexes of vector spaces and*
$$D_X(V^\bullet) = (V^{-\bullet})^\vee.$$
3. *For any $\mathcal{F}^\bullet$ in $\mathcal{D}^b(X)$, there is a natural map*
$$\mathcal{F}^\bullet \longrightarrow D_X^2\mathcal{F}^\bullet,$$
that is, there is a natural transformation $\mathrm{id} \to D_X \circ D_X$ *over $\mathcal{D}^b(X)$.*
4. *If $U \subseteq X$ is open, then*
$$D_X(\mathcal{F}^\bullet)|_U \cong D_U(\mathcal{F}^\bullet|_U).$$
5. *[Verdier duality] For any map $f : X \to Y$ and $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$, we have a natural isomorphism between the following composite of functors*
$$\begin{array}{ccc} \mathcal{D}^b(X)^{op} & \xrightarrow{D_X} & \mathcal{D}^b(X) \\ {\scriptstyle Rf_*}\big\downarrow & & \big\downarrow{\scriptstyle Rf_!} \\ \mathcal{D}^b(Y)^{op} & \xrightarrow[D_Y]{} & \mathcal{D}^b(Y) \end{array}.$$
That is,
$$D_Y Rf_*\mathcal{F}^\bullet \cong Rf_! D_X\mathcal{F}^\bullet.$$
Furthermore, for inverse images we also have the same isomorphisms:
$$D_X Rf^*\mathcal{G}^\bullet \cong Rf^! D_Y\mathcal{G}^\bullet.$$
6. *[Cohomological constructibility] The Verdier dual functor restricts to the constructible categpry for any stratification $\mathbb{S}$:*
$$D_X : \mathcal{D}^b_{\mathbb{S}}(X)^{op} \longrightarrow \mathcal{D}^b_{\mathbb{S}}(X)$$
and for any $\mathcal{F} \in \mathcal{D}^b_{\mathbb{S}}(X)$, the natural map $\mathcal{F}^\bullet \to D_X^2\mathcal{F}^\bullet$ is an isomorphism.

It follows that there is an intricate connection between compactly supported hypercohomology of $\mathcal{F}^\bullet$ and hypercohomology of the dual $D_X\mathcal{F}^\bullet$.

Theorem A.4.4. *Let X be an n-pseudomanifold and $U \subseteq X$ be an open set with $\mathcal{F}^\bullet$ be a complex of sheaves over X. Then,*
$$\mathbb{H}^i(U, D_X\mathcal{F}^\bullet) \cong \mathbb{H}_c^{-i}(U, \mathcal{F}^\bullet)^\vee.$$

Proof. Let $p_U : U \to \{\star\}$. Then note that $\Gamma(U, \mathcal{F}) = p_{U*}\mathcal{F}$ for any sheaf $\mathcal{F}$. We now have the following isomorphisms following Theorem A.4.3:
$$\begin{aligned} \mathbb{H}^i(U, D_X\mathcal{F}^\bullet) &= H^i(Rp_{U*}D_U\mathcal{F}^\bullet) \\ [A.4.3-5] &\cong H^i(D_\star Rp_{U!}\mathcal{F}^\bullet) \\ [A.4.3-2] &\cong H^i(Rp_{U!}\mathcal{F}^{-\bullet}) \\ &= H^{-i}(Rp_{U!}\mathcal{F}^\bullet)^\vee \\ &= \mathbb{H}_c^{-i}(U, \mathcal{F}^\bullet)^\vee. \end{aligned}$$
This completes the proof. ☐

Lemma A.4.5. *Let X be a space. Then,*

$$\mathcal{S}_X^\bullet \cong D_X(\underline{K}).$$

Lemma A.4.6. *Let X be an n-manifold and $\mathcal{L}$ be a local system over X of K-vector spaces. Then, (see 2.3.9)*

$$D_X \mathcal{L} \cong \mathcal{L}^\vee[n].$$

B Constructiblility

Constructibile sheaves are a mild generalization of local systems. To see why local systems are important in topology, recall the following theorem.

Theorem B.0.1. *Let X be a connected space. Then the following are equivalent:*

1. *$\mathcal{L}$ is a local system over X.*
2. *$p : L \to X$ is a covering fibration over X.*

Proof. (1. $\Rightarrow$ 2.) Pick a local system $\mathcal{L} \in \mathcal{S}h(X)$. This is a locally constant sheaf. Let $p : L \to X$ be the associated étale space of $\mathcal{L}$. Recall that étale space of a constant sheaf $\mathcal{A}$ over an abelian group A on a connected space is just $\amalg_{a\in A} X$. The étale space of a locally constant sheaf then simply is a covering space where evenly covered neighbors are obtained by looking at the open cover of X restricted to each one of them $\mathcal{L}$ is constant.

(2. $\Rightarrow$ 1.) Recall the construction which takes an étale space to a sheaf. Applying that construction on $p : L \to X$ which is a cover with fiber A, we yield a sheaf $\mathcal{L}$ whose sections at $U \subseteq X$ are all right sections of the map p; $\mathcal{L}(U) = \{s : U \to L \mid p \circ s = \mathrm{id}_U\}$. If U is an evenly covered neighborhood, we see that $\mathcal{L}|_U(V) = \{s : V \to L \mid p \circ s = \mathrm{id}_V\} = \{s : V \to A \mid s \text{ is continuous}\}$ where A is the discrete fiber at any point (the fiber of the bundle $p : L \to X$). Indeed, the above equality simply follows from the local trivial hypothesis of p; at U, $p : p^{-1}(U) \to U$ is a homeomorphism where $p^{-1}(U) \cong \coprod_{a\in A} U$. □

For more on local systems and homology with coefficients in them, see §2.3. As local systems appear everywhere, a generalization of them is a worthwhile endeavour. We will now fix an n-pseudomanifold X with a fixed stratification $\mathbb{S} : X = X_n \supseteq X_{n-1} \supseteq X_{n-2} \supseteq \cdots \supseteq X_1 \supseteq X_0$.

B.0.2 (Constructible sheaves over X)**.** A sheaf $\mathcal{F}$ over X is $\mathbb{S}$-**constructible** if on each difference $X_{n-k} - X_{n-k-1}$, the restricted sheaf

$$\mathcal{F}|_{X_{n-k}-X_{n-k-1}}$$

is a local system with finitely generated stalks on the $n-k$-dimensional manifold $X_{n-k} - X_{n-k-1}$. Denote the category of $\mathbb{S}$-constructible sheaves as $\mathrm{Cons}_{\mathbb{S}}(X) \subseteq \mathcal{S}h(X)$.

Theorem B.0.3 (Constructive category is abelian)**.** *The subcategory of $\mathbb{S}$-constructible sheaves in $\mathcal{S}h(X)$, i.e. $\mathrm{Cons}_{\mathbb{S}}(X)$, is abelian.*

Proof. Fix $0 \leq k \leq n$ and $i_k : Z_k = X_{n-k} - X_{n-k-1} \hookrightarrow X$ be the inclusion. Note that $\mathcal{F}|_{X_{n-k}-X_{n-k-1}} = i_k^*\mathcal{F}$. Pick any map $\varphi : \mathcal{F} \to \mathcal{G}$ of $\mathbb{S}$-constructible sheaves. Then we claim that kernel $\mathrm{Ker}(\varphi)$ is also $\mathbb{S}$-constructible. Indeed, as i_k^* is an exact functor we get that $i_k^*\mathrm{Ker}(\varphi)$ is the kernel of $i_k^*\varphi$. As $\mathrm{Ker}(i_k^*\varphi)$ is a local system on Z_k which is a subsystem of $i_k^*\mathcal{F}$, therefore by abelian nature of $\mathrm{LocSys}(X)$, we deduce that $\mathrm{Ker}(\varphi)$ is also $\mathbb{S}$-constructible. One can similarly show that cokernel of φ is also $\mathbb{S}$-constructible and so is direct sum. □

B.0.4 (Cohomologically $\mathbb{S}$-constructible complexes over X)**.** Let $\mathcal{F}^\bullet$ be a complex in the bounded derived category $\mathcal{D}^b(X)$. Then, $\mathcal{F}^\bullet$ is **cohomologically $\mathbb{S}$-constructible** if all the cohomology sheaves $\mathcal{H}^i(\mathcal{F}^\bullet) \in \mathcal{D}^b(X)$ are constructible sheaves over X.

B.0.5 (Cohomologically $\mathbb{S}$-constructible category)**.** We define a full subcategory of $\mathcal{D}^b(X)$ whose objects are cohomologically $\mathbb{S}$-constructible complexes and arrows arrows are those of $\mathcal{D}^b(X)$, called the *cohomologically* $\mathbb{S}$*-constructible category* denoted by $\mathcal{D}^b_{\mathbb{S}}(X)$.

B.0.6 (Constructible sheaves in $\mathcal{D}^b_{\mathbb{S}}(X)$)**.** Consider the inclusion of $\mathcal{S}h(X) \hookrightarrow \mathcal{D}^b(X)$ taking $\mathcal{F} \mapsto \mathcal{F}^\bullet$ which is the complex which is $\mathcal{F}$ at 0 and 0 elsewhere and sheaf maps as the corresponding map of complexes. Under this map, we claim that an $\mathbb{S}$-constructible sheaf goes to a cohomologically $\mathbb{S}$-constructible sheaf. Indeed, this is immediate, as the cohomology sheaves of complex concentrated at degree 0 is just the sheaf back.

By Theorem B.0.3, it hence follows that there is an abelian subcategory of $\mathcal{D}^b(X)$ consisting of $\mathbb{S}$-constructible sheaves over X. We denote this subcategory by $\mathcal{D}^b_{\mathbb{S}c}(X) \subseteq \mathcal{D}^b_{\mathbb{S}}(X)$.