Chow Rings & Applications in Enumerative Geometry

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Abstract

The goal of these notes is to study intersection theory of quasi-projective varieties over an algebraically closed field of characteristic 0. After studying basic notions on cycles and its functoriality, we study intersection product via Serre's intersection product, as developed in [Ser00]. Basic theorems of Chow ring are then developed, which aids computations for basic smooth varieties. Existence of affine stratification for certain smooth varieties gives rise to a generating set of its Chow ring. The example of Grassmannian is thus discussed and an application of its Chow ring computation is presented. An appendix contains many auxilliary results related to algebra and algebraic geometry which are used sometimes in the main text.

Contents

1 Chow groups of a scheme

We fix once and for all an algebraically closed field $\mathbf{k}$ whose characteristic is 0. By a *scheme* we will always mean a finite type separated scheme over $\mathbf{k}$, so that a variety is an integral scheme for us. A *point* will always mean a closed point. Some salient features of this hypothesis are as follows:

1. *All schemes are noetherian.* As $X \to \mathrm{Spec}\,(\mathbf{k})$ is a finite type map and $\mathrm{Spec}\,(\mathbf{k})$ is singleton, so X is covered by finitely many affine opens, each of which is spectrum of a finite type k-algebra.
2. *Every variety has finite dimension.* If X is a $\mathbf{k}$-variety, then $\dim X = \mathrm{trdeg}\, K(X)/\,\mathbf{k}$ where $K(X)$ is the function field. Since $K(X) \cong K(U)$ for some affine open U of X, thus $K(X)$ is isomorphic to fraction field of a finite type $\mathbf{k}$-algebra, which always has a finite transcendence basis.
3. *Every closed subscheme has finitely many irreducible components.* This is because X is noetherian. Consequently, every closed subscheme Y is union of finitely many of its subvarieties.
4. *Codimension is well-behaved for varieties.* If X is a variety and $Y \subseteq X$ is a subvariety, then $\mathrm{codim}\, Y = \dim X - \dim Y$.
5. *All points are rational.* If X is a scheme, then since $\kappa(p)$ for any (closed) point $p \in X$ is a finite extension of $\mathbf{k}$ (Zariski lemma), therefore by algebraic closure of $\mathbf{k}$ we have $\kappa(p) = \mathbf{k}$.

1.1 Algebraic cycles

We define algebraic cycles as higher codimension variants of divisors.

Definition 1.1.1 (Group of cycles). Let X be a scheme. The free abelian group generated by the collection of all subvarieties of X is called the group of cycles of X, denoted $Z(X)$. An element of $Z(X)$ is called a *cycle*. A cycle is *effective* if all its coefficients are positive. If X is of pure dimension n, then an $n-1$-cycle is also called a *Weil divisor*.

Remark 1.1.2 (Grading by dimension)**.** The group of cycles $Z(X)$ is graded by dimension of subvarieties. Indeed, if $Y \hookrightarrow X$ is a closed subvariety, then by our finite type hypothesis, it has finite dimension say k. Consequently, $Y \in Z_k(X)$ where we denote $Z_k(X)$ to be the free abelian group generated by all k-dimensional subvarieties of X. Hence we have the decomposition

$$Z(X) = \bigoplus_{k \geq 0} Z_k(X).$$

If X is pure of dimension n, then one also denotes $Z_{n-1}(X)$ by $\mathrm{Div}\,(X)$, the free abelian group generated by codimension 1 subvarieties.

1.1.1 Effective cycle of a coherent module

Recall that any closed subscheme $Y \hookrightarrow X$ is the support of the $\mathcal{O}_X$-module $\mathcal{O}_X/\mathcal{I}$ where $\mathcal{I}$ is the ideal sheaf of Y. One may then wish to associate a cycle to Y. Indeed, one can do this and we do this in the generality of a coherent $\mathcal{O}_X$-module.

Construction 1.1.3 (Effective cycle of a coherent $\mathcal{O}_X$-module)**.** Let X be a scheme and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Denote $Y = \mathrm{Supp}\,(\mathcal{F})$. This is a closed subset of X by Lemma A.2.1. By giving Y the reduced induced structure, we obtain that $Y \hookrightarrow X$ is a closed subscheme. Our hypotheses on schemes allows us to write

$$Y = \bigcup_{i=1}^{s} W_i$$

where W_i are irreducible components of Y. In particular, W_i are subvarieties of Y. Let $\eta_i \in W_i$ be the generic point of W_i. Consider the finitely generated $\mathcal{O}_{X,\eta_i}$-module $\mathcal{F}_{\eta_i}$. As $\mathcal{O}_{X,\eta_i}$ is a noetherian local ring, therefore $\mathcal{F}_{\eta_i}$ is a noetherian $\mathcal{O}_{X,\eta_i}$-module. By choosing an appropriate affine open around η_i, we get that for a noetherian ring R and a finitely generated R-module M and a minimal prime $\mathfrak{p} \in \mathrm{Supp}\,(M)$, we have $\mathcal{F}_{\eta_i} = M_{\mathfrak{p}}$ a finitely generated $R_{\mathfrak{p}}$-module. Thus $\mathcal{F}_{\eta_i}$ is a finite length $\mathcal{O}_{X,\eta_i}$-module by Proposition A.4.7. We may hence define the following algebraic cycle in $Z(X)$ corresponding to module $\mathcal{F}$:

$$\langle \mathcal{F} \rangle := \sum_{i=1}^{s} \mathrm{len}_{\mathcal{O}_{X,\eta_i}}(\mathcal{F}_{\eta_i}) W_i.$$

We will now construct an effective cycle of a closed subscheme. We will need the following lemma to this end.

Lemma 1.1.4. *Let $f : X \to Y$ will be a closed immersion of topological spaces and $\mathcal{F}$ a sheaf over X. Then, there is a natural isomorphism*

$$(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x.$$

Proof. From a straightforward unravelling of definitions of the two stalks, the result follows from the observation that each open set $U \ni x$ in X is in one-to-one correspondence with open set $f(U) \ni f(x)$ in Y. □

Remark 1.1.5 (Effective cycle of a closed subscheme)**.** Let $j : Y \hookrightarrow X$ to be a closed subscheme with ideal sheaf $\mathcal{I} \leq \mathcal{O}_X$. Write

$$Y = \bigcup_{i=1}^{s} Y_i$$

where Y_i are irreducible components of Y with generic points η_i and let $\mathcal{F} = \mathcal{O}_X/\mathcal{I}$. Note that $j_*\mathcal{O}_Y = \mathcal{F}$. By Lemma 1.1.4, we have $\mathcal{F}_{\eta_i} \cong \mathcal{O}_{Y,\eta_i}$ and hence we get an effective cycle by Construction 1.1.3 as in

$$\langle Y \rangle = \sum_{i=1}^{s} \operatorname{len}_{\mathcal{O}_{X,\eta_i}}(\mathcal{O}_{Y,\eta_i}) Y_i.$$

1.1.2 Map on cycles

Let $f : X \to Y$ be a map of schemes. Our goal is to define a group homomorphisms $f_* : Z(X) \to Z(Y)$ and $f^* : Z(Y) \to Z(X)$. As these are free abelian groups, therefore we need only define f_* and f^* on subvarieties of X.

Construction 1.1.6 (Pushforward map on cycles)**.** Let $f : X \to Y$ be a map of schemes. Let $W \hookrightarrow X$ be a k-dimensional subvariety. The scheme theoretic image $\overline{f(W)} \subseteq Y$ is a variety by Lemma A.3.1. By Remark A.3.3, it follows that $\dim W \geq \dim \overline{f(W)}$. This gives a dominant morphism of varieties

$$f : W \to \overline{f(W)}.$$

We may thus define a k-cycle on Y using Proposition A.3.2 as follows:

$$f_*(W) = \begin{cases} [K(W) : K(\overline{f(W)})] \cdot \overline{f(W)} & \text{if } \dim W = \dim \overline{f(W)} \\ 0 & \text{if } \dim W > \dim \overline{f(W)}. \end{cases}$$

This defines a group homomorphism by linear extension between group of cycles

$$f_* : Z(X) \to Z(Y).$$

This defines a map on cycles.

We can also construct pullback of cycles.

Construction 1.1.7 (Pullback map on cycles)**.** Let $f : X \to Y$ be a map of schemes. For each subvariety $V \subseteq Y$, we may define a cycle on X given by $\langle f^{-1}(V)\rangle$ where $f^{-1}(V)$ is the inverse image scheme. Consequently, we get the following map on cycles by linearity:

$$f^* : Z(Y) \to Z(X).$$

This is the pullback on cycles.

Recall the notion of constant relative dimension in Definition A.7.3.

Remark 1.1.8 (Pullback of relative dimension n)**.** Let $f : X \to Y$ be a map of relative dimension n. Then the pullback map on $Z_k(X)$ becomes the mapping

$$f^* : Z_k(Y) \longrightarrow Z_{k+n}(X).$$

When f is flat, then pullback is functorial as is visible from the following result.

Proposition 1.1.9. *If $f : X \to Y$ is flat, then for any subscheme $Z \subseteq Y$, we have*

$$f^*(\langle Z\rangle) = \langle f^{-1}(Z)\rangle.$$

Proof. See Lemma 1.7.1 of [Ful84]. □

An obvious question is how are these maps related. In one simple case, they are related by the following push-pull formula.

Proposition 1.1.10 (Push-pull formula)**.** *Let the following be a fiber product square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

where f is proper and g is flat. Then,

1. *f' is proper and g' is flat,*
2. *for any cycle $\alpha \in Z(X)$, we have*

$$f'_* g'^* \alpha = g^* f_* \alpha.$$

Proof. Item 1 is immediate as proper and flat maps are stable under base change. For item 2, see Proposition 1.7 of [Ful84]. □

1.2 Rational equivalence

We wish to now introduce an equivalence relation on each $Z_k(X)$, which will identify k-cycles which can be transformed from one to the other via a suitably parameterized family of cycles. We take our cue from the theory of linear equivalence of divisors.

Remark 1.2.1 (Linear equivalence of divisors)**.** Let X be a scheme regular in codimension 1 (for example, a non-singular variety). Define the following effective divisor on X:

$$\langle f \rangle := \sum_{Y \in \mathrm{PDiv}(X)} v_Y(f) \cdot Y$$

where $\mathrm{PDiv}(X)$ is the set of prinicipal divisors of X, i.e. codimension 1 subvarieties. We call $\langle f \rangle$ the principal divisor generated by f. This defines a group homomorphism

$$\begin{aligned} r^1 : K(X)^\times &\longrightarrow \mathrm{Div}\,(X) \\ f &\longmapsto \langle f \rangle. \end{aligned}$$

Indeed, $\langle fg \rangle = \langle f \rangle + \langle g \rangle$ follows from definition of valuations. Any principal divisor is said to be linearly equivalent to 0. One then defines the class group of X to be the cokernel of $r^1 : K(X)^\times \to \mathrm{Div}\,(X)$:

$$\mathrm{Cl}(X) := \mathrm{Div}\,(X)/\mathrm{Im}\left(r^1\right).$$

We will generalize the map r^1 to higher codimensions, so as to define higher codimension generalization of class groups, which are called Chow groups.

1.2.1 Order of rational functions

To generalize the linear equivalence, we first have to define orders for varieties which may not be regular in codimension 1. To this end, we begin by the observation made in Lemma A.4.10 and take it as a definition in the setting when we don't have regular in codimension 1.

Definition 1.2.2 (**Order function of a 1-dimensional local domain**)**.** Let K be a field and $R \subseteq K$ be a 1-dimensional noetherian local domain with $Q(R) = K$. Then the *order function on* K *defined by* R is the following map:

$$\begin{aligned} v_R : K^\times &\longrightarrow \mathbf{Z} \\ f = \frac{a}{b} &\longmapsto v_R(a) - v_R(b) \end{aligned}$$

where $v_R(a) = \mathrm{len}_R(R/aR)$. Note that R/aR is a finite length R-module by Theorem A.4.8 as R/aR is dimension 0.

The following shows that v_R is a group homomorphism.

Lemma 1.2.3. *Let K be a field, $R \subseteq K$ be a 1-dimensional noetherian local domain with $Q(R) = K$ and $v_R : K^\times \to \mathbf{Z}$ be the order function of R. If $f, g \in K^\times$, then*

$$v_R(fg) = v_R(f) + v_R(g).$$

Proof. Write $f = a/b$ and $g = c/d$ for $a, b, c, d \in R$. Then $v_R(fg) = v_R(ac) - v_R(bd)$. We may thus assume that $f, g \in R$. As $v_R(fg) = \mathrm{len}_R\, R/fgR = \mathrm{len}_R\, R/fR + \mathrm{len}_R\, R/gR$ (Lemma A.4.11), therefore we have $v_R(fg) = v_R(f) + v_R(g)$, as required. $\square$

Remark 1.2.4. If $\frac{a}{b} = \frac{a'}{b'}$ for $a, b, a', b' \in R$, then since R is a domain, we have $ab' = a'b$. It follows that $v_R(a) - v_R(b) = v_R(a') - v_R(b')$. This shows that the order function is well-defined.

We may then globalize above to generalize the discrete valuation associated to each prime divisor of a variety regular in codimension 1.

Definition 1.2.5 (Order function of a prime divisor). Let X be a variety and $Z \hookrightarrow X$ be a subvariety of codimension 1, that is, $Z \in \mathrm{PDiv}(X)$ is a prime divisor of X. Consider $f \in K(X)^\times$ a non-zero rational function on X. Then, define v_Z to be the order function of $\mathcal{O}_{X,Z} \subseteq K(X)$. That is,

$$\begin{aligned} v_Z : K(X)^\times &\longrightarrow \mathbf{Z} \\ \frac{a}{b} &\longmapsto v_{\mathcal{O}_{X,Z}}\left(\frac{a}{b}\right) = \mathrm{len}_{\mathcal{O}_{X,Z}} \frac{\mathcal{O}_{X,Z}}{a\mathcal{O}_{X,Z}} - \mathrm{len}_{\mathcal{O}_{X,Z}} \frac{\mathcal{O}_{X,Z}}{b\mathcal{O}_{X,Z}}. \end{aligned}$$

1.2.2 Chow groups

Definition 1.2.6 (Rational equivalence & Chow groups). Let X be a scheme. Denote $\mathrm{Sub}_l(X)$ to be the collection of all l-dimensional subvarieties of X. Let $W \in \mathrm{Sub}_{k+1}(X)$ and $f \in K(W)^\times$. We then define a k-cycle on X by

$$\langle f; W\rangle := \sum_{Y \in \mathrm{PDiv}(W)} v_Y(f) \cdot Y,$$

where $v_Y = v_{\mathcal{O}_{W,Y}}$ is the order function on $K(W)^\times$ for the one-dimensional local domain $\mathcal{O}_{W,Y} \subseteq K(W)^\times$ (see Definitions 1.2.2 and 1.2.5). This is well-defined for exactly the same reason why principal divisors are well-defined.

Note that $\mathrm{PDiv}(W) = \mathrm{Sub}_k(W)$. For each $k \geq 0$, define the following group homomorphism

$$\begin{aligned} r_k : \bigoplus_{W \in \mathrm{Sub}_{k+1}(X)} K(W)^\times &\longrightarrow Z_k(X) \\ (f_W)_W &\longmapsto \sum_W \langle f_W; W\rangle. \end{aligned}$$

We define $\mathrm{Rat}_k(X) := \mathrm{Im}\,(r_k) \subseteq Z_k(X)$ to be the group of cycles *rationally equivalent to 0*. Denote $\mathrm{Rat}(X) = \bigoplus_{k\geq 0} \mathrm{Rat}_k(X)$. One then defines the cokernel of r_k:

$$A_k(X) := Z_k(X)/\,\mathrm{Rat}_k(X).$$

Finally, the *Chow group* of X is defined to be all of these groups:

$$A(X) := \bigoplus_{k\geq 0} A_k(X).$$

We will also write $A^d(X) = A_{n-d}(X)$ for $0 \leq d \leq \dim X$.

Our first result shows that rational equivalence generalizes linear equivalence.

Lemma 1.2.7. *Let X be a variety of dimension n. Then,*

$$A_{n-1}(X) \cong \mathrm{Cl}(X).$$

Proof. We need only show that the map r_k actually reduces to r^1 as in Remark 1.2.1 when $k = n-1$. Indeed, observe that $\mathrm{Sub}_n(X) = \{X\}$ as X is integral. Thus, r_{n-1} is the map

$$\begin{aligned} r_{n-1} : K(X)^\times &\longrightarrow Z_{n-1}(X) \\ f &\longmapsto \langle f; X\rangle \end{aligned}$$

where $\langle f; X\rangle = \sum_{Y \in \mathrm{PDiv}(X)} v_Y(f) \cdot Y$, but this is just the principal divisor $\langle f\rangle$. Hence $r_{n-1} = r^1$, as required. □

Corollary 1.2.8 (Chow group of curves)**.** *If X is a curve, then*

$$A(X) \cong \mathrm{Cl}(X) \oplus \mathbf{Z}$$

Proof. By Lemma 1.2.7, we need only show that $A_0(X) \cong \mathbf{Z}$ as $A_k(X)$ for $k \geq 2$ is 0 since $Z_k(X) = 0$. For $k = 1$, we have

$$r_1 : \bigoplus_{W \in \mathrm{Sub}_2(X)} K(W)^\times \to Z_1(X)$$

and since $\mathrm{Sub}_2(X) = \emptyset$, hence r_1 is the 0-map. As $Z_1(X) = \mathbf{Z}$, generated by X, consequently, $A_1(X) = \mathbf{Z}$, as required. □

As every closed subscheme Y of X defines the cycle $\langle Y\rangle \in Z(X)$, we thus get a class of each subscheme in the Chow group.

Definition 1.2.9 (**Fundamental class of a subscheme**)**.** Let X be a scheme and $Y \subseteq X$ be a closed subscheme. The fundamental class of Y refers to the cycle class in $A(X)$ corresponding to $\langle Y\rangle \in Z(X)$, i.e., the image of $\langle Y\rangle$ under the quotient map

$$Z(X) \twoheadrightarrow A(X).$$

We denote the cycle class of Y by $[Y] \in A(X)$.

2 Intersection product

We wish to now give a product structure on $A(X) = \bigoplus_{k\geq 0} A_k(X)$ which will encode the intersection of subvarieties and will moreover induce a commutative ring structure on $A(X)$. In particular, we want a product structure on $A(X)$ such that for two subvarieties $A, B \subseteq X$, the product of cycle classes $[A], [B] \in A(X)$ must be such that

$$[A] \cdot [B] = [A \cap B]. \tag{2}$$

Unfortunately, in order to have such a multiplication, we need to restrict the class of subschemes for which the above identity must hold.

Remark 2.0.1. We cannot expect Eqn (2) to hold for any subvarieties $A, B \subseteq X$ as for well-definedness, we need to show that the rational equivalence class of the intersection, $[A \cap B]$, only depends on the rational equivalence classes of A and B.

We construct a product on $A(X)$ in two steps, we first give a product structure on certain type of cycles. Then by the famous moving lemma, we establish the invariance of the above defined product on cycle classes.

2.1 Proper intersection

The class of subvarieties on which we will define the product will be based on the following. This is where we shall critically use the assumption that $\mathbf{k}$ is algebraically closed as over such fields, regularity and smoothness of a variety are equivalent.

Definition 2.1.1 (Transverse intersection). Let X be a variety and $A, B \subseteq X$ be two subvarieties. We say that A and B intersect transversally at $p \in A \cap B$ if X, A and B are smooth at p[1] and T_pA and T_pB spans T_pX[2] (see Remark A.6.2).

Remark 2.1.2. We may express that T_pA and T_pB spans T_pX by saying that

$$\text{codim } T_pA \cap T_pB = \text{codim } T_pA + \text{codim } T_pB.$$

But since p is a regular point, we have $\dim X = \dim \mathcal{O}_{X,p} = \dim T_pX$. Consequently, we may further express this as

$$\dim T_pA \cap T_pB = \dim A + \dim B - \dim X.$$

Example 2.1.3. An example of a transverse intersection is given by $A = V(y - x^2)$ and $B = V(x)$ in $X = \mathbb{A}^2$ at the point $p = (0, 0)$, as shown below. As the map on local rings for A and B are respectively (let $\mathfrak{m} = \langle x, y\rangle$, the origin)

$$\mathbf{k}[x,y]_{\mathfrak{m}} \twoheadrightarrow \mathbf{k}[x]_{\langle x\rangle}$$
$$\mathbf{k}[x,y]_{\mathfrak{m}} \twoheadrightarrow \mathbf{k}[y]_{\langle y\rangle},$$

[1] i.e. $\mathcal{O}_{X,p}, \mathcal{O}_{A,p}$ and $\mathcal{O}_{B,p}$ are regular local rings.

[2] Our tangent spaces are over the base field $\mathbf{k}$.

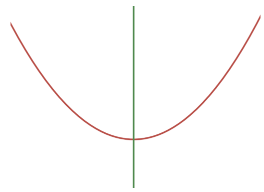

Figure 1: A transverse intersection.

therefore the map on cotangent spaces are

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \twoheadrightarrow \frac{\langle x \rangle}{\langle x^2 \rangle}$$
$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \twoheadrightarrow \frac{\langle y \rangle}{\langle y^2 \rangle}$$

where the codomains are clearly 1-dimensional $\mathbf{k}$-vector spaces. One sees easily that

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} = \{a\bar{x} + b\bar{y} \mid a, b \in \mathbf{k}\}.$$

The first map maps $\bar{y} \mapsto 0$ and the second onto $\bar{x} \mapsto 0$. Thus, $T_pA, T_pB \hookrightarrow T_pX$ are 1-dimensional subspaces which spans the whole of T_pX, as required.

Definition 2.1.4 (**Proper/dimensionally transverse intersection**). Let $A, B \subseteq X$ be subvarieties of a regular scheme X. Then A and B are said to intersection properly or dimensionally transverse if for each irreducible component $C \subseteq A \cap B$, we have

$$\text{codim } C = \text{codim } A + \text{codim } B.$$

Another way to say this is that

$$\dim C = \dim A + \dim B - \dim X.$$

Note we only require regularity of X. But of-course, our assumption that $\mathbf{k}$ is algebraically closed forces X to be smooth as soon as X is integral. We extend the above definition to cycles.

Definition 2.1.5 (**Properly intersecting cycles**). Let X be a regular scheme and $\alpha = \sum_i n_i A_i$ and $\beta = \sum_j m_j B_j$ be two cycles in $Z(X)$. We say that α and β intersect properly/have dimensionally transverse intersection, if each A_i and B_j intersect properly.

2.2 Product of proper cycles

We give the axioms that we require from a product operation on proper cycles which reflects the intersection. We will then give a candidate via Serre's Tor formula which will follow these axioms. Indeed, this is the only product structure on $A(X)$ if X is smooth quasi-projective.

Definition 2.2.1 (**Intersection product of proper cycles**). Let X be a variety. A pairing of two proper cycles $(\alpha, \beta) \mapsto \alpha \cdot \beta$ is called the intersection product of proper cycles if it satisfies the following axioms:

1. (Commutativity) The product $\cdot$ must be commutative.
2. (Associativity) The product $\cdot$ must be associative.
3. (Product formula) Let X, Y be two smooth varieties and $\alpha, \beta \in Z(X)$ and $\alpha', \beta' \in Z(Y)$ be properly intersecting cycles. Then we have cycles $\alpha \times \alpha'$ and $\beta \times \beta'$ in $Z(X \times Y)$. Then, we must have

$$(\alpha \times \alpha') \cdot (\beta \times \beta') = (\alpha \cdot \beta) \times (\alpha' \cdot \beta').$$

4. (Reduction to the diagonal) Let $\alpha, \beta \in Z(X)$ be two properly intersection cycles and let $\Delta : X \to X \times X$ be the diagonal map. Then, we must have following in $Z(X)$:

$$\alpha \cdot \beta = \Delta^*(\alpha \times \beta).$$

5. (Local intersection multiplicities) Let $A, B \subseteq X$ be subvarieties which intersect properly. Then there must exist non-negative integers $n_C \geq 0$ for each irreducible component $C \subseteq A \cap B$ such that

$$A \cdot B = \sum_{C \subseteq A \cap B \text{ irr. comp.}} n_C \cdot C.$$

Moreover, the integer n_C only depends on any affine open $U \subseteq X$ containing the generic point of C.

Serre gave the following candidate of intersection of proper cycles, which we shall use in this text.

Definition 2.2.2 (Intersection multiplicities and cycle). Let X be a regular scheme and $A, B \subseteq X$ be subvarieties which intersect properly in X. Let C be an irreducible component of $A \cap B$. We define intersection multiplicity of A and B along C by

$$i(A, B; C) := \sum_{i \geq 0} (-1)^i \operatorname{len}_{\mathcal{O}_{X,C}} \left(\operatorname{Tor}_i^{\mathcal{O}_{X,C}} (\mathcal{O}_{X,C}/\mathfrak{p}_A, \mathcal{O}_{X,C}/\mathfrak{p}_B) \right),$$

where $\mathfrak{p}_A$ and $\mathfrak{p}_B$ are primes of $\mathcal{O}_{X,C}$ corresponding to A and B, respectively[3]. Note that $\mathcal{O}_{X,C}/\mathfrak{p}_A = \mathcal{O}_{A,C}$ and $\mathcal{O}_{X,C}/\mathfrak{p}_B = \mathcal{O}_{B,C}$. Consequently, we define the intersection cycle of A and B, $A \cdot B \in Z(X)$, as

$$A \cdot B = \begin{cases} \sum_{C \subseteq A \cap B \text{ irr. comp.}} i(A, B; C) \cdot C & \text{if } A \cap B \neq \emptyset \\ 0 & \text{if } A \cap B = \emptyset. \end{cases}$$

Finally, for two properly intersecting cycles $\alpha = \sum_i n_i A_i$ and $\beta = \sum_j m_j B_j$, we define their product as

$$\alpha \cdot \beta = \sum_{i,j} n_i m_j A_i \cdot B_j.$$

[3] To see this, go to an open local affine patch.

Remark 2.2.3. We first need to see why each intersection multiplicity $i(A, B; C)$ is a non-negative quantity. To this end, we first see that the length

$$\mathrm{len}_{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/\mathfrak{p}_A \otimes_{\mathcal{O}_{X,C}} \mathcal{O}_{X,C}/\mathfrak{p}_B) = \mathrm{len}_{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/\mathfrak{p}_A + \mathfrak{p}_B)$$

is finite since $\mathcal{O}_{X,C}/\mathfrak{p}_A + \mathfrak{p}_B$ is a 0-dimensional ring as C corresponds to a minimal prime of $\mathfrak{p}_A + \mathfrak{p}_B$ in an open affine. Thus, by Theorem A.4.8 yields that $\mathrm{len}_{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/\mathfrak{p}_A + \mathfrak{p}_B) < \infty$. Hence, by Serre's Theorem A.8.1, $i(A, B; C) \geq 0$.

Serre's definition is indeed a product of proper cycles.

Theorem 2.2.4 (Serre)**.** *Let X be a variety. The product pairing defined in Definition 2.2.2 is indeed an intersection product of proper cycles.*

Our next goal is to carry over this product onto the Chow groups.

2.3 The Chow ring

Having defined product on cycles which intersect properly, we now wish to define a product on $A(X)$ using it:

$$A_p(X) \times A_q(X) \longrightarrow A_{p+q-n}(X)$$

where X is n-dimensional. But since we only have product for properly intersecting cycles, therefore we must answer the following two questions:

Q1. Does every pair of cycles $\alpha, \beta \in Z(X)$ admits a equivalence to a pair α', β' such that they intersect properly?

Q2. If answer to Q1 is affirmative, then does the cycle class of $\alpha' \cdot \beta'$ depend on the choice of α' and β'?

If the answer to Q1 is yes and Q2 is no, then we can safely define the intersection product as

$$\begin{aligned} \cdot : A_p(X) \times A_q(X) &\longrightarrow A_{p+q-n}(X) \\ ([\alpha], [\beta]) &\longmapsto [\alpha' \cdot \beta'], \end{aligned}$$

which would be well-defined.

The answer to Q1 is indeed affirmative, as is provided by Chow's moving lemma.

Theorem 2.3.1 (Moving lemma)**.** *Let X be a smooth quasi-projective variety and $\alpha, \beta \in Z(X)$ be two arbitrary cycles on X.*

1. *There exists a cycle $\beta' \in Z(X)$ rationally equivalent to β such that α and β' intersect properly.*
2. *If $\beta'' \in Z(X)$ is any other cycle such that α and β'' intersect properly, then $\alpha \cdot \beta'$ and $\alpha \cdot \beta''$ are rationally equivalent.*

Using moving lemma, we may now show that the pairing is indeed independent of representative taken.

Corollary 2.3.2. *Let X be a smooth quasi-projective variety. Then the pairing*

$$\begin{aligned} \cdot : A_p(X) \times A_q(X) &\longrightarrow A_{p+q-n}(X) \\ ([\alpha], [\beta]) &\longmapsto [\alpha' \cdot \beta'], \end{aligned}$$

where $[\alpha] = [\alpha']$, $[\beta] = [\beta']$ *and* α', β' *intersect properly, is well-defined. This is moreover a commutative associative and graded ring structure on* $A(X)$. *We call this the intersection product on* $A(X)$.

Proof. Let $[\alpha] = [\alpha'] = [\alpha'']$ and $[\beta] = [\beta'] = [\beta'']$ where $\alpha' \cdot \beta'$ and $\alpha'' \cdot \beta''$ are defined. By moving lemma (Theorem 2.3.1), let $\bar{\beta}$ be equivalent to β'' such that $\alpha' \cdot \bar{\beta}$ and $\alpha'' \cdot \bar{\beta}$ are defined. Then another application of moving lemma yields

$$[\alpha' \cdot \beta'] = [\alpha' \cdot \bar{\beta}] = [\alpha'' \cdot \bar{\beta}] = [\alpha'' \cdot \beta''],$$

as required. The associativity, commutativity follows from Theorem 2.2.4, that is, $\cdot$ is a product of proper intersections. $\square$

Hence, we have the Chow ring of a quasi-projective smooth variety.

Definition 2.3.3 (Chow ring). Let X be a smooth quasi-projective variety. Then the commutative associative graded ring $(A(X), \cdot)$ of cycles modulo rational equivalence is called the Chow ring of X.

Our next goal is to show the uniqueness of intersection product.

Theorem 2.3.4. *Let* X *be a smooth quasi-projective variety. Then there exists a unique commutative associative graded product on* $A(X)$ *with respect to the property that for each properly intersecting subvarieties* $A, B \subseteq X$, *we have*

$$[A] \cdot [B] = \sum_{C \subseteq A \cap B \textit{ irr. comp.}} i(A, B; C) \cdot [C].$$

Proof. Let $\star$ be another product satisfying this. Take any two cycles $\alpha, \beta \in Z(X)$. Observe that $[\alpha] \cdot [\beta]$ is determined by moving lemma (Theorem 2.3.1) by multiplication only on properly intersecting subvarieties A, B. We then conclude the proof as $[A] \star [B] = [A] \cdot [B]$. $\square$

2.4 Generic transversality

In this section, we isolate a situation which we will encounter frequently in our work. This is also useful for product calculations as under this condition, product of cycle classes will indeed correspond to the class of the intersection subscheme.

Definition 2.4.1 (Generically transverse intersection). Let X be a variety and $A, B \subseteq X$ be two subvarieties. We say that A and B intersect generically transversely if for each irreducible component $C \subseteq A \cap B$ we have that A and B meet transversely at any general point of C.

Our goal is to show that generically transverse intersection is a proper intersection, albeit with some special properties concerning the multiplicities.

Proposition 2.4.2. *Let* X *be a variety and* $A, B \subseteq X$ *be two subvarieties. Then the following are equivalent:*

1. *A and B are generically transverse.*
2. *A and B intersect properly and for each irreducible component $C \subseteq A \cap B$, there exists a smooth point of X in C and $A \cap B$ is reduced at that point.*

Proof. (1. $\Rightarrow$ 2.) Fix $C \subseteq A \cap B$ to be an irreducible component of $A \cap B$. We first wish to show that $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$. Indeed, for a general smooth point $p \in C$ over which we transverse intersection, we deduce from Remark 2.1.2 and Lemma A.6.5 that $\operatorname{codim} C = \operatorname{codim} T_pC = \operatorname{codim} T_pA \cap T_pB = \operatorname{codim} T_pA + \operatorname{codim} T_pB = \operatorname{codim} A + \operatorname{codim} B$, as required. By definition of generic transversality, we have that C contains a smooth point and moreover, $A \cap B$ is reduced at that point since it is smooth, and regular local rings are reduced.

(2. $\Rightarrow$ 1.) Fix an irreducible component C of $A \cap B$. The collection of smooth points of X is open and dense. As C contains a smooth point of X, therefore C contains an open dense set of smooth points of X. As points where $A \cap B$ is reduced is also open and dense, thus the collection of points of C which are smooth in X and reduced for A, B and C is open and dense. Pick any such point $p \in C$. We wish to show that A and B meet transversely at p. To this end, we first have to show that A and B are smooth at p. As we already have $\dim T_pA \geq \dim A$, we will show that $\dim T_pA \leq \dim A$ and similarly for B.

Observe that by smoothness of p at C and X, Lemma A.6.4 and proper intersection, we may write

$$\begin{aligned}\dim C &= \dim A + \dim B - \dim X \\ &= \dim T_pC = \dim T_pA + \dim T_pB - \dim T_pX \\ &= \dim T_pC = \dim T_pA + \dim T_pB - \dim X.\end{aligned}$$

Since $\dim T_pA \dim A$ and $\dim T_pB \geq \dim B$, therefore from above we get $\dim T_pA = \dim A$ and same for B. This also shows that A and B meet transversely at p, as required. □

Generically transverse intersections have multiplicity same as length.

Proposition 2.4.3. *Let X be a smooth variety and $A, B \subseteq X$ be two subvarieties. If they intersect generically transversely, then*

$$\operatorname{len}_{\mathcal{O}_{X,C}}(\mathcal{O}_{A\cap B,C}) = i(A, B; C).$$

Above result now tells us a condition when intersection product indeed corresponds to the class of intersection.

Corollary 2.4.4. *Let X be a smooth variety and $A, B \subseteq X$ be two subvarieties. If they intersect generically transversely, then their intersection product in $A(X)$ satisfies*

$$[A] \cdot [B] = [A \cap B].$$

Proof. Note that $[A \cap B] = \sum_{C \subseteq A\cap B} \operatorname{len}_{\mathcal{O}_{X,C}}(\mathcal{O}_{A\cap B,C}) \cdot [C]$ where the sum runs over irreducible components of $A \cap B$. The proof is now immediate from Proposition 2.4.3. □

Remark 2.4.5. Consider the hypotheses of Proposition 2.4.3. We have

$$i(A, B; C) = \sum_{i\geq 0} (-1)^i \operatorname{len}_{\mathcal{O}_{X,C}}(\operatorname{Tor}_i^{\mathcal{O}_{X,C}}(\mathcal{O}_{A,C}, \mathcal{O}_{B,C})).$$

We first have that

$$\mathrm{Tor}_0^{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/\mathfrak{p}_A, \mathcal{O}_{X,C}/\mathfrak{p}_B) \cong \frac{\mathcal{O}_{X,C}}{\mathfrak{p}_A + \mathfrak{p}_B}.$$

As $\mathcal{O}_{X,C}/\mathfrak{p}_A + \mathfrak{p}_B$ is isomorphic to $\mathcal{O}_{A\cap B,C}$, therefore we need only show that

$$\sum_{i\geq 1}(-1)^i \operatorname{len}_{\mathcal{O}_{X,C}} \mathrm{Tor}_i^{\mathcal{O}_{X,C}}(\mathcal{O}_{A,C}, \mathcal{O}_{B,C}) = 0.$$

Indeed, the proof of the theorem proceeds by showing that all higher ($i \geq 1$) Tor groups vanish.

An important result about generic transversality is that such intersections have multiplicity one.

Theorem 2.4.6 (Multiplicity one)**.** *Let X be a smooth variety and $A, B \subseteq X$ be two subvarieties in X. Then the following are equivalent:*

1. *For all components $C \subseteq A \cap B$, we have*

$$i(A, B; C) = 1.$$

2. *A and B intersect generically transversely.*

We give some computations of intersection multiplicities of properly intersecting subvarieties.

Example 2.4.7. Consider $A = V(y - x^2)$ and $B = V(y)$ in the affine plane $\mathbb{A}^2$. Then $C = A \cap B$ is the scheme $\operatorname{Spec}\left(\mathbf{k}[x,y]/\langle y - x^2, y\rangle\right) = \operatorname{Spec}\left(\mathbf{k}[x]/\langle x^2\rangle\right)$. This is a proper intersection as $\operatorname{codim} C = 2 = \operatorname{codim} A + \operatorname{codim} B = 1 + 1$. Consequently, the product of the classes $[A], [B] \in A^1(\mathbb{A}^2)$ is given by

$$[A] \cdot [B] = i(A, B; C) \cdot [C].$$

We need only calculate $i(A, B; C)$. Indeed, we claim that that all higher ($i \geq 1$) Tors vanish in the expression of multiplicity. To see this, first compute $\mathrm{Tor}_1^{\mathcal{O}_{\mathbb{A}^2,C}}(\mathcal{O}_{\mathbb{A}^2,C}/\mathfrak{p}_A, \mathcal{O}_{\mathbb{A}^2,C}/\mathfrak{p}_B)$, which is simply $\mathfrak{p}_A \cap \mathfrak{p}_B/\mathfrak{p}_A \cdot \mathfrak{p}_B$. This computation yields 0 module, thus $\mathrm{Tor}_1 = 0$, hence all higher Tors are 0. Consequently, we get

$$\begin{aligned}
i(A, B; C) &= \sum_{i\geq 0}(-1)^i \operatorname{len}_{\mathcal{O}_{\mathbb{A}^2,C}}\left(\mathrm{Tor}_i^{\mathcal{O}_{\mathbb{A}^2,C}}(\mathcal{O}_{A,C}, \mathcal{O}_{B,C})\right) \\
&= \operatorname{len}_{\mathcal{O}_{\mathbb{A}^2,C}}\left(\mathcal{O}_{\mathbb{A}^2,C}/\mathfrak{p}_A + \mathfrak{p}_B\right) \\
&= \operatorname{len}_{\mathcal{O}_{\mathbb{A}^2,C}}\left(\frac{\mathbf{k}[x,y]}{\langle y - x^2, y\rangle}\right)_{\langle x,y\rangle} \\
&= \operatorname{len}_{\mathcal{O}_{\mathbb{A}^2,C}}\left(\frac{\mathbf{k}[x]}{x^2}\right)_{\langle x,y\rangle} \\
&= 2.
\end{aligned}$$

Hence,

$$[A] \cdot [B] = 2 \cdot [C].$$

In particular, we have $[A] \cdot [B] = [A \cap B]$ in this instance.

There are two separate notions of multiplicity; multiplicity of intersection of two subvarieties at a component and multiplicity of a subscheme at a point. These two are closely related, but distinct notions.

For a $\mathbf{k}$ vector space V, we denote by $\mathrm{Sym}(V^*)$ the symmetric algebra on the dual vector space V^*. Recall that $\mathrm{Sym}(V^*) = T(V^*)/I$ where $T(V^*) = \mathbf{k} \oplus V^* \oplus (V^* \otimes V^*) \oplus \dots$ is the tensor algebra and I is the ideal generated by $x \otimes y - y \otimes x$. This is the polynomial algebra generated on the basis of V^*, as can be seen easily by the universal property of $\mathrm{Sym}(V^*)$.

Definition 2.4.8 (**Multiplicity at a point**)**.** Let X be a scheme and $p \in X$ be a point. The multiplicity of X at p is given by the degree of the projectivized tangent cone (see §A.6.2) as a subscheme of $\mathbb{P}(T_p X)$:

$$m_p(X) := \deg \mathbb{P}TC_p(X)$$

where $\mathbb{P}TC_p X \subseteq \mathbb{P}(T_p X)$.

3 Properties of Chow ring

Having established the ring structure on $A(X)$ for quasi-projective varieties, we now aim to develop results which will allow us to compute them. We begin with some computations which are already relatively straightforward. From now on, we will also occasionally follow the cohomological notation; $A^d(X) = A_{n-d}(X)$ for $n = \dim X$.

3.1 Functoriality

3.1.1 Proper pushforward

Our goal in this section is to show that the Chow groups construction is covariantly functorial with respect to proper maps of schemes. Here's the theorem.

Theorem 3.1.1. *Let $\varphi : X \to Y$ be a proper map of schemes. Then for all $k \geq 0$, the map on cycles*

$$\varphi_* : Z_k(X) \longrightarrow Z_k(Y)$$

maps $\mathrm{Rat}_k(X)$ into $\mathrm{Rat}_k(Y)$[4]*.*

The main strategy of the proof is to first reduce to the case of a surjective proper morphism and then check case by case on difference of dimensions of X and Y. The most important case is that when $\dim X = \dim Y$. The relevant result here is as follows.

Proposition 3.1.2. *Let $\varphi : X \to Y$ be a proper surjective map of varieties. If $\dim X = \dim Y$, then for all $f \in K(X)^\times$, we have*

$$\varphi_*(\langle f; X \rangle) = \langle N_{K(X)/K(Y)}(f); Y \rangle$$

where $N_{K(X)/K(Y)}(f) \in K(Y)$ is the norm of element of $f \in K(X)^\times$.

[4] This holds even if $\mathrm{char}\, \mathbf{k} > 0$.

Proof. Firstly, the extension $K(X)/K(Y)$ is a finite extension by Proposition A.3.2, so that norm of an element of this extension is well-defined. We first pass to the normalizations of the varieties X and Y. By universal property of normalization, we get the following commutative diagram:

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "\tilde{\varphi}"] \arrow[d, "p_X"'] & \tilde{Y} \arrow[d, "p_Y"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}.$$

Observe that $K(X) = K(\tilde{X})$ as fraction field of integral closure is same as that of original domain. Similarly, $K(Y) = K(\tilde{Y})$. This also shows that $\dim X = \dim \tilde{X}$ and $\dim Y = \dim \tilde{Y}$. Finally, by valuative criterion, it follows that $\tilde{\varphi}$ is also proper. Consequently, it suffices to show the result for the induced map $\tilde{\varphi}$ and for the normalization map p_Y as then we can write

$$\begin{aligned} \varphi_*(\langle f; X\rangle) &= \varphi_*(p_{X*}(\langle f; \tilde{X}\rangle)) = p_{Y*}(\tilde{\varphi}_*(\langle f; \tilde{X}\rangle)) = p_{Y*}(\langle N_{K(X)/K(Y)}(f); \tilde{Y}\rangle) \\ &= \langle N_{K(X)/K(Y)}(f); Y\rangle. \end{aligned}$$

We first show that the normalization map $p_Y : \tilde{Y} \to Y$ also satisfies the said equality. As the normalization map of varieites is finite, therefore it is enough to show that if $\varphi : X \to Y$ is a finite map, then it satisfies the said equality. Let $A = \mathcal{O}_{Y,Z}$. We claim that there exists a finite A-algebra B such that for each subvariety V_i of X mapping onto Z corresponds to a maximal ideal $\mathfrak{m}_i$ of B such that $B_{\mathfrak{m}_i} = \mathcal{O}_{X,V_i}$. Having proved this, by Lemma A.4.16 we would get that LHS of Eq. (1) is $\operatorname{len}_A(B/fB)$ (assuming $f \in B$ by linearity of order functions and norm) and Theorem A.4.17 would yield the RHS to be equal to $\operatorname{len}_A(B/fB)$ as well. This would hence complete the proof.

Indeed, if $\operatorname{Spec}(\Gamma)$ is an open affine containing the generic point of Z in Y, say $\mathfrak{p}$, then by finiteness there is an open affine $\operatorname{Spec}(\Lambda)$ of X such that Λ is a finite Γ-algebra and we may set $B = \Lambda \otimes_\Gamma \Gamma_{\mathfrak{p}} = \Lambda \otimes_\Gamma A$. We claim that this B satisfies the condition. Clearly, B is a finite A-algebra. Moreover, the maximal ideals of B correspond to those primes of Λ whose inverse image under $\psi : \Gamma \to \Lambda$ is exactly $\mathfrak{p}$, i.e. the local rings $\mathcal{O}_{X,V_i}$, as needed.

To complete the proof, we cover the other case. Assume that X and Y are normal and $\varphi : X \to Y$ is a proper surjective map. One can then conclude just as above using Stein factorization (Theorem A.3.5) of φ. $\square$

We may now prove the main theorem.

Proof of Theorem 3.1.1. Let $\alpha \in \operatorname{Rat}_k(X)$ be given by $\alpha = \sum_W \langle f_W; W\rangle$. It suffices to show that $\varphi_*(\langle f_W; W\rangle) \in \operatorname{Rat}(Y)$. As $\langle f_W; W\rangle = \sum_{Z \in \operatorname{PDiv}(W)} v_Z(f_W) Z$, therefore we get

$$\varphi_*(\langle f_W; W\rangle) = \sum_{Z \in \operatorname{PDiv}(W)} v_Z(f_W)\varphi_*(Z).$$

It is then sufficient to show that $\varphi_*(Z) \in \operatorname{Rat}(Y)$ for each $Z \in \operatorname{PDiv}(X)$. Note that as φ is proper, therefore φ is closed. It follows that $\overline{\varphi(Z)} = \varphi(Z)$. Consequently, by replacing X by W and Y by $\varphi(W)$, we may assume that $\varphi : X \to Y$ is a proper surjective map where X is a variety. Hence by Lemma A.3.1 and Remark A.3.3, Y is a variety and $\dim Y \leq \dim X$. We thus need to show that for any $f \in K(X)^\times$, the map $\varphi_* : Z(X) \to Z(Y)$ takes the divisor $\langle f; X\rangle$ of X to a cycle in $\mathrm{Rat}(Y)$.

We have

$$\varphi_*(\langle f; X\rangle) = \sum_{Z \in \mathrm{PDiv}(X)} v_Z(f) \cdot \varphi_*(Z)$$

where

$$\varphi_*(Z) = \begin{cases} [K(Z) : K(\varphi(Z))] \cdot \varphi(Z) & \text{if } \dim Z = \dim \varphi(Z) \\ 0 & \text{if } \dim Z > \dim \varphi(Z). \end{cases}$$

Note that $\dim Z = \dim X - 1$. Now if $\dim Y < \dim X - 1$, then $\dim \varphi(Z) \leq \dim Y < \dim X - 1$, therefore $\varphi_*(Z) = 0$. We thus have two cases that either $\dim Y = \dim X$ or $\dim Y = \dim X - 1$. If we have that $\dim Y = \dim X$, then by Proposition 3.1.2 we can dispense this case off.

Finally, suppose that $\dim Y = \dim X - 1$. Observe that

$$\varphi_*(\langle f; X\rangle) = \sum_{W \in \mathrm{PDiv}(X)} v_W(f) \cdot \varphi_*(W)$$

and

$$\varphi_*(W) = \begin{cases} [K(W) : K(\varphi(W))] \cdot \varphi(W) & \text{if } \dim W = \dim \varphi(W) \\ 0 & \text{if } \dim W > \dim \varphi(W). \end{cases}$$

As a consequence of $\dim Y = \dim X - 1$, we have

$$\varphi_*(\langle f; X\rangle) = \sum_{W \in \mathrm{PDiv}(X) \text{ s.t. } \varphi(W) = Y} v_W(f) \cdot [K(W) : K(Y)] \cdot Y.$$

We claim that $\sum_{W \in \mathrm{PDiv}(X) \text{ s.t. } \varphi(W)=Y} v_W(f) \cdot [K(W) : K(Y)] = 0$. This follows from the argument in pp 13 of [Ful84]. □

Definition 3.1.3 (**Degree map**)**.** Let $f : X \to \mathbf{k}$ be a complete scheme and suppose $\mathbf{k}$ is not necessarily algebraically closed. The degree map on the group of 0-cycle classes $A_0(X)$ is defined as

$$\begin{aligned} \deg : A_0(X) &\longrightarrow \mathbf{Z} \\ [p] &\longmapsto [\kappa(p) : \mathbf{k}], \end{aligned}$$

that is, it is the proper pushforward of f at A_0 as $f_*([p]) = [\kappa(p) : \kappa(f(p))] \cdot f(p)$. Hence it is a well-defined homomorphism.

Remark 3.1.4. If $\mathbf{k}$ is algebraically closed, then the degree map is simply given as follows: on a class of a point $[p] \in A_0(X)$, it maps $[p]$ to 1.

3.1.2 Rational equivalence as parametrized family

We wish to give an alternate definition of rational equivalence which is more geometric in nature. We begin with the following observation.

Lemma 3.1.5. *Let V be a $k+1$-dimensional variety and $\pi : V \to \mathbb{P}^1$ be a dominant morphism. Denote $V(0) = \pi^{-1}(0)$ and $V(\infty) = \pi^{-1}(\infty)$. Then the following are true.*

1. *Both $V(0)$ and $V(\infty)$ are pure k-dimensional subschemes.*
2. *π determines a rational function $\pi \in K(V)$ which satisfies the following equality of cycles:*

$$\langle V(0)\rangle - \langle V(\infty)\rangle = \langle \pi; V\rangle.$$

Proof. Note that $\pi : V \to \mathbb{P}^1$ is flat by Theorem A.7.5, . By Theorem A.7.4, the fibers are moreover of pure dimension k. This proves item 1. Next, we show that the map $\pi : V \to \mathbb{P}^1$ determines a unique element of $K(V)$. Indeed, on $U_0 = \operatorname{Spec}\left(\mathbf{k}[T_{1/0}]\right)$ of $\mathbb{P}^1$ and an open affine $U = \operatorname{Spec}(A) \subseteq \pi^{-1}(U_0)$, we get a map $\mathbf{k}[T_{1/0}] \to A$. As $K(V) = Q(A)$, so the image of $T_{1/0}$ in A determines an element of $K(V)$ induced by π, which we denote by π as well.

Consider the cycle $\langle \pi; V\rangle = \sum_{W \in \mathrm{PDiv}(V)} \upsilon_W(\pi) \cdot W$. Denote $\pi = f/g$ for regular functions f, g on V. By comparison of dimensions, it is clear that if for any $W \in \mathrm{PDiv}(V)$ we have $0 \in \pi(W)$, then W is an irreducible component of $V(0) = \pi^{-1}(0)$. Similarly for ∞. Thus, if $W \in \mathrm{PDiv}(V)$ is such that $0, \infty \notin \pi(W)$, then $f \in \mathcal{O}_{V,W}$ and $g \in \mathcal{O}_{V,W}$ are units. It then follows by definition of multiplicities $\upsilon_W(\pi)$ that $\upsilon_W(f) = 0 = \upsilon_W(g)$. Hence, $\langle \pi; V\rangle = \sum_{W \subseteq V(0) \text{ irr. comp.}} \upsilon_W(\pi) \cdot W + \sum_{W \subseteq V(\infty) \text{ irr. comp.}} \upsilon_W(\pi) \cdot W$. If W is an irreducible component of $V(0)$, then $\upsilon_W(g) = 0$ as $g \in \mathcal{O}_{V,W}$ is invertible. Similarly, if W is an irreducible component of $V(\infty)$, then $\upsilon_W(f) = 0$ as $f \in \mathcal{O}_{V,W}$ is invertible. Consequently, we may reduce to showing that $\upsilon_W(f) = \operatorname{len} \mathcal{O}_{V,W}$ for $W \subseteq V(0)$ an irreducible component. This is equivalent to showing that the regular function f is 0 on W, which is evident as $\pi(W) = 0$. $\square$

Theorem 3.1.6. *Let X be a scheme and $\alpha \in Z_k(X)$ be a k-cycle. Then, the following are equivalent:*

1. *$\alpha \in Z_k(X)$ is rationally equivalent to 0, i.e. $\alpha \in \mathrm{Rat}_k(X)$.*
2. *There exists $k+1$-dimensional subvarieties $V_1, \dots, V_r$ of $X \times \mathbb{P}^1$ such that the projections $\pi_i : V_i \to \mathbb{P}^1$ are dominant and*

$$\alpha = \sum_{i=1}^{r} \langle V_i(0)\rangle - \langle V_i(\infty)\rangle.$$

where $V_i(0) = V_i \cap (X \times 0) = \pi_i^{-1}(0)$ and $V_i(\infty) = V_i \cap (X \times \infty) = \pi_i^{-1}(\infty)$.

Example 3.1.7. Using this more geometric version of rational equivalence, we can construct explicitly examples of cycles which are rationally equivalent. Consider $X = \mathbb{A}^2$, Y the hyperbola $x^2 - y^2 - 1$ and Z the subscheme $x^2 - y^2$ which is the union of two lines. Observe that as Y is irreducible, therefore the cycle generated by $[Y]$ is just $1 \cdot [x^2 - y^2 - 1]$. Moreover, one can calculate that the cycle generated by $y^2 - x^2$ is $1 \cdot [y - x] + 1 \cdot [y + x]$. We thus wish to find a rational equivalence between these two cycles. Consider V to be the subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ obtained by vanishing of $x^2 - y^2 - t$ in $\mathbb{A}^2$ where $[t : 1 - t] \in \mathbb{P}^1$ for all $t \in \mathbf{k}$. We get a surjective projection $\pi : V \to \mathbb{P}^1$, whose fibers at $[1 : 0]$ is $x^2 - y^2 - 1$ and at $[0 : 1]$ is $x^2 - y^2$, as required.

One can define $V \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ more formally as follows. First note that $\mathbb{A}^2 \times \mathbb{P}^1$ is the relative projective space $\mathbb{P}^1_{\mathbb{A}^2}$ given by the fiber square

$$\begin{tikzcd}
\mathbb{P}^1_{\mathbb{A}^2} = \mathbb{A}^2 \times_{\mathbf{k}} \mathbb{P}^1 \arrow[r, "\pi"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{P}^1 \arrow[d] \\
\mathbb{A}^2 \arrow[r] & \mathbf{k}
\end{tikzcd}.$$

Consequently, $\mathbb{P}^1_{\mathbb{A}^2} = \mathrm{Proj}(A[s,t])$ where $A = \mathbf{k}[x,y]$. We then define V as the subvariety of $\mathbb{P}^1_{\mathbb{A}^2}$ given by the homogeneous prime ideal $\langle sx^2 - sy^2 - t\rangle \leq A[s,t]$. This defines a homogeneous prime ideal in $A[s,t]$, hence a subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$. The projection π restricts to the closed subvariety V of $\mathbb{P}^1_{\mathbb{A}^2}$ to give a map $\pi : V \to \mathbb{P}^1$ such that the fiber at $[a : b]$ is the subvariety of V given by $\langle ax^2 - ay^2 - b\rangle$ in V. So the fiber at $[1 : 0]$ is $x^2 - y^2$ and the fiber at $[1 : 1]$ is $x^2 - y^2 - 1$, as required.

3.1.3 Pullback & flat pullback

We next show the contravariance of $X \mapsto A(X)$. We omit the proof of the case of interest to us but we provide a proof for flat pullback.

Definition 3.1.8 (**Generic transversality to a map**)**.** Let $f : X \to Y$ be a map of smooth varieties and $A \subseteq Y$. Then A is said to be generically transverse to f if the scheme theoretic inverse $f^{-1}(A)$ is generically reduced and $\operatorname{codim}_X f^{-1}(A) = \operatorname{codim}_Y A$, where the equality means that $f^{-1}(A)$ is pure and every irreducible component C of $f^{-1}(A)$ has $\operatorname{codim}_X C = \operatorname{codim}_Y A$.

Lemma 3.1.9. *For two smooth subvarieties $A, B \subseteq X$ of a smooth variety X, the following are equivalent:*

1. *A and B intersect generically transversely.*
2. *A is generically transverse to $i : B \hookrightarrow X$.*

Proof. Suppose A and B are generically transverse. Note that $i^{-1}(A) = A \cap B$. From Lemma A.6.3, $A \cap B$ is generically reduced. As $\operatorname{codim}_B(A \cap B) = \dim B - \dim C$ for any irreducible component $C \subseteq A \cap B$, it suffices to show that this is equal to codim A. By Proposition 2.4.2, this is immediate. For converse, we show the statement 2 of Proposition 2.4.2. Indeed, we have A and B intersect properly and every irreducible component $C \subseteq A \cap B$ is generically reduced by assumption. We need only show that there exists a point of C which is smooth in X, but X is already smooth, so we are done. □

Theorem 3.1.10 (Pullback)**.** *Let $f : X \to Y$ be a map of smooth quasi-projective varieties. Then for each $k \geq 0$, there exists a map of groups $f^* : A^k(Y) \to A^k(X)$ unique w.r.t. the property that for any subvariety $A \subseteq Y$ which is generically transverse to f, we have*

$$f^*([A]) = [f^{-1}(A)].$$

While the above required smooth quasi-projective hypotheses, the following only requires the map to be flat and has a more simpler proof.

Theorem 3.1.11 (Flat pullback)**.** *Let $f : X \to Y$ be a flat map of schemes of relative dimension n. Then the pullback of cycles map $f^* : Z_k(Y) \to Z_{k+n}(X)$ maps $\mathrm{Rat}_k(Y)$ into $\mathrm{Rat}_{k+n}(X)$. Consequently, we get a group homomorphism*

$$f^* : A_k(Y) \longrightarrow A_{k+n}(X).$$

Proof. Pick a cycle $\alpha \in Z_k(X)$ which is rationally equivalent to 0. As f^* is linear on cycles, therefore by Theorem 3.1.6, we may assume that $\alpha = \langle V(0)\rangle - \langle V(\infty)\rangle$ for some $k+1$-dimensional subvariety $V \subseteq X \times \mathbb{P}^1$ where the projection map $\pi : V \to \mathbb{P}^1$ is dominant. We now show that

$f^*(\langle V(0)\rangle - \langle V(\infty)\rangle)$ is rationally equivalent to 0.

Consider the following diagram

$$\begin{tikzcd}
 & W \arrow[r, hook] \arrow[dl, "\xi"'] \arrow[d, "f\times\mathrm{id}"] \arrow[dr, phantom, "\lrcorner", very near start] & X\times\mathbb{P}^1 \arrow[r, "p"] \arrow[d, "f\times\mathrm{id}"] & X \arrow[d, "f"] \\
\mathbb{P}^1 & V \arrow[l, "\pi"] \arrow[r, hook] & Y\times\mathbb{P}^1 \arrow[r, "q"'] & Y
\end{tikzcd}$$

where p and q are projection (hence proper maps) and $W = (f\times\mathrm{id})^{-1}(V)$ is a closed subscheme of $X\times\mathbb{P}^1$. Also note that ξ is dominant. By Theorem A.7.5, π is a flat map and by Theorem A.7.4, $V(0)$ and $V(\infty)$ are pure of dimension k. Observe that $V(0)\times 0 \subseteq Y\times\mathbb{P}^1$ is a closed subscheme with $q_*(V(0)\times 0) = V(0)$ (Construction 1.1.6). Same remark holds for $V(\infty)$. Consequently, we get from the above diagram that

$$\langle V(0)\rangle - \langle V(\infty)\rangle = q_*\left(\langle \pi^{-1}(0)\rangle - \langle \pi^{-1}(\infty)\rangle\right).$$

We thus obtain that

$$f^*\left(\langle V(0)\rangle - \langle V(\infty)\rangle\right) = f^*q_*\left(\langle \pi^{-1}(0)\rangle - \langle \pi^{1}(\infty)\rangle\right).$$

By Proposition 1.1.10, we have $f^*q_* = p_*(f\times\mathrm{id})^*$. We thus get that

$$\begin{aligned}
f^*q_*\left(\langle \pi^{-1}(0)\rangle - \langle \pi^{1}(\infty)\rangle\right) &= p_*(f\times\mathrm{id})^*(\langle \pi^{-1}(0)\rangle - \langle \pi^{1}(\infty)\rangle) \\
&= p_*(f\times\mathrm{id})^*\pi^*(\langle 0\rangle - \langle \infty\rangle) \\
&= p_*\xi^*(\langle 0\rangle - \langle \infty\rangle) \\
&= p_*\left(\langle W(0)\rangle - \langle W(\infty)\rangle\right)
\end{aligned}$$

where the last three equalities follows from definition of pullback map on cycles (Construction 1.1.7), which is functorial by Proposition 1.1.9. By proper pushforward (Theorem 3.1.1), it is sufficient to show that $\langle W(0)\rangle - \langle W(\infty)\rangle$ is a cycle on X rationally equivalent to 0.

Denote $W_1,\dots,W_r$ be the irreducible components of W and let $\langle W\rangle = \sum_{i=1}^r m_i\cdot W_i$ where $m_i = \operatorname{len}\mathcal{O}_{W,W_i}$. We claim that

$$\langle W(0)\rangle - \langle W(\infty)\rangle = \sum_{i=1}^r m_i\left(\langle W_i(0)\rangle - \langle W_i(\infty)\rangle\right). \tag{3}$$

To this end, we first show that each $\xi : W_i \to \mathbb{P}^1$ is dominant. Indeed, we will show that the image $\xi(W_i)$ contains an open subset of $\mathbb{P}^1$. To this end, consider the open subset $W_i^o = W - \bigcup_{j\neq i} W_j$. This is an open subset of W inside W_i^o. Since flat maps are open by Proposition A.7.7, it follows that $\xi(W_i^o)$ is an open subset of $\mathbb{P}^1$, hence dense, as required.

By Lemma 3.1.5, it follows that $\langle W_i(0)\rangle - \langle W_i(\infty)\rangle = \langle \xi; W_i\rangle$. Consequently, if Eqn (3) holds, then we will have

$$\langle W(0)\rangle - \langle W(\infty)\rangle = \sum_{i=1}^r m_i\cdot\langle \xi; W_i\rangle,$$

that is a cycle rationally equivalent to zero. To show Eqn (3), it is sufficient to show the equality

$$\langle W(0)\rangle = \sum_{i=1}^{r} m_i \langle W_i(0)\rangle.$$

To this end, observe that the closed subscheme $W(0)$ of W is such that it is locally vanishing of one non-zero divisor and W is pure. The claim then follows from Lemma 1.7.2 of [Ful84]. □

Remark 3.1.12. If $f : X \to Y$ is a flat map of smooth quasi-projective varieties, then the flat pullback agrees with the unique map given by Theorem 3.1.10 as it satisfies the uniqueness condition by Proposition 1.1.10.

3.2 Excision & Mayer-Vietoris

We next derive two important calculational tools for computing Chow rings. We begin by observing the data provided by a closed subscheme of a scheme.

Remark 3.2.1. Let $i : Z \hookrightarrow X$ be a closed subscheme. Recall that i is a proper map as it is a closed immersion. For $U = X - Z$, consider the inclusion $j : U \hookrightarrow X$. Recall that j is a flat map (Lemma A.7.8). Consequently, we have proper pushforward for i and flat pullback for j of cycle classes.

Theorem 3.2.2 (Excision)**.** *Let X be a scheme and $Z \subseteq X$ be a closed subscheme. Denote $U = X - Z$ to be an open subscheme. Then for all $k \geq 0$, the following is an exact sequence where maps are induced by the inclusions $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$*

$$A_k(Z) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \longrightarrow 0\ .$$

Proof. By Remark 3.2.1, the above maps are well-defined. We first show the exactness of the above at the cycle level; the following is exact:

$$0 \longrightarrow Z_k(Z) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0\ .$$

By extension theorem of varieties (Lemma A.2.3), j^* above is surjective. The map i_* is injective by definition. We next show that $j^* \circ i_* = 0$. Indeed, pick a cycle $\alpha \in Z_k(Z)$. By linearity of i_* and j^*, we may assume $\alpha = W$ a k-dimensional subvariety of Z. As $i_*(Z) = Z$ as a subvariety of X contained in Z, therefore $j^*(i_*(Z)) = j^*(Z) = \langle j^{-1}(Z)\rangle = 0$ as $j^{-1}(Z) = \emptyset$. Finally, consider a cycle $\alpha = \sum_i n_i \cdot V_i \in Z_k(X)$ such that $j^*(\alpha) = \sum_i n_i \cdot \langle j^{-1}(V_i)\rangle = 0$. As $j^{-1}(V_i) = U \cap V_i$ is an open subscheme of V_i, therefore $U \cap V_i$ is a subvariety of U, so that $\langle j^{-1}(V_i)\rangle = V_i \cap U$ as a cycle. Now, if $V_i \cap U = V_j \cap U$ as varieties in U, then $V_i = V_j$ by taking closures. It follows that $\sum_i n_i \cdot V_i \cap U = 0$ and hence either $n_i = 0$ or $V_i \cap U = \emptyset$. In the latter case $V_i \subseteq Z$. Consequently, $\alpha = \sum_i n_i \cdot V_i$ is a cycle of X where each non-zero term is a k-dimensional subvariety of Z, that is, $\alpha \in Z_k(Z)$, as required.

Next, we show that the above exact sequence descends to cycle class level. Surjectivity of j^* is clear. It is also clear that $j^* \circ i_* = 0$ as it is so on the cycle level. We need only show that

$\operatorname{Ker}(j^*) \supseteq \operatorname{Im}(i_*)$. To this end, we first show that j^* induces a surjective map at the level of rational equivalence; the following is surjective:

$$j^* : \operatorname{Rat}_k(X) \longrightarrow \operatorname{Rat}_k(U).$$

Indeed, as $\operatorname{Rat}_k(U)$ is generated by prinicipal divisors of the form $\langle f; W\rangle$ where $W \subseteq U$ is a $k+1$-dimensional subvariety of U and $f \in K(W)^\times$, therefore it suffices to show that $\langle f; W\rangle$ can be extended to $\operatorname{Rat}_k(X)$. To this end, first extend W to a $k+1$-dimensional subvariety $\overline{W}$ of X and $\bar{f} \in K(\overline{W})$ (Lemma A.2.3). We thus have $j^*(\langle \bar{f}, \overline{W}\rangle) = \langle f; W\rangle$, which shows the surjectivity.

We now complete the proof. Consider the following diagram where the top row is exact and all verticals are exact:

$$\begin{tikzcd}
0 \arrow[r] & Z_k(Z) \arrow[r, "i_*"] \arrow[d] & Z_k(X) \arrow[r, "j^*"] \arrow[d] & Z_k(U) \arrow[r] \arrow[d] & 0 \\
& A_k(Z) \arrow[r, "i_*"'] \arrow[d] & A_k(X) \arrow[r, "j^*"'] \arrow[d] & A_k(U) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}.$$

Pick an element $[\alpha] \in A_k(X)$ such that $j^*([\alpha]) = 0$ for some $\alpha \in Z_K(X)$. Consequently $\alpha' := j^*(\alpha) \in Z_k(U)$ is rationally equivalent to 0, thus $\alpha' \in \operatorname{Rat}_k(U)$. By surjectivity of $j^* : \operatorname{Rat}_k(X) \to \operatorname{Rat}_k(U)$, there exists $\beta \in \operatorname{Rat}_k(X)$ such that $j^*(\beta) = \alpha'$. Consider $\alpha - \beta \in Z_k(X)$. We have that $j^*(\alpha-\beta) = 0$ in $Z_k(U)$ and thus by exactness lies in the image of i_*, say $i_*(\gamma) = \alpha - \beta$ for $\gamma \in Z_k(Z)$. As $i_*([\gamma]) = [\alpha - \beta] = [\alpha] - [\beta] = [\alpha]$ in $A_k(X)$, as required. □

Corollary 3.2.3. *If X is a smooth quasi-projective variety and $j : U \hookrightarrow X$ is an open subscheme, then the map*

$$j^* : A(X) \longrightarrow A(U)$$

is a surjective ring homomorphism.

Proof. Surjectivity is clear from Theorem 3.2.2. For ring homomorphism, we must show that $j^*([A] \cdot [B]) = j^*([A]) \cdot j^*([B])$ for $A, B \subseteq X$ two generically transverse subvarieties. As $[A] \cdot [B] = [A \cap B]$ (Corollary 2.4.4), hence we have $j^*([A] \cdot [B]) = [j^{-1}(A \cap B)]$ (Proposition 1.1.9). On the other hand, we also have $j^*([A]) \cdot j^*([B]) = [j^{-1}(A)] \cdot [j^{-1}(B)]$. We first claim that $j^{-1}(A)$, $j^{-1}(B)$ are generically transverse. Supposing that it is true, we will get $[j^{-1}(A)] \cdot [j^{-1}(B)] = [j^{-1}(A \cap B)]$, which will complete the proof.

To this end, we have $j^{-1}(A) = A \cap U$ and $j^{-1}(B) = B \cap U$. We have that A, B are transverse in an open dense set of each component $C \subseteq A \cap B$. It is an easy exercise to see that irreducible components of $A \cap B \cap U$ are $C \cap U$, where C varies over irreducible components of $A \cap B$. Consequently, $A \cap U$ and $B \cap U$ are transverse in any general point of $C \cap U$, as required. □

We next quickly cover the Mayer-Vietoris sequence, before moving on to calculations.

Theorem 3.2.4 (Mayer-Vietoris)**.** *Let X be a scheme and $X_1, X_2 \subseteq X$ be two closed subschemes. Consider the following fiber square*

$$\begin{tikzcd} X_1 \cap X_2 \arrow[r, "j_2"] \arrow[d, "j_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_2 \arrow[d, hook, "i_2"] \\ X_1 \arrow[r, hook, "i_1"'] & X_1 \cup X_2 \end{tikzcd}$$

Then we have the following exact sequence for each $k \geq 0$:

$$A_k(X_1 \cap X_2) \xrightarrow{\begin{bmatrix} j_{1*} - j_{2*} \end{bmatrix}} A_k(X_1) \oplus A_k(X_2) \xrightarrow{\begin{bmatrix} i_{1*} \\ i_{2*} \end{bmatrix}} A_k(X_1 \cup X_2) \longrightarrow 0 \ .$$

Proof. We may replace X by $X_1 \cup X_2$ so to assume that $X_1 \cup X_2 = X$. We first show the exactness at the cycle level; we claim that the following is exact:

$$0 \longrightarrow Z_k(X_1 \cap X_2) \xrightarrow{\begin{bmatrix} j_{1*} - j_{2*} \end{bmatrix}} Z_k(X_1) \oplus Z_k(X_2) \xrightarrow{\begin{bmatrix} i_{1*} \\ i_{2*} \end{bmatrix}} Z_k(X) \longrightarrow 0 \ .$$

For surjectivity on the right, we need only observe that any k-dimensional subvariety of X is by irreducibility either in X_1 or X_2. For injectivity on the left, if $Y \subseteq Z_k(X_1 \cap X_2)$ is a subvariety such that $(j_{1*}(Y), -j_{2*}(Y)) = (Y, -Y) = 0$, then $Y = 0$. It is also immediate to see that the the composite of both maps is 0. Finally, if $(V_1, V_2) \in Z_k(X_1) \oplus Z_k(X_2)$ where V_i are k-dimensional subvarieties such that $V_1 + V_2 = 0$, then $V_2 = -V_1$ and hence $V_1 \subseteq X_1 \cap X_2$. Consequently, $(j_{1*}(V_1), -j_{2*}(V_1)) = (V_1, V_2)$, as required.

We next show that the above exact sequence descends to cycle classes. Consider the following diagram where we claim that the bottom row is exact:

$$\begin{tikzcd} 0 \arrow[r] & Z_k(X_1 \cap X_2) \arrow[r, "{\begin{bmatrix} j_{1*} - j_{2*} \end{bmatrix}}"] \arrow[d] & Z_k(X_1) \oplus Z_k(X_2) \arrow[r, "{\begin{bmatrix} i_{1*} \\ i_{2*} \end{bmatrix}}"] \arrow[d] & Z_k(X) \arrow[r] \arrow[d] & 0 \\ & A_k(X_1 \cap X_2) \arrow[r, "a"'] \arrow[d] & A_k(X_1) \oplus A_k(X_2) \arrow[r, "b"'] \arrow[d] & A_k(X) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

By exactness of the top row, we are reduced to only showing that $\mathrm{Ker}\,(a) \subseteq \mathrm{Im}\,(b)$. To this end, by following the same idea as proof of excision, one reduces to showing that the following map is surjective:

$$\mathrm{Rat}_k(X_1) \oplus \mathrm{Rat}_k(X_2) \xrightarrow{\begin{bmatrix} i_{1*} \\ i_{2*} \end{bmatrix}} \mathrm{Rat}_k(X) \ .$$

Indeed, pick any $W \subseteq X$ a $k+1$-dimensional subvariety of X and $f \in K(W)$ a non-zero rational function. Then, we wish to find a preimage of the element $\langle f; W \rangle \in \mathrm{Rat}_k(X)$. As W is irreducible in $X = X_1 \cup X_2$, therefore W lies in either X_1 or X_2 scheme theoretically as if a prime contains $I \cap J$, then it contains either I or J. The surjectivity now is clear. $\square$

3.3 Affine bundles & $\mathbb{A}^n$

For affine bundles, we see that the map on Chow groups is surjective. This allows us to calculate Chow group of affine spaces, which will be building block for projective spaces and Grassmannians. We first need a result surrounding points of $X \times \mathbb{A}^1$ of codimension 1.

Lemma 3.3.1. *Let X be a variety and denote $p : X \times \mathbb{A}^1 \to X$ to be the projection map. Then any codimension 1 subvariety V of $X \times \mathbb{A}^1$ with generic point η is of one of the following two types:*
1. *$V = p^{-1}(\overline{p(\eta)})$ where $\overline{p(\eta)}$ is a codimension 1 subvariety of X,*
2. *$p(\eta)$ is the generic point of X.*

Proof. Pick a codimension 1 subvariety V of $X \times \mathbb{A}^1$ with generic point $\eta \in X \times \mathbb{A}^1$. Consider a finite type integral open affine $U = \operatorname{Spec}(A)$ of X so that $U \times \mathbb{A}^1 = \operatorname{Spec}(A[t])$ is an open affine containing η. Hence, η corresponds to a prime $\mathfrak{p}$ in $A[t]$ of height 1. There are two cases:
1. $\mathfrak{p} \cap A \neq 0$,
2. $\mathfrak{p} \cap A = 0$

Note that in case 1, $\mathfrak{p} \cap A$ is a prime ideal of A of height atmost 1 as any prime $\mathfrak{q} \leq A$ gives a distinct prime $\mathfrak{q}A[t]$ of A. As A is a domain, therefore height of $\mathfrak{p} \cap A$ is exactly 1. Consequently, $\mathfrak{p} \cap A$ gives a codimension 1 subvariety of X. Moreover, note that $p(\eta) = \mathfrak{p} \cap A$. Let $W = \overline{p(\eta)}$ be the codimension 1 subvariety of X. We claim that $p^{-1}(W) = V$. Indeed, $p^{-1}(W) = W \times \mathbb{A}^1$ which is a codimension 1 subvariety of $X \times \mathbb{A}^1$ which contains V, so $V = W \times \mathbb{A}^1$, as required.

On the other hand, we have $\mathfrak{p} \cap A = 0$ in case 2. As $p(\eta) = \mathfrak{p} \cap A = 0$, therefore p maps η to the generic point of X, as required. $\square$

Theorem 3.3.2. *Let $p : E \to B$ be an affine bundle of rank n over B. Then the flat pullback map*

$$p^* : A_k(B) \longrightarrow A_{k+n}(E)$$

is surjective for all $k \geq 0$.

Proof. We first show the claim for the trivial rank n-affine bundle over B. Indeed, let $E = B \times \mathbb{A}^n$ and $p : E \to B$ be projection onto first coordinate. Note that p factors as $B \times \mathbb{A}^n \xrightarrow{r} B \times \mathbb{A}^{n-1} \xrightarrow{q} B$, hence we get the following triangle

$$\begin{tikzcd} A_{k+n}(B \times \mathbb{A}^n) & \\ A_k(B) \arrow[u, "p^*"] \arrow[r, "q^*"'] & A_{k+n-1}(B \times \mathbb{A}^{n-1}) \arrow[ul, "r^*"'] \end{tikzcd}.$$

Consequently, it is sufficient to show that for $p : B \times \mathbb{A}^1 \to B$, the induced map

$$p^* : A_k(B) \longrightarrow A_{k+1}(B \times \mathbb{A}^1)$$

is surjective. To this end, pick any $k+1$-dimensional subvariety V of $B \times \mathbb{A}^1$. We wish to find k-dimensional subvarieties W_i of B such that

$$[V] = \sum_i n_i \cdot p^*[W_i].$$

It is sufficient to find W_i in the scheme theoretic image $\overline{p(V)}$, so we may replace B by $\overline{p(V)}$ so that we may now assume that B is a variety and $p : V \to B$ is dominant (Lemma A.3.1). By Remark A.3.3, $k + 1 = \dim V \geq \dim B$. If $\dim B < k$, then $A_k(B) = 0 = A_{k+1}(B \times \mathbb{A}^1)$ and there is nothing to prove. If $\dim B = k$, then $B \times \mathbb{A}^1$ is a $k+1$-dimensional variety and hence $V = B \times \mathbb{A}^1$. Thus, we may take $W_i = B$ so that $[V] = p^*([B])$. Hence we may assume that $\dim B = k + 1$. So $\dim B \times \mathbb{A}^1 = k + 2$ and V is a codimension 1 subvariety of $B \times \mathbb{A}^1$. Let η be the generic point of V.

By Lemma 3.3.1, either $p(\eta)$ is a codimension 1 subvariety of B (say, type 1) or V dominates B (say, type 2). If of type 1, then $V = p^{-1}(W)$ for $W \in Z_k(B)$ and hence $[V] = p^*[W]$ in $A_{k+1}(B \times \mathbb{A}^1)$. So it is sufficient to show that if V is of type 2, then it is rationally equivalent to sum of type 1 subvarieties. To this end, fix a finite type integral open affine $U = \mathrm{Spec}\,(A)$ of variety B. Then consider the following fiber squares ($\mathfrak{o}$ denotes the generic point of B in U):

$$\begin{tikzcd}
\mathrm{Spec}\,(K[t]/\mathfrak{p}K[t]) \arrow[r, hook, "i"] \arrow[d] & p^{-1}(\mathfrak{o}) = \mathrm{Spec}\,(K[t]) \arrow[r] \arrow[d] & \mathfrak{o} = \mathrm{Spec}\,(K) \arrow[d, hook] \\
V \cap (U \times \mathbb{A}^1) = \mathrm{Spec}\,(A[t]/\mathfrak{p}) \arrow[r, hook, "i"'] & U \times \mathbb{A}^1 = \mathrm{Spec}\,(A[t]) \arrow[r, "p"'] & U = \mathrm{Spec}\,(A)
\end{tikzcd}$$

where $K = Q(A)$ is the function field of B. The affine $\mathrm{Spec}\,(K[t]/\mathfrak{p}K[t])$ is the fiber of $V \cap (U \times \mathbb{A}^1)$ at the generic point $\mathfrak{o}$ of B. This fiber is non-empty as V dominates B. Consequently $\mathfrak{p}K[t]$ is a prime ideal of $K[t]$ and hence is a principal ideal generated by $f(t) \in K[t]$. We may assume $f(t) \in \mathfrak{p} \leq A[t]$ is a regular function on $U \times \mathbb{A}^1$ by clearing the denominators.

We claim that the principal divisor of f over $B \times \mathbb{A}^1$ contains only one codimension 1 subvariety of type 2 which is V itself. Indeed, if $\mathfrak{p}'$ is a height 1 prime of $A[t]$ of type 2 such that $f \in \mathfrak{p}'$, then as $f \in \mathfrak{p}'K[t]$ therefore $\mathfrak{p}K[t] \subseteq \mathfrak{p}'K[t]$. By maximality of $\mathfrak{p}K[t]$, we deduce $\mathfrak{p}K[t] = \mathfrak{p}'K[t]$. Intersecting with $A[t]$, we get $\mathfrak{p} = \mathfrak{p}'$, as required. This shows that

$$\langle f; B \times \mathbb{A}^1 \rangle = v_V(f) \cdot V + \sum_i n_i \cdot V_i.$$

where V_i are type 1 subvarieties of $B \times \mathbb{A}^1$. To complete the proof, it is sufficient to show that $v_V(f) = 1$, that is, $\mathrm{len}\,\frac{\mathcal{O}_{E,V}}{f\mathcal{O}_{E,V}} = 1$. As $\mathcal{O}_{E,V} = A[t]_{\mathfrak{p}}$, therefore it is sufficient to show that $fA[t]_{\mathfrak{p}}$ is the unit ideal. Recall that $\mathfrak{p}$ is a type 2 codimension 1 point of $A[t]$, hence $\mathfrak{p} \cap A = 0$. Thus, $A[t]_{\mathfrak{p}} = K[t]_{\mathfrak{p}K[t]}$, that is,

$$v_V(f) = \mathrm{len}\,\frac{K[t]_{\mathfrak{p}K[t]}}{fK[t]_{\mathfrak{p}K[t]}} = \mathrm{len}\left(\frac{K[t]}{fK[t]}\right)_{\mathfrak{p}K[t]} = 1$$

as $fK[t]$ is the maximal ideal $\mathfrak{p}K[t]$ of $K[t]$ and hence the quotient is a field. This completes the proof when E is the trivial affine bundle.

Now consider $p : E \to B$ to be a rank n-affine bundle over B. Let $U \subseteq B$ be any open affine which is a local trivialization of the bundle p. Then by excision (Theorem 3.2.2), we have the following commutative diagram

$$\begin{tikzcd}
A_k(Z) \arrow[r] \arrow[d, "p^*"'] & A_k(B) \arrow[r] \arrow[d, "p^*"'] & A_k(U) \arrow[r] \arrow[d, "p^*"] & 0 \\
A_{k+n}(p^{-1}(Z)) \arrow[r] & A_{k+n}(E) \arrow[r] & A_{k+n}(U \times \mathbb{A}^n) \arrow[r] & 0
\end{tikzcd}$$

whose commutativity follows from the fiber square of $U \hookrightarrow B \leftarrow E$. By four lemma, it suffices to show that $p : U \times \mathbb{A}^n \to U$ and $p : p^{-1}(Z) \to Z$ induces a surjective map on Chow groups. By noetherian induction on Z, we may assume $Z = \emptyset$. So we may replace B by U and thus assume that E is trivial, as required. $\square$

Corollary 3.3.3 (Chow ring of $\mathbb{A}^n$)**.** *For any $n \in \mathbf{N}$, we have $A^k(\mathbb{A}^n) = 0$ for all $1 \leq k \leq n$ and $A^0(\mathbb{A}^n) \cong \mathbf{Z}$ generated by class $[\mathbb{A}^n]$. That is we have a ring isomorphism*

$$A(\mathbb{A}^n) \cong \mathbf{Z}.$$

Proof. First observe that $A_0(\mathbb{A}^n) = 0$ for any $n \geq 1$ as for any two points $p, q \in \mathbb{A}^n$, we have the subvariety defined by the homogeneous prime ideal

$$I = \langle tx_i - sp_i - (t-s)q_i \mid 1 \leq i \leq n \rangle$$

in $A[s,t]$ where $A = \mathbf{k}[x_1, \dots, x_n]$ which then gives the subvariety $W = \mathrm{Proj}(A[s,t]/I) \hookrightarrow \mathrm{Proj}(A[s,t]) = \mathbb{P}^1_{\mathbb{A}^n} = \mathbb{A}^n \times_{\mathbf{k}} \mathbb{P}^1$. Thus it is a subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$. Note that $\pi : W \to \mathbb{P}^1$ is dominant and the fiber $W([1:1]) = p$ and $W([0:1]) = q$. Hence by Theorem 3.1.6, $p - q \in \mathrm{Rat}_0(\mathbb{A}^n)$, as required.

Fix $n \in \mathbf{N}$. Observe that $p : \mathbb{A}^n \to \mathbb{A}^{n-1}$ is an affine bundle of rank 1. By Proposition 3.3.2, we have

$$p^* : A_k(\mathbb{A}^{n-1}) \longrightarrow A_{k+1}(\mathbb{A}^n)$$

is surjective for all $k \geq 0$. For $k = 0$, by previous we'll have $\mathbb{A}_0(\mathbb{A}^{n-1}) = 0$, and thus $A_1(\mathbb{A}^n) = 0$. For $k = 1$, we get $A_2(\mathbb{A}^n) = 0$ and so on. By induction, we get $A_k(\mathbb{A}^n) = 0$ for all $0 \leq k \leq n-1$ and $A_n(\mathbb{A}^n) = \mathbf{Z}$ generated by $\mathbb{A}^n$. $\square$

Example 3.3.4. Let $U \subseteq \mathbb{A}^n$ be an open subscheme. Then, $A_k(U) = 0$ for all $0 \leq k < n$. This follows immediately from excision (Theorem 3.2.2). On the other hand, as U is irreducible, therefore we have $A_n(U) = \mathbf{Z}$ generated by class of U.

3.4 Stratified schemes & $\mathbb{P}^n$

Chow rings become helpful for enumerative problems after their calculation in some known parameter spaces, like Grassmannians. This calculation is usually done by finding a generating set of the Chow ring and then showing that these generators are torsion free. The main result that helps in this calculation is what we cover in this section. We then show its use by calculating the Chow ring of the prototypical parameter space, $\mathbb{P}^n$.

Definition 3.4.1 (**Stratified schemes**)**.** A scheme X is said to be stratified if there exists finitely many disjoint irreducible locally closed subschemes U_i of X of dimension n_i such that $X = \coprod_i U_i$ and $\overline{U_i} - U_i \subseteq \bigcup_j U_{i_j}$ for some U_{i_j} of dimension lower than n_i. The subsets U_i are called open strata of X and $Y_i = \overline{U_i}$ are called closed strata where we fix a scheme structure on each Y_i. Note that

$$U_i = Y_i - \bigcup_{Y_j \subsetneq Y_i} Y_j.$$

Note that U_i may not be actually open. The dimension of X is defined to be $\max_i n_i$. We moreover say that the stratification $\{U_i\}$ is an *affine stratification* if each open stratum U_i is isomorphic to $\mathbb{A}^n$ for some n. We say it is a *quasi-affine stratification* if each U_i is isomorphic to an open subset of $\mathbb{A}^n$.

Lemma 3.4.2. *Let X be a scheme. The following are equivalent:*

1. *X has a quasi-affine stratification.*
2. *There exists a filtration of X by closed subschemes*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

such that each complement $X_i - X_{i-1}$ is disjoint union of finitely many schemes U_{ij} which are isomorphic to open subsets of $\mathbb{A}^i$ of dimension i.

Proof. (1. $\Rightarrow$ 2.) Consider $X_i = \bigcup_j U_j$ where U_j is of dimension $\leq i$. Then, clearly $X_i - X_{i-1}$ is a disjoint union of U_j whose dimension is i. As each U_j is isomorphic to an open subset in $\mathbb{A}^n$, hence we are done.

(2. $\Rightarrow$ 1.) Consider the collection $\{U_j\}$ of quasi-affine opens of X formed from each difference $X_i - X_{i-1}$. We claim that this forms a quasi-affine stratification of X. Indeed, each U_j is disjoint and covers X by induction. Moreovoer, if $U_j \subsetneq X_i - X_{i-1}$ and if the closure $Y_j = \overline{U_j}$ intersects X_{i-1}, then Y_j intersects U_k where $U_k \subseteq X_l$ for $l \leq i-1$. The union over all such U_k covers $\overline{U_j}$. $\square$

An important example of this is the projective n-space.

Proposition 3.4.3. *For each $n \geq 1$, the projective n-space $\mathbb{P}^n$ has an affine stratification.*

Proof. Consider any complete flag of linear subspaces in $\mathbb{P}^n$ given by $0 = L^0 \subset L^1 \subset \cdots \subset L^n = \mathbb{P}^n$ where $L^i = V(x_0, \ldots, x_{n-i-1})$. Note that each $L^k \cong \mathbb{P}^k$ (Lemma A.2.4), obtained by identifying L^k by $\mathrm{Proj}(\mathbf{k}[x_{n-k}, \ldots, x_n])$. We claim that these form the closed strata of an affine stratification on $\mathbb{P}^n$. Indeed, let $U_i = L^i - L^{i-1}$. Observe that $U_i = D_+(x_{n-i}) \cong \mathbb{A}^i$ and these form a disjoint open cover of $\mathbb{P}^n$ together with $\overline{U_i} = \mathbb{P}^i = L^i$, as required. $\square$

A simple but useful result to keep in mind is the fact that any two k-dimensional linear subspace of $\mathbb{P}^n$ are rationally equivalent to each other. The main result here is that for a quasi-affine stratification of a scheme X, the class of closed strata forms a generating set of $A(X)$.

Proposition 3.4.4. *Let X be a scheme with a quasi-affine stratification with closed strata $\{Y_i\}$. Then $A(X)$ is generated as an abelian group by the classes of closed strata $\{[Y_i]\}$.*

Proof. We proceed by induction on the number of strata U_i of X. If X has only one strata, then $X = U$ is an open subset of $\mathbb{A}^n$, whose Chow ring is $\mathbf{Z}$ as shown in Example 3.3.4. This dispenses the base case. Consider now the inductive case where we assume that $X = \coprod_{i=1}^n U_i$ and for any union of less than $n-1$ open strata satisfies the claim. Consider U_1 to be the minimal dimension open strata of X. Then by definition $\overline{U_1} = U_1$ as there is no lower dimensional lower strata. Hence, $U_1 \subseteq X$ is closed. Observe that $X - U_1$ is is union of $n-1$ open strata and thus satisfies the claim. By excision (Theorem 3.2.2), we get the following exact sequence induced by inclusions:

$$\mathbf{Z} = A(U_1) \longrightarrow A(X) \longrightarrow A(X - U_1) \longrightarrow 0 \ .$$

Thus we have that $A(X - U_1)$ is isomorphic to quotient of $A(X)$ with the image of $\mathbf{Z}$. The image of $\mathbf{Z}$ is generated by the class of U_1 and the quotient is generated by the classes of closed strata of $X - U_1$, thus, $A(X)$ has a generating set given by all closed strata of X, as required. □

We now wish to calculate the Chow ring of $\mathbb{P}^n$ using the above result. To this end, we first calculate each $A_k(\mathbb{P}^n)$ as a group.

Lemma 3.4.5. *For any $n \geq 1$, the group $A_k(\mathbb{P}^n)$ is generated by the class $[L^k]$ of the k-dimensional linear subspace of $\mathbb{P}^n$. In particular*

$$A(\mathbb{P}^n) = \bigoplus_{k=0}^{n} A_k(\mathbb{P}^n) = \mathbf{Z} \cdot [L^n] \oplus \cdots \oplus \mathbf{Z} \cdot [L^0].$$

Proof. Fix the affine stratification of $\mathbb{P}^n = \mathrm{Proj}(\mathbf{k}[x_0, \dots, x_n])$ given in Proposition 3.4.3; we have $0 = L^0 \subset \cdots \subset L^n = \mathbb{P}^n$:

$$L^k = V(x_0, \dots, x_{n-k-1}) = \mathrm{Proj}(\mathbf{k}[x_{n-k}, \dots, x_n]) \cong \mathbb{P}^k.$$

The closed strata of this affine stratification is L^k itself as $L^k - L^{k-1} = \mathbb{A}^k$. Consequently, the cycle classes of L^k generates $A(\mathbb{P}^n)$ (Proposition 3.4.4). As each L^k is a k-dimensional subvariety of $\mathbb{P}^n$, therefore $[L^k] \in A_k(\mathbb{P}^n)$. Hence, $A_k(\mathbb{P}^n)$ is generated by $[L^k]$, as required. □

Our next aim is to show that $\mathbf{Z} \cdot [L^k] = \mathbf{Z}$, i.e. each $[L^k]$ is a torsion free element in $A_k(\mathbb{P}^n)$.

Lemma 3.4.6. *For any $n \geq 1$ and $0 \leq k \leq n$, we have $A_k(\mathbb{P}^n) = \mathbf{Z} \cdot [L^k] \cong \mathbf{Z}$.*

Proof. It suffices to show that $[L^k] \in A(\mathbb{P}^n)$ is a torsion free element. We first show that $A_0(\mathbb{P}^n) \cong \mathbf{Z}$. Indeed, observe that as $\mathbb{P}^n$ is a complete variety over $\mathbf{k}$, therefore the degree map (Definition 3.1.3)

$$\deg : A_0(\mathbb{P}^n) \to \mathbf{Z}$$

maps $[L^0] \mapsto 1$. Thus, $[L^0] \in A_0(\mathbb{P}^n)$ has no torsion, showing that $A_0(\mathbb{P}^n) = \mathbf{Z} \cdot [L^0] \cong \mathbf{Z}$.

Consider the intersection product on $A(\mathbb{P}^n)$, which in particular gives the following bilinear map for each $k \geq 0$:

$$A_k(\mathbb{P}^n) \times A_{n-k}(\mathbb{P}^n) \longrightarrow A_0(\mathbb{P}^n).$$

Fixing a general $n-k$ linear subspace $[M] \in A_{n-k}(\mathbb{P}^n)$ which intersects L^k in precisely one point transversely, we get the following linear map (Corollary 2.4.4)

$$\begin{aligned} A_k(\mathbb{P}^n) &\longrightarrow A_0(\mathbb{P}^n) \\ [L^k] &\longmapsto [M \cap L^k] = [p]. \end{aligned}$$

Composing by the isomorphism $\deg : A_0(\mathbb{P}^n) \to \mathbf{Z}$, we get the following map:

$$\begin{aligned} \theta : A_k(\mathbb{P}^n) &\longrightarrow \mathbf{Z} \\ [L^k] &\longmapsto 1 \end{aligned}$$

If $[L^k]$ has torsion, that is, $d[L^k] = 0$ in $A_k(\mathbb{P}^n)$, then $0 = \theta(d[L^k]) = d\theta([L^k]) = d$. Hence, $A_k(\mathbb{P}^n) = \mathbf{Z} \cdot [L^k]$. □

We finally compute the intersection product on $A(\mathbb{P}^n)$.

Theorem 3.4.7. *For any $n \geq 1$, the Chow ring of $\mathbb{P}^n$ is given by*

$$A(\mathbb{P}^n) = \frac{\mathbf{Z}[\xi]}{\xi^{n+1}}$$

where $[\xi] \in A^1(\mathbb{P}^n)$ is the class of a hyperplane.

Proof. By Lemma refL-3.4.6, we have $A^{n-k}(\mathbb{P}^n) = A_k(\mathbb{P}^n) = \mathbf{Z} \cdot [L^k] \cong \mathbf{Z}$. To compute the product, we need only calculate the product for linear subspaces of $\mathbb{P}^n$. First observe that any two k-dimensional linear subspace of $\mathbb{P}^n$ are rationally equivalent. As $[L^k] = [L^{n-1}]^k$, therefore, we have $A^{n-k}(\mathbb{P}^n) = \mathbf{Z} \cdot [L^{n-1}]^k$. The result now follows. □

Recall we computed that the group $A(X) = \mathrm{Cl}(X) \oplus \mathbf{Z}$ for a curve X (Corollary 1.2.8). We now compute the ring structure on $A(X)$.

Example 3.4.8 (Chow ring of curves)**.** For X a smooth quasi-projective curve we have $A^1(X) = \mathrm{Cl}(X)$ and $A^0(X) = \mathbf{Z}$. We claim that the map

$$\mathrm{Cl}(X) \times \mathbf{Z} \longrightarrow \mathrm{Cl}(X)$$

is given by $[\alpha] \cdot n[X] \mapsto n[\alpha]$. Indeed, we need only show that for a prime divisor $Y \subseteq X$, we have $[Y] \cdot [X] = [Y]$. To this end, we first see that Y and X intersect generically transversely as Y represents a point in X so $Y \cap X = Y$ and since $T_Y X + T_Y Y = T_Y X$, so Y and X indeed intersect transversely at point Y. It follows by Corollary 2.4.4 that $[Y] \cdot [X] = [Y \cap X] = [Y]$, as required.

4 Grassmannians

In order to use intersection theory to solve a given enumerative problem, we will have to construct a parameter space in which the enumerative problem gives cycles. Then we study the geometry of these cycles in this parameter space. Usually this will require an understanding of the geometry of this parameter space, and that usually means understanding its tangent bundle. The simplest of such parameter spaces is the Grassmannian, which parameterizes linear subspaces of a vector space of a fixed dimension. In this section we study the basic geometry of Grassmannians.

Definition 4.0.1 (**Grassmannian as a set**)**.** Let V be a vector space of dimension n over $\mathbf{k}$ and let $k \geq 0$. Then, $\mathrm{Gr}(k, V)$ is the set of all k-dimensional linear subspaces of V. As k-dimensional linear subspaces of V are same as $k-1$-dimensional linear subspaces of $\mathbb{P}V = \mathrm{Proj}(\mathrm{Sym}(V^*))$, the projectivization of V, which is isomorphic to $\mathbb{P}^{n-1}_{\mathbf{k}}$, thus there is a natural bijection between $\mathbb{G}\mathrm{r}(k-1, \mathbb{P}V)$, the set of all $k-1$-dimensional linear subspaces of $\mathbb{P}V$, and $\mathrm{Gr}(k, V)$ and we henceforth identify them as same sets. We will also write $\mathbb{G}\mathrm{r}(k-1, \mathbb{P}V)$ as $\mathbb{G}\mathrm{r}(k-1, n-1)$.

We wish to give a variety structure on $\mathrm{Gr}(k, V)$. We do this by giving an inclusion in a projective space, showing it is a closed irreducible subspace of it by showing it is vanishing of some polynomials and then show that these give a reduced structure on $\mathrm{Gr}(k, V)$, thus showing that $\mathrm{Gr}(k, V)$ is a projective variety.

Before moving, we will have to cover an important technical point. In our approach to enumerative problems, we will end up constructing cycles in a given parameter space whose intersection corresponds to the set we wish to count. However, intersection theory as developed above will be applicable in our case only if we can say that all our cycles that we will end up constructing actually intersect generically transverse. Indeed, this is what the following result of Kleiman asserts. This theorem can be seen as a more explicit version of moving lemma. Recall $\mathbf{k}$ is of characteristic 0 and is algebraically closed.

Theorem 4.0.2 (Kleiman transversality)**.** *Let G be an algebraic group acting transitively on a variety X. Let $A \subseteq X$ be a subvariety.*

1. *If $f : Y \to X$ is a morphism of varieties, then for any general $g \in G$, the inverse image $f^{-1}(gA)$ is generically reduced and* $\mathrm{codim}\,_X A = \mathrm{codim}\,_Y f^{-1}(gA)$[5]. *In particular if $B \hookrightarrow X$ is a subvariety, then by Proposition 2.4.2, gA and B intersect generically transversely for any general $g \in G$.*
2. *If G is an affine algebraic group, then for all $g \in G$*

$$[gA] = [A] \text{ in } A^*(X).$$

For a proof, see Theorem 1.7 of [**?**].

Remark 4.0.3. Note that $\mathrm{Gr}(k, V)$ has a natural action of $\mathrm{GL}(V)$, which is an affine algebraic group, given by $g \cdot \Lambda := g(\Lambda)$. Thus for any two subvarieties $A, B \subseteq \mathrm{Gr}(k, V)$ and any general $g \in \mathrm{GL}(V)$, gA and B are generically transverse and $[gA] = [A]$ in $A^*(\mathrm{Gr}(k, V))$.

4.1 Plücker coordinates

Let V be an n-dimensinal $\mathbf{k}$-vector space. Our goal in this section is to show that $\mathrm{Gr}(k, V)$ is a projective variety of dimension $k(n-k)$ and to find an affine open covering of it. As $\mathbf{k}$ is algebraically closed, it would thus follow at once that $\mathrm{Gr}(k, V)$ is regular and hence smooth.

Remark 4.1.1. We first observe that $\mathrm{Gr}(k, V)$ is in bijection with rank k matrices of shape $k \times n$. Indeed, fix a basis $\{e_1, \ldots, e_n\}$ of V and consider $M^k_{k\times n}$ the set of $k \times n$ matrices of rank k upto row

[5]Recall that this notation means that $f^{-1}(gA)$ is pure and the equality holds for each irreducible component of $f^{-1}(gA)$.

equivalence (i.e. two rank k matrices are identified if they have the same row space). Then there is a bijection

$$\begin{aligned}\mathrm{Gr}(k,V) &\longrightarrow M^k_{k\times n}\\ \Lambda &\longmapsto A_\Lambda\end{aligned}$$

where the rows of A_Λ are basis vectors of Λ expanded in the basis of V, to yield a $k\times n$ matrix of rank k. Converse is obtained by taking row space of a matrix $A\in M^k_{k\times n}$.

We next construct the Plücker coordinates.

Construction 4.1.2 (Plücker embedding)**.** Consider the function

$$\begin{aligned}P:\mathrm{Gr}(k,n) &\longrightarrow \mathbb{P}\wedge^k V\\ \Lambda &\longmapsto [v_1\wedge\cdots\wedge v_k]\end{aligned}$$

where Λ has basis $\{v_1,\dots,v_k\}$. This is well-defined as if $\{w_1,\dots,w_k\}$ forms another basis of Λ, then $w_1\wedge\cdots\wedge w_k = d\cdot(v_1\wedge\cdots\wedge v_k)$ where d is the determinant of the change of basis matrix, and thus they determine same point in $\mathbb{P}\wedge^k V$.

We next wish to write P in projective coordinates of $\mathbb{P}\wedge^k V$. To this end, fix a basis $\{e_1,\dots,e_n\}$ of V. Writing each v_i in this basis, we deduce that the k-plane Λ is the row space of the $k\times n$ matrix A_Λ whose rows are v_i. We can then write

$$v_1\wedge\cdots\wedge v_k = \sum_{I\in\mathrm{Inc}(k,n)} p_I e_I$$

where $I=(i_1,\dots,i_k)$ is an increasing sequence of elements from $\{1,\dots,n\}$, $e_I = e_{i_1}\wedge\cdots\wedge e_{i_k}$ forms the basis of $\wedge^k V$ and $p_I = \det A_\Lambda[I]$, the $k\times k$-minor of A_Λ determined by columns with index I. In projective coordinates (of which there are ${}^nC_k$ many), the map P is merely

$$P:\Lambda\mapsto [p_I]_{I\in\mathrm{Inc}(k,n)}$$

where $p_I=\det A_\Lambda[I]$ is a polynomial in the entries of a general $k\times n$ matrix.

We first wish to show that this function is injective. Indeed, if $P(\Lambda)=P(\Lambda')$, then $v_1\wedge\cdots\wedge v_k = d\cdot w_1\wedge\cdots\wedge w_k$ for $d\in\mathbf{k}^\times$ where $\{v_1,\dots,v_k\}$ is a basis of Λ and $\{w_1,\dots,w_k\}$ is a basis of Λ'. If $[p_I]_I$ and $[q_I]_I$ are projective coordinates of $v_1\wedge\cdots\wedge v_k$ and $w_1\wedge\cdots\wedge w_k$ respectively, then $p_I = d\cdot q_I$. It follows that every $k\times k$ minor of A_Λ is a common multiple of the same minor of $A_{\Lambda'}$. Consequently, A_Λ and $A_{\Lambda'}$ have same row space, as required.

The map P embeds $\mathrm{Gr}(k,V)$ as a subspace of $\mathbb{P}(\wedge^k V)$. We next claim that $\mathrm{Gr}(k,n)$ is in-fact a closed subspace.

Lemma 4.1.3. *The Plücker map is a closed embedding of* $\mathrm{Gr}(k,V)$ *into* $\mathbb{P}(\wedge^k V)$*.*

Proof. We need only show that the image of P is closed. To this end, we first claim that

$$\mathrm{Im}\,(P) = \left\{[\eta]\in\mathbb{P}(\wedge^k V)\mid \dim\mathrm{Im}\left(V\overset{\wedge\eta}{\to}\wedge^{k+1}V\right)\le n-k\right\}.$$

Indeed, image of P consists of classes of all those $\eta\in\wedge^k V$ where $\eta = v_1\wedge\cdots\wedge v_k$ for $v_i\in V$, i.e. η is a pure tensor. The vector η is of this form if and only if $\dim\mathrm{Ker}\left(V\overset{\wedge\eta}{\to}\wedge^k V\right)\ge k$ and hence the desired claim follows.

As $\wedge\eta$ is a linear map, therefore $\dim\mathrm{Im}\left(V\overset{\wedge\eta}{\to}\wedge^{k+1}V\right)\le n-k$ if and only if all $n-k+1$ minors of $\wedge\eta$ are 0. This is a closed condition, as required. $\square$

4.2 Universal bundles of $\mathrm{Gr}(k, V)$

We wish to study the two universal bundles over $\mathrm{Gr}(k, V)$ and its tangent bundle. We will later be interested in Chern classes of these bundles and hence studying their behaviour will be fruitful later on.

Construction 4.2.1 (Universal sub-bundle over $\mathrm{Gr}(k, V)$)**.** Let V be an n-dimensional $\mathbf{k}$-vector space. We construct a rank k-bundle over $\mathrm{Gr}(k, V)$ whose fiber at $\Lambda \in \mathrm{Gr}(k, V)$ is the subspace Λ itself. We will from now on have to view V as an affine n-space, so we make the following notation:

$$\hat{V} := \mathrm{mSpec}\,(\mathrm{Sym}(V^*)),$$

that is, the closed points of $\mathrm{Spec}\,(\mathrm{Sym}(V^*))$, which corresponds to vectors in V and hence $\hat{V}$ has a vector space structure which is isomorphic to V. Thus we will freely think of a vector in V as a point in $\hat{V}$.

Now define the following subset of the trivial bundle $\mathrm{Gr}(k, V) \times \hat{V}$:

$$\mathcal{V}_n^k = \{(\Lambda, v) \in \mathrm{Gr}(k, V) \times \hat{V} \mid v \in \Lambda\}.$$

We wish to show that $p : \mathcal{V}_n^k \to \mathrm{Gr}(k, V)$ mapping $(\Lambda, v) \mapsto \Lambda$ is a rank k-vector bundle over $\mathrm{Gr}(k, V)$. To this end, we first have to show that $\mathcal{V}_n^k$ is a scheme. Indeed, we show that $\mathcal{V}_n^k$ is a closed subscheme over $\mathrm{Gr}(k, V) \times \hat{V}$ by showing that it is obtained as vanishing of certain polynomials in the coordinates of $\mathrm{Gr}(k, V) \times \hat{V}$.

Fix a basis $\{e_1, \ldots, e_n\}$ of V. Pick a point $\Lambda \in \mathrm{Gr}(k, V)$ and $v \in \hat{V}$. Let A_Λ be the $k \times n$ matrix whose row space is Λ. We may write $v = (v_1, \ldots, v_n)$ in the basis of V. Thus, $v \in \Lambda$ if and only if the augmented $(k+1) \times n$ matrix

$$A_{\Lambda,v} = \begin{bmatrix} A_\Lambda \\ v \end{bmatrix}$$

whose last row is v is such that all $k+1$-minors of $A_{\Lambda,v}$ are 0. Let $I \in \mathrm{Inc}(k+1, n)$ and $m_I = \det A_{\Lambda,v}[I]$ be one such minor. Clearly, m_I is a bilinear combination of v_i and k-minors of A_Λ, i.e. a bilinear combination of coordinates of v and Plücker coordinates of Λ in $\mathbb{P}(\wedge^k V)$. Hence, we get the ideal sheaf on $\mathcal{V}_n^k$, giving us the required scheme structure on it.

To complete the proof that $\mathcal{V}_n^k$ is a vector bundle, we have to show that it is locally trivial. Indeed, we claim that for any linear subspace of dimension $n-k$, $\Gamma \subseteq V$, the open affine chart $U_\Gamma \subseteq \mathrm{Gr}(k, V)$ is a trivializing open neighborhood of $\mathcal{V}_n^k$. To this end, fix $\Omega \in U_\Gamma$ to be origin of U_Γ so that we have projection maps

$$\pi_\Omega : V = \Omega \oplus \Gamma \to \Omega$$
$$\pi_\Gamma : V = \Omega \oplus \Gamma \to \Gamma.$$

Recall that if $\Lambda \in \mathrm{Gr}(k, V)$, then $\pi_\Omega|_\Lambda : \Lambda \to \Omega$ is an isomorphism. We may thus consider the following map

$$\begin{aligned} \varphi : p^{-1}(U_\Gamma) &\longrightarrow U_\Gamma \times \hat{\Omega} \\ (\Lambda, v) &\longmapsto (\Lambda, \pi_\Omega(v)). \end{aligned}$$

As the above map is coordinatewise linear and invertible, therefore φ is an isomorphism over U_Γ. This shows that

$$p : \mathcal{V}_n^k \to \mathrm{Gr}(k, V)$$

is a rank k-vector bundle which we call the *universal sub-bundle of* $\mathrm{Gr}(k, V)$. If k and n are clear from context, we will simply write it as $\mathcal{V}$.

Construction 4.2.2 (Universal quotient bundle over $\mathrm{Gr}(k, V)$)**.** Let V be an n-dimensional $\mathbf{k}$-vector space. We have constructed a rank k-bundle over $\mathcal{V}_n^k$ over $\mathrm{Gr}(k, V)$. This is a sub-bundle of the trivial bundle $\mathcal{E} = \mathrm{Gr}(k, V) \times \hat{V}$. Hence, we may consider the quotient bundle $\mathcal{Q}_n^{n-k} = \mathcal{E}/\mathcal{V}_n^k$ which is of rank $n - k$. We call this the universal quotient bundle over $\mathrm{Gr}(k, V)$.

We next see why we call them universal.

Theorem 4.2.3. *Let X be a scheme, V be an n-dimensional $\mathbf{k}$-vector space and $k \geq 0$. There is a natural bijection*

$$\mathrm{Hom}_{\mathcal{S}ch}\left(X, \mathrm{Gr}(k, V)\right) \cong \{\textit{Rank } k \textit{ subbundles of trivial bundle } V \otimes \mathcal{O}_X \textit{ over } X\}$$

given by $f \mapsto f^*\mathcal{V}_n^k$ where $\mathcal{V}_n^k$ is the universal k-plane bundle over $\mathrm{Gr}(k, V)$.

4.3 Tangent bundle of $\mathrm{Gr}(k, V)$

In order to do geometry over Grassmannians, we need an understanding of its tangent bundle. To this end, we need to answer the following questions.

Q1. Let $\Lambda \in \mathrm{Gr}(k, V)$. What is $T_\Lambda \mathrm{Gr}(k, V)$ both algebraically and geometrically?

Q2. What is the tangent bundle of $\mathrm{Gr}(k, V)$ in terms of universal bundles?

Recall that the tangent bundle to a smooth variety is given by $TX = \mathbf{Spec}(\mathrm{Sym}\, \check{\Omega}_{X/\mathbf{k}})$ where $\Omega_{X/\mathbf{k}}$ is the cotangent sheaf of differentials over X.

Theorem 4.3.1. *Let V be an n-dimensional $\mathbf{k}$-vector space and $k \geq 0$. The tangent bundle of* $\mathrm{Gr}(k, V)$ *is given by the hom bundle of $\mathcal{V}$ and $\mathcal{Q}$ over it:*

$$T\,\mathrm{Gr}(k, V) \cong \mathcal{H}om\left(\mathcal{V}, \mathcal{Q}\right).$$

Proof. We denote $G = \mathrm{Gr}(k, V)$ and $p : TG \to G$ and $q : \mathcal{H}om(\mathcal{V}, \mathcal{Q}) \to G$ be the two given rank $k(n - k)$ bundles. Let $\Gamma \subseteq V$ be an $n - k$ plane of V and consider the open affine patch U_Γ of all k-planes linearly disjoint to Γ. For a fixed $\Omega \in U_\Gamma$, we have $U_\Gamma = \mathrm{Hom}\left(\Omega, \Gamma\right)$. Then, $TG|_{U_\Gamma} = U_\Gamma \times \mathrm{Hom}\left(\Omega, \Gamma\right)$ since TG is trivial over any affine chart of G. Our first claim is that fibers of $\mathcal{H}om(\mathcal{V}, \mathcal{Q})$ at $\Omega \in U_\Gamma$ is isomorphic to $(TG)_\Omega$. Indeed, as $(TG)_\Omega = \mathrm{Hom}\left(\Omega, \Gamma\right)$, therefore we need only show that $\mathcal{V}_\Omega = \Omega$ and $\mathcal{Q}_\Omega = \Gamma$. To this end, by construction $\mathcal{V}_\Omega = \Omega$ and $\mathcal{Q}_\Omega = V/\Omega = \Gamma$ since $V = \Omega \oplus \Gamma$. Consequently we have isomorphism

$$\varphi_\Omega : (TG)_\Omega \longrightarrow \mathcal{H}om(\mathcal{V}, \mathcal{Q})_\Omega$$

for each $\Omega \in G$. We claim that these define a bundle isomorphism. To this end, we need only show that transition maps $U_\Gamma \cap U_{\Gamma'} \to \mathrm{GL}_k(\mathbf{k})$ that both the bundle induces are isomorphic for any two affine open patches $U_\Gamma, U_{\Gamma'}$ of G. Indeed, as G is a variety so every bundle over G is a variety

(Lemma ??) so that we may work pointwise. We first observe the transition maps for TG. Recall that transitions for tangent bundle comes from differential of the variety. As the transition of the G from U_Γ to U'_Γ is (denote $U = U_\Gamma \cap U_{\Gamma'}$)

$$\psi : U_\Gamma = \mathrm{Hom}\,(\Omega, \Gamma) \longrightarrow U_{\Gamma'} = \mathrm{Hom}\,\left(\Omega, \Gamma'\right)$$

which is obtained by the composite linear isomorphisms $\Gamma \xrightarrow{\alpha} V/\Omega \xrightarrow{\beta^{-1}} \Gamma'$. Consequently, the transition map of TG is the differential of ψ:

$$d\psi : U \times \mathrm{Hom}\,(\Omega, \Gamma) \to U \times \mathrm{Hom}\,\left(\Omega, \Gamma'\right)$$

which is again linear. We next wish to show that $\mathcal{H}om(\mathcal{V}, \mathcal{Q})$ has the same transitions. Indeed, by Theorem ??, it is immediate that the transition of $\mathcal{H}om(\mathcal{V}, \mathcal{Q})$ on U is same as $d\psi$. □

Corollary 4.3.2. *Let $G = \mathrm{Gr}(k, V)$ for some n-dimensional $\mathbf{k}$-vector space V. For $\Lambda \in G$, we have*

$$T_\Lambda G = \mathrm{Hom}\,(\Lambda, V/\Lambda).$$

Proof. We may take $\Gamma = V/\Lambda$ and $\Omega = \Lambda$ in Theorem 4.3.1. □

4.4 Schubert cycles

In order to solve enumerative problems, it is essential for us to a) be able to represent the problems by some cycles in the Grassmannian, and b) to be able to calculate the Chow ring of Grassmannians. Schubert cycles will allow us to achieve a). We will achieve b) by showing that $\mathrm{Gr}(k, V)$ has an affine stratification using Schubert cells, so by Proposition 3.4.4, we will immediately know a generating set of the Chow ring of Grassmannians. We will then show, in-fact, that it forms a free basis.

We begin by setting up some definitions.

Definition 4.4.1 (**Schubert symbols & V-chains**)**.** Let V be an n-dimensional $\mathbf{k}$-vector space and $k \geq 0$. Denote $G = \mathrm{Gr}(k, V)$. Fix a complete flag V of V:

$$\mathsf{V} : 0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = V.$$

For any k-plane $\Lambda \in G$, define the V-chain of Λ to be the induced flag on Λ:

$$0 \subseteq V_1 \cap \Lambda \subseteq V_2 \cap \Lambda \subseteq \cdots \subseteq V_{n-1} \cap \Lambda \subseteq V_n \cap \Lambda = \Lambda.$$

Define a Schubert symbol on G to be $\vec{a} = (a_1, \ldots, a_k)$ where

$$n - k \geq a_1 \geq \cdots \geq a_k \geq 0.$$

Remark 4.4.2. To understand what a Schubert symbol signifies, we consider the following case. Let Λ be a general k-plane in V. Then, we have

1. $V_i \cap \Lambda = 0$ for $i \leq n - k$.
2. $\dim V_{n-k+i} \cap \Lambda = i$ for $n \geq n - k + i > n - k$.

Consequently, the V-chain of Λ will have dimensions

$$\dim V_j \cap \Lambda = \begin{cases} 0 & \text{if } 0 \leq j \leq n-k, \\ i & \text{if } n-k < j = n-k+i \leq n. \end{cases}$$

Let $g_i = n-k+i$. Note that the i^{th}-jump in dimension of V-chain of Λ cannot occur after g_i, that is, g_i is the maximum possible index on which i^{th}-jump can take place.

Now assume that Λ is any k-plane, not necessarily general. Then the dimension of its V-chain may not be same as above. In particular, the i^{th}-jump may happen earlier. Consequently, if we denote $\sigma_i \in \{1,\dots,n\}$ the index where the first jump actually happens, that is,

$$\dim V_{\sigma_i} \cap \Lambda = i \ \&\ \dim V_{\sigma_i - 1} \cap \Lambda = i-1,$$

then, we must have

$$\sigma_i \leq g_i \text{ for all } 1 \leq i \leq k.$$

A Schubert symbol $\vec{a} = (a_1,\dots,a_k)$ then specifies the exact way in which the given k-plane fails to be "dimensionally general" by telling an upper bound on each σ_i; we say that Λ has Schubert symbol $\vec{a}$ if $\sigma_i \leq g_i - a_i$, equivalently, that $\dim V_{g_i - a_i} \cap \Lambda \geq i$. Consequently a same plane may have many Schubert symbols.

With the above remark, we define a Schubert cycle as all k-planes with the given Schubert symbol.

Definition 4.4.3 (**Schubert cycle & class**)**.** Let V be an n-dimensional $\mathbf{k}$-vector space and $k \geq 0$. Denote $G = \mathrm{Gr}(k,V)$. Fix a complete flag V of V. Let $\vec{a} = (a_1,\dots,a_k)$ be a Schubert symbol. Denote $g_i = n-k+i$, for $1 \leq i \leq k$. The associated Schubert cycle of $\vec{a}$ is the subset

$$\Sigma_{\vec{a}}(\mathsf{V}) = \{\Lambda \in G \mid \dim V_{g_i - a_i} \cap \Lambda \geq i, \ \forall 1 \leq i \leq k\}.$$

If the flag V is clear from context, we will drop it to simply write $\Sigma_{\vec{a}}$.

For a Schubert cycle $\Sigma_{\vec{a}}$, we define its Schubert class in the Chow ring as

$$\sigma_{\vec{a}} := [\Sigma_{\vec{a}}(\mathsf{V})] \in A^*(G).$$

If $\vec{a} = (i,i,\dots,i)$ for some $0 \leq i \leq n-k$, then we write $\vec{a} = i^k$ and the corresponding cycle and class as Σ_{i^k} and σ_{i^k} respectively.

Remark 4.4.4. The subspace $\Sigma_{\vec{a}}(\mathsf{V})$ is closed because if $W \subseteq V$ is any linear subspace of dimension l, then

$$S = \{\Lambda \in G \mid \dim \Lambda \cap W \geq i\}$$

is a closed subspace. To see this, first observe that we have a short exact sequence

$$0 \to \Lambda \cap W \to \Lambda \oplus W \to \Lambda + W \to 0.$$

Hence, if $\dim \Lambda \cap W \geq i$, then $\dim \Lambda + W \leq k + l - i$. If the matrix of Λ, W are A_Λ, A_W respectively, then the matrix of $\Lambda + W$ is

$$A_{\Lambda+W} = \begin{bmatrix} A_\Lambda \\ A_W \end{bmatrix}.$$

Hence, we want $\mathrm{rank} A_{\Lambda \cap W} \leq k + l - i$. This is equivalent to demanding all $> k + l - i$-minors of $A_{\Lambda+W}$ to be zero, which are polynomials in Plücker coordinates of G, as required.

Remark 4.4.5. We will have to check why the Schubert class is independent of the flag chosen. Indeed, if V and V' are two complete flags of V, then by induction, we may constrtuct a $g \in \mathrm{GL}(V)$ such that $g(V_i) = V_i'$ for all $1 \leq i \leq n$. By Kleiman's transversality (Theorem 4.0.2), we have $g\Sigma_{\vec{a}}(\mathsf{V}) = \Sigma_{\vec{a}}(\mathsf{V}')$ and thus $[\Sigma_{\vec{a}}(\mathsf{V})] = [\Sigma_{\vec{a}}(\mathsf{V}')]$, as required.

Example 4.4.6. If $\vec{a} = (0, 0, \ldots, 0)$, then it is clear that $\Sigma_{\vec{a}}(\mathsf{V}) = \mathrm{Gr}(k, V)$.

Example 4.4.7. Schubert symbols allows us to represent enumerative problems into Schubert cycles. For example, let V_l be an l-plane in V. The set of all k-planes intersecting V_l non-trivially is given by the Schubert cycle associated to the symbol

$$\vec{a} = (n - k + 1 - l, 0, 0, \ldots, 0).$$

Indeed, we have the equivalences (notations of Remark 4.4.2)

$$V_l \cap \Lambda \neq 0 \iff \dim V_l \cap \Lambda \geq 1 \iff \sigma_1 \leq l \iff g_1 - a_1 = l.$$

Thus $a_1 = g_1 - l = n - k + 1 - l$, as required. If $l = n - k$, then

$$\Sigma_{(1,0,\ldots,0)} = \{\Lambda \in G \mid V_{n-k} \cap \Lambda \neq 0\}$$

Remark 4.4.8. For another example, consider the problem of finding all k-planes contained in a given l-plane V_l:

$$\{\Lambda \in G \mid \Lambda \subseteq V_l\}.$$

Observe the following equivalences:

$$\begin{aligned} \Lambda \subseteq V_l &\iff V_l \cap \Lambda = \Lambda \iff \dim V_l \cap \Lambda = k \iff \dim V_i \cap \Lambda = k \ \forall i \geq l \iff \sigma_k \leq l \\ &\iff \sigma_i \leq l \ \forall 1 \leq i \leq k. \end{aligned}$$

Hence $g_k - a_k = l$ so that $a_k = n - l$. Thus the symbol is $\vec{a} = (n - l, n - l, \ldots, n - l)$ so that

$$\Sigma_{\vec{a}} = \{\Lambda \in G \mid \Lambda \subseteq V_l\},$$

as required.

Next, we wish to order Schubert symbols.

Definition 4.4.9 (Ordering on Schubert symbols). For two Schubert symbols $\vec{a}, \vec{b}$, we write $\vec{a}\vec{b}$ if $a_i \leq b_i$ for all $1 \leq i \leq k$.

We then have the following simple observation.

Lemma 4.4.10. *Let $\vec{a}, \vec{b}$ be two Schubert symbols for $G = \mathrm{Gr}(k, V)$. Then the following are equivalent:*

1. $\vec{a} \geq \vec{b}$.
2. $\Sigma_{\vec{a}} \subseteq \Sigma_{\vec{b}}$.

Proof. The (1. $\Rightarrow$ 2.) is immediate. For (2. $\Rightarrow$ 1.), we proceed as follows. Pick any $\Lambda \in \Sigma_{\vec{a}}$. Thus, Λ satisfies

$$\dim V_{g_i - a_i} \cap \Lambda \geq i, \ \forall 1 \leq i \leq k.$$

By hypothesis, we then have $\Lambda \in \Sigma_{\vec{b}}$. Consider the k-plane spanned by the vectors $\{b_1, \ldots, b_k\}$ where we choose $b_i \in V_{g_i - a_i} \setminus V_{g_i - a_i - 1}$ to be a basis vector $V_{g_i - a_i}$, for each $1 \leq i \leq k$. Then, $V_{g_i - a_i} \cap \Lambda$ contains $\{b_1, \ldots, b_i\}$, hence has dimension $= i$. This shows that $\Lambda \in \Sigma_{\vec{a}}$. Moreover, for this Λ, we have $\sigma_i = g_i - a_i$. It then follows that $\Lambda \in \Sigma_{\vec{b}}$. Consequently, $= g_i - a_i = \sigma_i \leq g_i - b_i$ for each $1 \leq i \leq k$. It follows that $\vec{b} \leq \vec{a}$, as required. $\square$

Corollary 4.4.11. *If $\vec{a} \neq \vec{b}$ are two distinct Schubert symbols, then the corresponding Schubert cycles $\Sigma_{\vec{a}}, \Sigma_{\vec{b}}$ are different.* $\square$

Lemma 4.4.12. *Let V be an $n+1$-dimensional vector space with a basis $\{e_1, \ldots, e_{n+1}\}$ and a corresponding flag V. Then we have canonical open inclusions*

$$\begin{aligned} i : \mathrm{Gr}(k, V/V_1) &\hookrightarrow \mathrm{Gr}(k+1, V) \\ \Lambda &\longmapsto \Lambda + \langle e_{n+1} \rangle \\ j : \mathrm{Gr}(k, V_n) &\longrightarrow \mathrm{Gr}(k, V) \\ \Lambda &\longmapsto \Lambda. \end{aligned}$$

If $\vec{a}$ and $\vec{b}$ are Schubert symbols over $\mathrm{Gr}(k+1, V)$ and $\mathrm{Gr}(k, V)$, then

$$\begin{aligned} i^*(\sigma_{\vec{a}}) &= \sigma_{\vec{a}} \\ j^*(\sigma_{\vec{b}}) &= \sigma_{\vec{b}} \end{aligned}$$

where we consider $\sigma_{\vec{a}}, \sigma_{\vec{b}}$ as Schubert classes in $A^*(\mathrm{Gr}(k, V_n))$ by setting $\sigma_{\vec{a}} = 0$ if $a_{k+1} > 0$, $\sigma_{\vec{b}} = 0$ if $b_1 > n-k$, and if not, then as the Schubert class of the symbol in $\mathrm{Gr}(k, V)$ it denotes.

Proof. Suppose $\vec{a}$ and $\vec{b}$ by restriction are also Schubert symbols over $\mathrm{Gr}(k, V)$. Then the proof is immediate. Otherwise, say $a_{k+1} > 0$ so that $\vec{a}$ by restriction is not a Schubert symbol over $\mathrm{Gr}(k, V)$. Then,

$$\begin{aligned} i^{-1}(\Sigma_{\vec{a}}) &= \{\Lambda \in \mathrm{Gr}(k, V_n) \mid \Lambda + \langle e_{n+1} \rangle \in \Sigma_{\vec{a}}\} \\ &= \{\Lambda \in \mathrm{Gr}(k, V_n) \mid \dim V_{g_i - a_i} \cap (\Lambda + \langle e_{n+1} \rangle) \geq i \ \forall 1 \leq i \leq k+1\} \\ &\subseteq \{\Lambda \in \mathrm{Gr}(k, V_n) \mid \dim V_{g_{k+1} - a_{k+1}} \cap (\Lambda + \langle e_{n+1} \rangle) \geq k+1\} \\ &= \{\Lambda \in \mathrm{Gr}(k, V_n) \mid V_{g_{k+1} - a_{k+1}} \supseteq \Lambda + \langle e_{n+1} \rangle\}. \end{aligned}$$

But $V_{g_{k+1} - a_{k+1}} = V_{n-k+k+1-a_{k+1}} = V_{n+1-a_{k+1}}$. As $a_{k+1} > 0$, therefore $V_{n+1-a_{k+1}}$ has dimension $< n+1$, in particular, it doesn't contain e_{n+1}. Hence, the above subspace is empty, as required. A similar proof works for $\vec{b}$. $\square$

Out goal now is to show that the closed subspace $\Sigma_{\vec{a}}$ is actually a variety. Moreover, it will follow from this, that the collection of $\Sigma_{\vec{a}}$ forms a closed strata of $\mathrm{Gr}(k, V)$.

4.5 Schubert variety & affine stratification

Our goal in this section is to (finally) show that $\mathrm{Gr}(k,V)$ has an affine stratification. Moreover, in the process we will end up showing that the Schubert cycle $\Sigma_{\vec{a}}$ is a subvariety of $\mathrm{Gr}(k,V)$, consequently its tangent space at $\lambda \in \mathrm{Gr}(k,V)$ is a subspace of $T_\Lambda \mathrm{Gr}(k,V)$. We will characterize this subspace.

Notation 4.5.1. For this section, we fix an n-dimensional $\mathbf{k}$-vector space, V a complete flag of V, $k \geq 0$ and denote $G = \mathrm{Gr}(k,V)$.

Definition 4.5.2 (Schubert cell). Let $\vec{a}$ be a Schubert symbol over G. Denote (see Definition 4.4.9 and Lemma 4.4.10)

$$\Sigma^{\circ}_{\vec{a}} := \Sigma_{\vec{a}} \setminus \left(\bigcup_{\vec{b} > \vec{a}} \Sigma_{\vec{b}} \right).$$

We call $\Sigma^{\circ}_{\vec{a}}$ to be the Schubert cell associated to symbol $\vec{a}$. As $\Sigma^{\circ}_{\vec{a}}$ is an open subset of a closed set, therefore it is locally closed.

Our first goal is to show that $\Sigma^{\circ}_{\vec{a}}$ is isomorphic to an affine space, so that it forms open strata of an affine stratification on G.

Theorem 4.5.3. *Let $\vec{a}$ be a Schubert symbol over G. Then,*

$$\Sigma^{\circ}_{\vec{a}} \cong \mathbb{A}^{k(n-k)-|a|}$$

where $|a| = \sum_{i=0}^{k} a_i$.

4.6 Chern classes of universal bundle

We will see that total Chern classes of universal k-plane bundle over $\mathrm{Gr}(k,V)$ is an alternating sum of special Schubert cycles. Using this observation as a technical tool, we will give a complete description of generators and relations of the Chow ring $A^*(\mathrm{Gr}(k,V))$.

Proposition 4.6.1. *Let V be a vector space over $\mathbf{k}$ of dimension n and $k \geq 0$. Let $\mathcal{V}^k_n$ denote the universal k-plane bundle over $\mathrm{Gr}(k,V)$. Then,*

$$c(\mathcal{V}^k_n) = 1 - \sigma_1 + \sigma_{1,1} - \cdots + (-1)^k \sigma_{1^k}$$

in the ring $A^{\Pi}(\mathrm{Gr}(k,V))$.

Proof. □

4.7 Chow ring of $\mathrm{Gr}(k,V)$

We now prove the general form of Chow ring of finite Grassmannian.

Theorem 4.7.1. *Let V be a an n-dimensional $\mathbf{k}$-vector space and $k \geq 0$. Then there is an isomorphism of rings given by*

$$\varphi : \frac{\mathbf{Z}[c_1, \ldots, c_k]}{I} \longrightarrow A^*(G(k,V))$$
$$c_i \longmapsto c_i(\mathcal{V}_n^k)$$

where the ideal I is generated by k relations obtained by putting all the terms of degree $n-k+1, \ldots, n$ to be zero in the expansion

$$\frac{1}{1 + c_1 + \cdots + c_k} = 1 - (c_1 + \cdots + c_k) + (c_1 + \cdots + c_k)^2 - \ldots .$$

Moreover, $A^*(\mathrm{Gr}(k,V))$ is a complete intersection ring.

4.8 Enumerative problems

An enumerative problem is a question of the following type:

Q1. Count the number of subvarieties satisfying property P in a variety X.

For example, the problem of counting the number of lines on a smooth cubic surface S in $\mathbb{P}^3$ is an enumerative problem where $X = S$ and P is the property that subvarieties are lines on S. A more simpler enumerative problem, which we will focus on in these notes is the following:

Q2. Count the number of lines in $\mathbb{P}^3$ intersecting four general lines.

We begin by showcasing a general method to solve problems of the type given in Q1.

1. Find a suitable parameter space parameterizing the object to be counted, say $\mathcal{M}$. Study its geometry and show its complete over $\mathbf{k}$.
2. Compute its intersection ring, say $A^*(\mathcal{M})$.
3. Find the cycles that the property P induces in $\mathcal{M}$, say $\Sigma_1, \ldots, \Sigma_r$.
4. Compute the product of cycle classes in $A^*(\mathcal{M})$:
$$\sigma = [\Sigma_1] \cdot \cdots \cdot [\Sigma_r].$$
5. Establish generic transversality of cycles $\Sigma_1, \ldots, \Sigma_r$, so that $\sigma \in A_0(\mathcal{M})$.
6. Calculate $\deg \sigma$. This is the count.

Usually, it is the step 5 which is the most difficult. In our case, this will be verified by the Kleiman's result (Theorem 4.0.2).

Here's an example of an enumerative problem.

Theorem 4.8.1. *The number of lines in $\mathbb{P}^3$ that intersect four general lines is 2.*

Proof. Our parameter space for this problem is $\mathbb{G}\mathrm{r}(1,3)$. By Theorem 4.7.1, we may calculate its Chow ring as

$$A^*(\mathbb{G}\mathrm{r}(1,3)) \cong \frac{\mathbf{Z}[c_1, c_2]}{\langle 2c_1c_2 - c_1^3, c_2^2 - 3c_1^2c_2 + c_1^4 \rangle} = \frac{\mathbf{Z}[c_1, c_2]}{\langle 2c_1c_2 - c_1^3, c_1^2c_2 - c_2^2 \rangle}$$

where under the isomorphism, $-\sigma_{10} \leftarrow\!\shortmid c_1$ and $\sigma_{11} \leftarrow\!\shortmid c_2$ by Proposition 4.6.1. Next, we have to find the cycles that the problem induces in $\mathbb{G}\mathrm{r}(1,3)$. We may fix a complete flag of $\mathbb{G}\mathrm{r}(1,3)$ as

$$0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = \mathbf{k}^4$$

which projectively is

$$p \subset L \subset H \subset \mathbb{P}^3$$

with p a point, L a line and H a hyperplane in $\mathbb{P}^3$. A simple expansion of definition of Schubert cycles gives us the following:

$$\begin{aligned}
\Sigma_{00} &= \mathbb{G}\mathrm{r}(1,3) \\
\Sigma_{10} &= \{l \mid l \cap L \neq \emptyset\} \\
\Sigma_{20} &= \{l \mid p \in l\} \\
\Sigma_{11} &= \{l \mid l \subset H\} \\
\Sigma_{21} &= \{l \mid p \in l \subset H\} \\
\Sigma_{22} &= \{l \mid l = L\}.
\end{aligned}$$

Recall that $\mathrm{GL}_4(\mathbf{k})$ acts on $\mathbb{G}\mathrm{r}(1,3)$. Hence four general lines in $\mathbb{P}^3$ may be written as

$$g_1L, g_2L, g_3L, g_4L$$

for $g_i \in \mathrm{GL}_4(\mathbf{k})$ being four general linear isomorphisms. Note that we are interested in the cycles

$$g_1\Sigma_{10}, g_2\Sigma_{10}, g_3\Sigma_{10}, g_4\Sigma_{10}$$

and in particular, we want to count the set

$$\bigcap_{i=1}^{4} g_i\Sigma_{10}.$$

By Kleiman's transversality (Theorem 4.0.2), we have that each of $g_i\Sigma_{10}$ intersects generically transverse to each other. Consequently, by Corollary 2.4.4 and another use of Kleiman's transversality, it follows that in $A^*(\mathbb{G}\mathrm{r}(1,3))$ we have

$$\begin{aligned}
[g_1\Sigma_{10} \cap g_2\Sigma_{10} \cap g_3\Sigma_{10} \cap g_4\Sigma_{10}] &= [g_1\Sigma_{10}] \cdot [g_2\Sigma_{10}] \cdot [g_3\Sigma_{10}] \cdot [g_4\Sigma_{10}] \\
&= [\Sigma_{10}]^4 = \sigma_{10}^4.
\end{aligned}$$

Note that $\sigma_{10}^4 \in A^4(\mathbb{G}\mathrm{r}(1,3))$, i.e. the Chow group of dimension 0-cycles. As $\mathbb{G}\mathrm{r}(1,3)$ is complete since it is a closed subvariety of a projective space, therefore we have the degree map

$$\deg : A^4(\mathbb{G}\mathrm{r}(1,3)) \to \mathbf{Z}$$

which is well-defined by proper pushforward. As $A^4(\mathbb{G}\mathrm{r}(1,3))$ is generated by a single Schubert class, namely σ_{22}, therefore we have $\sigma_{10}^4 = c \cdot \sigma_{22}$ for $c \in \mathbf{Z}$. Hence

$$\deg \sigma_{10}^4 = c.$$

Hence we need only calculate c and that will be our count.

From the isomorphism in the beginning, we deduce

$$\sigma_{10}^4 = \sigma_{10}^3 \cdot \sigma_{10} = 2\sigma_{10}^2 \cdot \sigma_{11} = 2\sigma_{11}^2.$$

We finally claim that $\sigma_{11}^2 = \sigma_{22}$. Indeed, for a general $g \in \mathrm{GL}_4(\mathbf{k})$ we have by Kleiman's transversality and Corollary 2.4.4 that

$$\sigma_{11}^2 = [\Sigma_{11}] \cdot [\Sigma_{11}] = [\Sigma_{11}] \cdot [g\Sigma_{11}] = [\Sigma_{11} \cap g\Sigma_{11}].$$

Since

$$\Sigma_{11} \cap g\Sigma_{11} = \{l \in \mathbb{G}\mathrm{r}(1,3) \mid l \subseteq H \cap gH\}$$

and $H \cap gH$ intersects in a line in $\mathbb{P}^3$, therefore

$$\Sigma_{11} \cap g\Sigma_{11} = \{H \cap gH\}.$$

As

$$[\Sigma_{11} \cap g\Sigma_{11}] = [H \cap gH] = [L] = [\Sigma_{22}] = \sigma_{22},$$

hence $\sigma_{10}^4 = 2\sigma_{22}$ and thus $\deg \sigma_{10}^4 = 2$, i.e. there are two lines in $\mathbb{P}^3$ which intersects the four general lines. $\square$

A Results from scheme theory

We collect here general results from scheme theory which we need in the main text.

A.1 Irreducible components

In this short section, we describe the decomposition of a closed subscheme of a locally noetherian scheme into finitely many irreducible components. Let us begin by the following basic observation.

Remark A.1.1. Recall that a closed subset of an integral scheme is closed under specialization, whereas an open subset of an integral scheme is closed under generization.

Remark A.1.2 (Integral closed subschemes by points)**.** Let X be a scheme and $x \in X$ be a point and $Z = \overline{\{x\}}$ to be the closed irreducible subspace of X. Giving Z the reduced induced subscheme structure on Z, thus making $Z \hookrightarrow X$ an integral closed subscheme of X.

The following proposition shows that a minimal prime in an affine open subset gives an irreducible component of the whole scheme!

Proposition A.1.3. *Let X be a scheme, $U = \operatorname{Spec}(A) \subseteq X$ an open affine and $\mathfrak{p} \in U$. Denote $Y = \overline{\{\mathfrak{p}\}}$ to be the closed irreducible subspace of X. Then the following are equivalent:*

1. *$\mathfrak{p}$ is a minimal prime of A.*
2. *Y is an irreducible component of X.*

Proof. (L $\Rightarrow$ R) Suppose $Y \subsetneq Z$ is a closed irreducible set of X properly containing Y. Denote $\eta \in Z$ to be its unique generic point. As $U \cap Z$ is a non-empty open subset of Z, therefore it is dense in Z. Consequently, $\eta = \mathfrak{q} \in U$. If $Y \cap U \subsetneq Z \cap U$, then $Z \cap U = V(\mathfrak{q}) \subseteq U$ properly contains $Y \cap U = V(\mathfrak{p})$. We get $\sqrt{\mathfrak{q}} \subsetneq \sqrt{\mathfrak{p}}$, so that $\mathfrak{q} \subsetneq \mathfrak{p}$, contradicting the minimality of $\mathfrak{p}$.

(R $\Rightarrow$ L) Let Y be a maximal closed irreducible set. If $\mathfrak{q} \subsetneq \mathfrak{p}$, then $V(\mathfrak{q}) \supseteq V(\mathfrak{p})$ in U. Denote $Z = \overline{\{\mathfrak{q}\}}$. If $V(\mathfrak{p}) = V(\mathfrak{q})$, then $Z \cap U = Y \cap U$ and hence $Y = Z$. Hence we may assume $Y \cap U \subsetneq Z \cap U$. As $\mathfrak{p} \in Y \cap U \subseteq Z \cap U$, therefore $\overline{\{\mathfrak{p}\}} = Y \subseteq Z$. But since $Y \cap U \neq Z \cap U$, therefore $Y \subsetneq Z$, a contradiction to maximality of Y. $\square$

Let X be a scheme and $Z \subseteq X$ be a closed subset. We always consider Z as a closed subscheme of X, where the subscheme structure is given as follows.

Remark A.1.4 (Reduced induced subscheme structure)**.** Let X be a scheme and $Z \subseteq X$ be a closed subset. The reduced induced structure on Z is the ideal sheaf $\mathcal{I}_Z$ of $\mathcal{O}_X$ unique with the property that for any open affine $U = \operatorname{Spec}(A) \subseteq X$, the ideal sheaf $\mathcal{I}_Z$ on U gives $\mathcal{I}_Z(U) = \mathfrak{a}_U$ where $\mathfrak{a}_U$ is the intersection of all the primes of $\operatorname{Spec}(A)$ contained in $Z \cap U$. In particular, if $Z \cap U = V(I)$ as a set for some ideal $I \leq A$, then $\mathfrak{a}_U = \sqrt{I}$, the radical of I. Hence, $Z \cap U$ has the scheme structure of $\operatorname{Spec}(A/\mathfrak{a}_U)$. Note that since each $\operatorname{Spec}(A/\mathfrak{a}_U)$ is reduced, therefore $(Z, \mathcal{O}_X/\mathcal{I}_Z)$ is a reduced scheme.

The following is another way to state the property that reduced induced scheme structure is the smallest possible scheme structure on Z.

Lemma A.1.5. *Let X be a scheme and $Z \subseteq X$ a closed set. The reduced induced scheme structure on Z is given by the largest ideal $\mathcal{I}$ of $\mathcal{O}_X$ such that* $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) = Z$.

Proof. Let $\mathcal{I}_Z$ be the ideal corresponding to reduced induced structure on Z and $\mathcal{I}$ be any other ideal such that $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I}) = Z$. For any open affine $U = \mathrm{Spec}\,(A)$ of X, we have by definition that $\mathcal{I}_Z(U) = \sqrt{\mathcal{I}(U)}$. Thus on U, we have an inclusion $\mathcal{I}|_U \hookrightarrow \mathcal{I}_Z|_U$. By locality of injective map of sheaves, we have $\mathcal{I} \hookrightarrow \mathcal{I}_Z$, as required. $\square$

Lemma A.1.6. *Let $Z \subseteq X$ be a closed subset of a scheme X. Then the reduced induced structure on Z is the unique reduced scheme structure on Z.*

Proof. Let $\mathcal{I}$ be a reduced subscheme structure on Z. Then by definition of reduced induced structure, it is sufficient to show that for any open affine $U = \mathrm{Spec}\,(A)$ of X, the ideal $\mathcal{I}(U) \leq A$ is radical of itself. This is immediate, since $A/\mathcal{I}(U)$ is reduced, so $\sqrt{\mathcal{I}(U)}$ is indeed radical of itself. $\square$

Remark A.1.7. If X is a scheme and Z is an irreducible component, then the reduced induced structure on Z gives it a structure of a subvariety of X, which we again call the *irreducible component* of X.

The following observation is elementary but is important to study intersections.

Lemma A.1.8. *Let X be a scheme and $Y, Z \subseteq X$ be two closed subschemes. If $\mathcal{I}_Y$ and $\mathcal{I}_Z$ are ideal sheaves of Y and Z respectively, then the ideal sheaf of the intersection $Y \cap Z$ is $\mathcal{I}_A + \mathcal{I}_B$.*

Proof. Note that $Y \cap Z$ is a closed subscheme of X. Pick any affine open $U = \mathrm{Spec}\,(A)$ of X. Then $U \cap Y = \mathrm{Spec}\,(A/I_Y) = V(I_Y)$ and $U \cap Z = \mathrm{Spec}\,(A/I_Z) = V(I_Z)$ where $I_Y = \mathcal{I}_Y(U)$ and $I_Z = \mathcal{I}_Z(U)$. Thus, $U \cap Y \cap Z = V(I_Y) \cap V(I_Z) = V(I_Y + I_Z)$. Since we have $(\mathcal{I}_Y + \mathcal{I}_Z)(U) = I_Y + I_Z$, thus by uniqueness of closed subschemes and ideal sheaves, we conclude that $\mathcal{I}_Y + \mathcal{I}_Z$ is the ideal sheaf of $Y \cap Z$. $\square$

Theorem A.1.9 (Generalized principal ideal theorem)**.** *Let $f : X \to Y$ be a map of varieties where Y is smooth. If $B \subseteq Y$ is a subvariety, then every irreducible component C of $f^{-1}(B)$ satisfies*

$$\mathrm{codim}\,_X C \leq \mathrm{codim}\,_Y B.$$

An important corollary of the above principal ideal theorem is the following.

Corollary A.1.10. *Let X be a smooth variety, $A, B \subseteq X$ subvarieties and $C \subseteq A \cap B$ be an irreducible component of $A \cap B$. Then*

$$\mathrm{codim}\, C \leq \mathrm{codim}\, A + \mathrm{codim}\, B.$$

Proof. Take f to be the inclusion $i : B \hookrightarrow X$ in Theorem A.1.9. $\square$

A.2 Subvarieties of a scheme

We first wish to see that the support of a coherent $\mathcal{O}_X$-module is closed.

Lemma A.2.1. *Let X be a scheme and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Y = \mathrm{Supp}\,(\mathcal{F})$ is a closed subset of X.*

Proof. As being closed is a local property, therefore for any open affine $U = \operatorname{Spec}(A)$, we wish to show that $Y \cap U$ is closed in U. As $Y \cap U = \{x \in U \mid \mathcal{F}_x \neq 0\}$, therefore $Y \cap U = \operatorname{Supp}(\mathcal{F}|_U)$. It follows from coherence of $\mathcal{F}$ that $\mathcal{F}|_U \cong \widetilde{M}$ where M is a finitely generated A-module. As $\operatorname{Supp}\left(\widetilde{M}\right) = \operatorname{Supp}(M)$ and M is finitely generated, therefore $\operatorname{Supp}(M) = V(\mathfrak{a}_M)$ where $\mathfrak{a}_M$ is annihilator of M, as required. □

Recall that any coherent module of $\mathcal{O}_X$ admitting an injection to $\mathcal{O}_X$ is an ideal sheaf.

Lemma A.2.2. *Let X be a scheme and $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module with an injective map $f : \mathcal{M} \to \mathcal{O}_X$. Then, $\mathcal{M}$ is an ideal sheaf of $\mathcal{O}_X$.*

Proof. It suffices to show that for any open affine $U = \operatorname{Spec}(A)$ of X, the $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ is an ideal of $\mathcal{O}_X(U)$. Indeed, $f_U : \mathcal{M}(U) \hookrightarrow \mathcal{O}_X(U)$ is injective by definition of an injective map of sheaves. Consequently, $\mathcal{M}(U)$ is an ideal of $\mathcal{O}_X(U)$, hence $\mathcal{M}$ is an ideal sheaf of $\mathcal{O}_X$. □

We next show the following result, which shows the uniqueness of a variety structure on a closed irreducible set.

Lemma A.2.3 (Extension of varieties)**.** *Let X be a scheme and $U \subseteq X$ be an open subscheme. If $V \subseteq U$ is a k-dimensional subvariety of U, then the closure $\overline{V}$ in X is a k-dimensional subvariety of X with $K(V) = K(\bar{V})$.*

Proof. Let $\mathcal{I} \leq \mathcal{O}_U$ be the ideal sheaf of V in U and $j : U \hookrightarrow X$ be the inclusion. Then, $j_*\mathcal{I} \leq X$ is an ideal sheaf of X. We claim that $\operatorname{Supp}(\mathcal{O}_X/j_*\mathcal{I}) = \overline{V}$. Indeed, for $x \in X$, we have that $(\mathcal{O}_X/j_*\mathcal{I})_x \neq 0$ if and only if $(j_*\mathcal{I})_x \subsetneq \mathcal{O}_{X,x}$ is a proper ideal. Hence, clearly $\operatorname{Supp}(\mathcal{O}_X/j_*\mathcal{I}) \supseteq V$ as for any $x \in V$, $(j_*\mathcal{I})_x = \mathcal{I}_x$. As $\mathcal{O}_X/j_*\mathcal{I}$ is coherent and support of coherent sheaves are closed (Lemma A.2.1), therefore we further have $\operatorname{Supp}(\mathcal{O}_X/j_*\mathcal{I}) \supseteq \overline{V}$. If $x \in X - \overline{V}$, then there is an open affine $x \in U \subseteq X - \overline{V}$. Consequently, $(j_*\mathcal{I})_x = 0$ as $j_*\mathcal{I}|_U = 0$. This shows that $\operatorname{Supp}(\mathcal{O}_X/j_*\mathcal{I}) = \overline{V}$. As V is irreducible, hence so is $\overline{V}$ and of same dimension as V. Thus, $(\overline{V}, \mathcal{O}_X/j_*\mathcal{I})$ is a subvariety of X of dimension k.

To see the last assertion, observe that the generic point of V is in U, sa y η. Thus for $\mathcal{O}_{\bar{V}} = \mathcal{O}_X/j_*\mathcal{I}$, localizing at the generic point gives $K(\bar{V}) = \mathcal{O}_{X,\eta}/\mathcal{I}_\eta = \mathcal{O}_{V,V} = K(V)$. □

Lemma A.2.4. *Let L^k be a k-dimensional linear subspace of $\mathbb{P}^n$. Then $L^k \cong \mathbb{P}^k$.*

Proof. Note that L^k is given by $\operatorname{Proj}(\mathbf{k}[x_0, \dots, x_n]/I)$ where $I = \langle f_1, \dots, f_{n-k}\rangle$ where each f_i is a linear homogeneous polynomial in $K[x_0, \dots, x_n]$. In particular, $L^k = V(I)$. We may write

$$\begin{bmatrix} f_1 \\ \vdots \\ f_{n-k} \end{bmatrix} = A \cdot \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} = A \cdot \vec{x}$$

where $f_i = \sum_{j=0}^n a_{ij} x_i$. Thus A is of size $n - k \times n + 1$ and it is of rank $n - k$. It is clear that the ideal generated by $EA \cdot \vec{x}$ is same as I where E is an elementary row matrix. We may thus reduce A to its row reduced echelon form, which will be of form

$$EA = \begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix},$$

where I is of size $n-k$. Denoting $\vec{g} = EA \cdot \vec{x}$, we get that

$$g_i = x_i - h_i$$

for $0 \leq i \leq n-k-1$ and h_i homogeneous polynomial in coordinates $x_{n-k}, \ldots, x_n$. As we have $\langle f_1, \ldots, f_{n-k} \rangle = \langle g_0, \ldots, g_{n-k-1} \rangle$, hence by above, we have that the map

$$\begin{aligned} \psi : \mathbf{k}[x_0, \ldots, x_n] &\longrightarrow \mathbf{k}[x_0, \ldots, x_n] \\ x_i &\longmapsto \begin{cases} x_i - h_i(x_{n-k}, \ldots, x_n) & \text{if } 0 \leq i \leq n-k-1 \\ x_i & \text{if } n-k \leq i \leq n. \end{cases} \end{aligned}$$

has kernel I and image isomorphic to $\mathbf{k}[x_{n-k}, \ldots, x_n]$. Thus, we have produced an isomorphism $\mathbf{k}[x_0, \ldots, x_n]/I \cong \mathbf{k}[x_{n-k}, \ldots, x_n]$, as required. □

The following correspondence is useful for geometric intuition.

Proposition A.2.5. *Let X be a variety. Then there is a natural isomorphism between elements of function field of X and dominant rational maps $X \dashrightarrow \mathbb{P}^1$.*

$$K(X) \cong \mathrm{Hom}_{\mathcal{V}ar^{\mathrm{DR}}_{\mathbf{k}}}\left(X, \mathbb{P}^1\right).$$

Proof. By equivalence between dominant rational maps and finitely generated field extensions, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{V}ar^{\mathrm{DR}}_{\mathbf{k}}}\left(X, \mathbb{P}^1\right) \cong \mathrm{Hom}_{\mathcal{F}ld^{\mathrm{fg}}_{\mathbf{k}}}\left(K(\mathbb{P}^1), K(X)\right).$$

As $K(\mathbb{P}^1) = \mathbf{k}(T)$ where T is the coordinate of an open affine patch $\mathrm{Spec}\,(\mathbf{k}[T])$ and since any k-linear field homomorphism $\alpha : \mathbf{k}(T) \to K(X)$ equivalently is determined by any non-constant function in $K(X)$, hence we have the natural isomorphism

$$\mathrm{Hom}_{\mathcal{F}ld^{\mathrm{fg}}_{\mathbf{k}}}\left(\mathbf{k}(T), K(X)\right) \cong K(X).$$

This completes the proof. □

A.3 Scheme theoretic image of varieties

Lemma A.3.1. *Let $f : X \to Y$ be a map of schemes and $W \hookrightarrow X$ be a subvariety of X. Then $\overline{f(W)}$ is a subvariety of Y. This is called the scheme theoretic image of map f[6].*

Proof. By putting reduced induced structure, we have that $\overline{f(W)}$ is a closed reduced subscheme. We need only show that $\overline{f(W)}$ is irreducible. Indeed, consider the generic point $\eta \in W$ of W. We claim that $\overline{f(\eta)} = \overline{f(W)}$. By continuity, we have $f(W) = f(\overline{\eta}) \subseteq \overline{f(\eta)} \subseteq \overline{f(W)}$. Taking closures again in the above inclusion, we get

$$\overline{f(W)} \subseteq \overline{f(\eta)} \subseteq \overline{f(W)},$$

which shows that $\overline{f(\eta)} = \overline{f(W)}$, as required. □

[6] see Ex.II.3.11 of [Har77].

One wonders the relationship of function field and dimension of the scheme theoretic image with the domain. For our purposes, the following relation is sufficient.

Proposition A.3.2. *Let $f : X \to Y$ be a map of varieties where $\dim X = \dim Y$. Then the induced map on function fields $f^\flat : K(Y) \to K(X)$ is a finite extension.*

Proof. Note that as X and Y are finite type $\mathbf{k}$-schemes, therefore $K(X)$ and $K(Y)$ are fraction fields of finite type k-algebras so they are finitely generated field extensions of k. As $\dim X = \operatorname{trdeg} K(X)/\mathbf{k} = \operatorname{trdeg} K(Y)/\mathbf{k} = \dim Y$, therefore by additive tower law of transcendence degree, we deduce that

$$\operatorname{trdeg} K(X)/K(Y) = 0.$$

It follows that $K(X)/K(Y)$ is an algebraic extension. As $K(X)$ and $K(Y)$ are finitely generated extensions of $\mathbf{k}$, therefore by tower law, $K(X)/K(Y)$ is a finitely generated extension. By algebraicity of $K(X)/K(Y)$, we deduce that $K(X)/K(Y)$ is finite, as required. □

Remark A.3.3 (Dimension of scheme theoretic image)**.** Let $f : X \to Y$ be a dominant morphism of varieties. Consequently, there is an induced map on function fields as generic point maps to generic point by dominance. Let $f^\flat : K(Y) \to K(X)$ be this map. As $f^\flat$ is an injection, we thus have the inequality

$$\operatorname{trdeg} K(Y)/\mathbf{k} \le \operatorname{trdeg} K(X)/\mathbf{k}.$$

We deduce that

$$\dim Y \le \dim X.$$

Remark A.3.4 (Inverse image)**.** Consider a map of schemes $f : X \to Y$ and $Z \subseteq Y$ be a closed subscheme of Y. Then the inverse image $f^{-1}(Z)$ is defined to be the fiber product:

$$\begin{tikzcd} f^{-1}(Z) \arrow[r] \arrow[d, hook, "c.i."'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, hook, "c.i."] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $f^{-1}(Z) \hookrightarrow X$ is a closed subscheme as closed immersions are stable under base change.

The following is an important decomposition of a proper surjective map of varieties.

Theorem A.3.5 (Stein factorization)**.** *If $f : X \to Y$ is a proper surjective map of varieties, then f factors via maps $f' : X \to Y'$ which has connected fibers and $g : Y' \to Y$ which is finite. Moreover, if $\dim X = \dim Y$, then there exists a non-empty open $U \subseteq X$ such that $f'|_U$ is an isomorphism.*

A.4 Length

As intersection multiplicity is defined as the length of a certain module, hence we give here some basic properties of length for reference.

Definition A.4.1 (**Length of a module**)**.** Let R be a ring and M be an R-module. Then the length of M is given by the length of the longest ascending chain of submodules of M:

$$\operatorname{len}_R(M) := \sup\{r \in \mathbf{N} \mid M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r \text{ is a chain of submodules of } M\}.$$

A finite chain $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r$ is called a maximal length chain if it cannot be extended, that is, each factor M_i/M_{i-1} is a simple module. A maximal length chain is also called a composition series. Consequently, length of a module M is defined to be the length of the longest composition series.

An important result about length of modules is the fact that over a local ring R, any two composition series have the same length and composition factors.

Theorem A.4.2 (Jordan-Hölder)**.** *Let R be a local ring and M be an R-module which contains a composition series. Then any other composition series has the same length and composition factors. That is, length of M is equal to length of any composition series.*

The following are essential properties of length which one uses while dealing with maps.

Lemma A.4.3. *Let $f : R \to S$ be a map of rings and M be an S-module. Then $\operatorname{len}_R(M) \geq \operatorname{len}_S(M)$ and equality holds if f is surjective.*

Proof. Follows from correspondence of submodules via a quotient map. □

The following is an easy exercise.

Lemma A.4.4 (Additivity of length)**.** *If M_i are finite length R-modules and the following is exact:*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to 0,$$

then

$$\sum_{i=0}^{n} (-1)^i \operatorname{len}_R(M_i) = 0.$$

We wish to characterize finite length modules over a noetherian ring. We begin with a lemma.

Lemma A.4.5. *Any finite length R-module is finitely generated.*

Proof. If M is not finitely generated, then let $\{f_\alpha\}_{\alpha \in I}$ be a generating set of M and let $\{f_n\}_n$ be a subsequence. Then, the chain

$$0 \subsetneq \langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \dots$$

is a chain of submodules of M which doesn't stabilizes, a contradiction to finite length. □

Using results on artinian rings, we see an important characterization of artinian rings and finite length rings.

Theorem A.4.6. *Let R be a ring. The following are equivalent:*

1. *R is artinian.*

2. *R has finite length.*

Proof. (1. $\Rightarrow$ 2.) By structure theorem of artinian rings, we reduce to assuming R is local artinian, $(R, \mathfrak{m})$. Recall that for an artinian ring, the Jacobson radical of R is nilpotent, which is just $\mathfrak{m}$. We construct a chain of ideals of R, where each subquotient has finite length. Indeed, consider the chain

$$0 = \mathfrak{m}^n \subsetneq \mathfrak{m}^{n-1} \subsetneq \cdots \subsetneq \mathfrak{m}^2 \subsetneq \mathfrak{m} \subsetneq R.$$

Note that $\mathfrak{m}^{i-1}/\mathfrak{m}^i$ is an $\kappa = R/\mathfrak{m}$-module. If any one of $\mathfrak{m}^i/\mathfrak{m}^{i-1}$ is infinite dimensional as an κ-vector space, then the above chain of ideals can be refined to an infinite chain of strictly decreasing ideals, a contradiction to artinian condition. Hence each subquotient is a finite dimensional κ-module and hence its length as an R-module is equal to its dimension as a κ-module (Lemma A.4.3).

(2. $\Rightarrow$ 1.) Take any descending chain of ideals $I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \dots$. If it doesn't stabilize, then we have an infinite length chain, so that $\operatorname{len}(R)$ is not finite, a contradiction. $\square$

The following is an essential result which we'll use later.

Proposition A.4.7. *Let R be a noetherian ring and M be a finitely generated R-module. If $\mathfrak{p} \in \operatorname{Supp}(M)$ is a minimal prime of M, then $M_\mathfrak{p}$ is a finite length $R_\mathfrak{p}$-module.*

Proof. As $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$, therefore a minimal prime $\mathfrak{p} \in \operatorname{Supp}(M)$ is an isolated/minimal prime of $\operatorname{Ann}(M)$. As $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module, therefore it suffices to construct a composition series of $M_\mathfrak{p}$. Let M be generated by $f_1, \dots, f_n \in M$, so that $M_\mathfrak{p}$ is also generated by their respective images. We thus get the following chain:

$$0 \subseteq \langle f_1 \rangle \subseteq \langle f_1, f_2 \rangle \subseteq \cdots \subseteq \langle f_1, \dots, f_n \rangle = M_\mathfrak{p}.$$

It suffices to show that $\frac{\langle f_1, \dots, f_i \rangle}{\langle f_1, \dots, f_{i-1} \rangle}$ is a finite length $R_\mathfrak{p}$-module. Indeed, we have a surjection

$$\langle f_i \rangle \twoheadrightarrow \frac{\langle f_1, \dots, f_i \rangle}{\langle f_1, \dots, f_{i-1} \rangle},$$

hence it suffices to show that $\langle f_i \rangle$ is a finite length $R_\mathfrak{p}$-module. To this end, pick any $x \in M$. We'll show that $\langle x \rangle = xR_\mathfrak{p}$ is a finite length $R_\mathfrak{p}$-module. Observe that $\langle x \rangle = xR_\mathfrak{p}$ is isomorphic to $R_\mathfrak{p}/I$ where I is the annihilator of x in $R_\mathfrak{p}$. We may write $I = \mathfrak{a}R_\mathfrak{p}$ where $\mathfrak{a} \leq R$ is contained in $\mathfrak{p}$. Hence, we wish to show that $S = R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}$ is a finite length $R_\mathfrak{p}$-module, that is S is a finite length ring. Indeed, as $S = (R/\mathfrak{a})_\mathfrak{p}$ and $\mathfrak{p}$ is a minimal prime in $\operatorname{Supp}(M)$, that is, minimal prime containing $\operatorname{Ann}(M)$, and since $\operatorname{Ann}(M) \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, therefore $\mathfrak{p}$ is a minimal prime of $\mathfrak{a}$ as well. It follows that $S = (R/\mathfrak{a})_\mathfrak{p}$ is a dimension 0 ring. Since R is noetherian and noetherian property is inherited by quotients and localizations, therefore S is a noetherian ring of dimension 0, hence artinian. From Theorem A.4.6, it follows that S is of finite length, as required. $\square$

Theorem A.4.8. *Let R be a noetherian ring and M be an R-module. Then the following are equivalent:*

1. *M has finite length.*

2. *M is finitely generated and* $\dim R/\mathrm{Ann}(M) = 0$*, i.e.* $R/\mathrm{Ann}(M)$ *is an artinian ring.*

Proof. (1. $\Rightarrow$ 2.) By Lemma A.4.5, M is finitely generated. Let $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M$ be a composition series of M, which exists as $\mathrm{len}(M) < \infty$. We thus get that $M_i/M_{i-1} \cong R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$, as these subquotients are simple. Note that $\dim R/\mathrm{Ann}(M) = 0$ if and only if $\mathrm{Supp}\,(M)$ consists only of maximal ideals. So let $\mathfrak{p} \in \mathrm{Supp}\,(M)$. Thus $M_\mathfrak{p} \neq 0$. It follows that for some i, $(M_i/M_{i-1})_\mathfrak{p} \neq 0$. As $(M_i/M_{i-1})_\mathfrak{p} = (R/\mathfrak{m}_i)_\mathfrak{p}$, therefore this can only happen if $\mathfrak{m}_i \subseteq \mathfrak{p}$, i.e. $\mathfrak{m}_i = \mathfrak{p}$, as required. This also shows that $\mathrm{Supp}\,(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$.

(2. $\Rightarrow$ 1.) We need only construct a composition series of M. We have $\mathrm{Supp}\,(M)$ consists only of maximal ideals. Consider $\mathrm{Supp}\,(M) \subseteq \mathrm{Spec}\,(R)$. As M is finitely generated, say by $f_1, \ldots, f_n$. Then we get a chain of submodules

$$0 \subsetneq \langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \cdots \subsetneq \langle f_1, \ldots, f_n \rangle = M.$$

We need only show that each subquotient is a finite length R-module. Indeed, as we have a surjection

$$\langle f_i \rangle \twoheadrightarrow \frac{\langle f_1, \ldots, f_i \rangle}{\langle f_1, \ldots, f_{i-1} \rangle},$$

so it suffices to show that $\langle f_i \rangle$ is a finite length R-module. To this end, it suffices to show that for each $x \in M$, the submodule Rx is of finite length. Indeed, we have $Rx \cong R/I$ where $I = \mathrm{Ann}(x)$. As $I \supseteq \mathrm{Ann}(M)$, therefore

$$R/I \cong \frac{R/\mathrm{Ann}(M)}{I/\mathrm{Ann}(M)}.$$

As $R/\mathrm{Ann}(M)$ is an artinian ring and any quotient of artinian ring is an artinian ring, it follows at once that R/I is an artinian ring. By Theorem A.4.6, $R/I \cong Rx$ is of finite length, as required. ☐

From the above proof, we can deduce the following.

Corollary A.4.9. *Let R be a noetherian ring and M be a finitely generated R-module. Then the following are equivalent:*

1. *M is of finite length.*
2. *There exists a chain*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

where $M_i/M_{i-1} \cong R/\mathfrak{m}_i$ *and* $\mathrm{Supp}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$.

3. *Support of M consists of finitely many maximal ideals.*

Proof. (1. $\Leftrightarrow$ 2.) If M is a finite length module, then the maximal chain has each subquotient a simple module. Let $R/\mathfrak{m}_i$ be the subquotients. We wish to show that $\mathrm{Supp}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. This is what we showed in the proof of Theorem A.4.8. Conversely, suppose M has the a chain as above with each subquotient a field. Then this is a maximal chain, as required. Note that (2. $\Rightarrow$ 3.) is immediate.

(3. $\Rightarrow$ 1.) Let $\mathrm{Supp}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. As showed in the forward part of proof of Theorem A.4.8, $R/\mathrm{Ann}(M)$ is of dimension 0 if and only if $\mathrm{Supp}(M)$ consists of finitely many maximals. The result follows from the other part of the theorem. ☐

The following lemma shows how one should generalize the valuation of a rational function at a codimension 1 subvariety.

Lemma A.4.10. *Let R be a DVR with valuation $v : K \to \mathbf{Z}$ and fix $a \in R$. Then,*

$$v(a) = \operatorname{len}_R R/aR.$$

Proof. Let $t \in R$ be the local parameter of R. Then $a = ut^n$ where $u \in R^\times$ is a unit and $n \geq 0$. As $v(a) = n$, therefore we must show $\operatorname{len}_R R/aR = n$. Indeed, as R is a DVR, therefore every ideal of R is a principally generated by a power of t. Hence, $R/aR = R/t^nR$. We first show that $M = R/t^nR$ is a finite length R-module. Indeed, as $\operatorname{Ann}(R) = t^nR$, therefore by Theorem A.4.8 we must show R/t^nR is a dimension 0 ring. Indeed, as R has only two primes, therefore R/t^nR has only one prime. It follows that $\dim R/t^nR = 0$, as required.

The only chain of ideals in R/t^nR is the image of the following chain in R:

$$t^nR \subsetneq t^{n-1}R \subsetneq \cdots \subsetneq t^2R \subsetneq tR = \mathfrak{m} \subsetneq R.$$

Thus, $\operatorname{len}_R R/t^nR = n$, as required. □

The following is an essential property of lengths.

Lemma A.4.11. *Let R be a ring such that for $f, g \in R$, the R-modules $R/fR, R/gR$ and R/fgR has finite length. Then,*

$$\operatorname{len}_R R/fgR = \operatorname{len}_R R/fR + \operatorname{len}_R R/gR.$$

Proof. Follows from simple comparions of chains of ideals. □

Here's a more general result.

Lemma A.4.12. *Let M be a finite length R-module. Then,*

$$\operatorname{len}_R(M) = \sum_{\mathfrak{p} \in \operatorname{Spec}(R)} \operatorname{len}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \sum_{\mathfrak{m} \in \operatorname{Supp}(M)} \operatorname{len}_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. We first have to show that the sum is finite. Indeed, observe that as M has finite length, therefore M is finitely generated and $R/\operatorname{Supp}(M)$ is an artinian ring by Theorem A.4.8. Consequently, $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$ has only finitely many maximal ideals of R. This shows that the sum is finite.

To show the equality, it suffices to show that each maximal $\mathfrak{m} \in \operatorname{Supp}(M)$ occurs $\operatorname{len}_{R_\mathfrak{m}}(M_\mathfrak{m})$ many times as a composition factor in any composition series of M. Take a composition series $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of M with $M_i/M_{i-1} \cong R/\mathfrak{m}_\mathfrak{i}$ where $\mathfrak{m}_i \in \operatorname{Supp}(M)$ and fix $\mathfrak{m} \in \operatorname{Supp}(M)$. Localizing at $\mathfrak{m}$, we get the chain $(M_0)_\mathfrak{m} \subseteq (M_1)_\mathfrak{m} \subseteq \cdots \subseteq (M_r)_\mathfrak{m}$ where the subquotient $(M_i/M_{i-1})_\mathfrak{m} \cong (R/\mathfrak{m}_i)_\mathfrak{m} \cong R/\mathfrak{m}_i \cong R_\mathfrak{m}/\mathfrak{m}_i R_\mathfrak{m}$. Hence, $\mathfrak{m}_i R_\mathfrak{m}$ is the maximal ideal of $R_\mathfrak{m}$, showing that $\mathfrak{m}_i = \mathfrak{m}$. Hence $\mathfrak{m}$ appears $\operatorname{len}_{R_\mathfrak{m}}(M_\mathfrak{m})$-many times as a composition factor of the chain of M, as required. □

We deal exclusively with length of modules over local rings. The following therefore shows the effect on length of a module under a map of local rings.

Proposition A.4.13. *Let $\varphi : A \to B$ be a local homomorphism of local rings. We thus have a field extension of residue fields $\kappa(B)/\kappa(A)$. Then the following are equivalent for a B-module M:*

1. *As an A-module via φ, we have $\operatorname{len}_A(M) < \infty$.*
2. *We have $\operatorname{len}_B(M) < \infty$ and $[\kappa(B) : \kappa(A)] < \infty$.*

Moreover, if any of the above is satisfied, then

$$\operatorname{len}_A(M) = [\kappa(B) : \kappa(A)] \cdot \operatorname{len}_B(M).$$

Proof. Let $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$ be a composition series of M as a B-module. Hence $\operatorname{len}_B(M) = n$. As B is local, therefore each subquotient is isomorphic to $\kappa(B)$. Now for any $1 \leq i \leq n$, we have the following exact sequence of A-modules:

$$0 \to M_{i-1} \to M_i \to \kappa(B) \to 0.$$

By Lemma A.4.4, we have the following equalities:

$$\begin{aligned} \operatorname{len}_A(M) &= \operatorname{len}_A(M_{n-1}) + \operatorname{len}_A(\kappa(B)) \\ &\vdots \\ &= \operatorname{len}_A(M_0) + n \cdot \operatorname{len}_A(\kappa(B)). \end{aligned}$$

Consequently, $\operatorname{len}_A(M) = \operatorname{len}_B(M) \cdot \operatorname{len}_A(\kappa(B))$. We need only show that $\operatorname{len}_A(\kappa(B)) = [\kappa(B) : \kappa(A)]$. As $\operatorname{len}_A(\kappa(B)) = \operatorname{len}_{A/\mathfrak{m}_A} \kappa(B)$ since $\mathfrak{m}_A \cdot \kappa(B) = 0$, therefore $\operatorname{len}_A(\kappa(B)) = \operatorname{len}_{\kappa(A)} \kappa(B) = [\kappa(B) : \kappa(A)]$, as required. □

Corollary A.4.14. *Let $\varphi : A \to B$ be a local homomorphism of local rings. We thus have a field extension of residue fields $\kappa(B)/\kappa(A)$. Let $f \in A$ be a non zero-divisor. Let M be a B-module be a finite length modules and $\kappa(B)/\kappa(A)$ is a finite extension. Then*

$$\operatorname{len}_A\left(\frac{M}{fM}\right) = [\kappa(B) : \kappa(A)] \cdot \operatorname{len}_B\left(\frac{M}{fM}\right).$$

Proof. Immediate from Proposition A.4.13. □

Corollary A.4.15. *Let A be a ring, M be a finite length A-module and $f \in A$ be a non zero-divisor. Then,*

$$\operatorname{len}_A\left(\frac{M}{fM}\right) = \sum_{\mathfrak{p} \in \operatorname{Supp}(M/fM)} \operatorname{len}_{A_\mathfrak{p}}\left(\frac{M_\mathfrak{p}}{fM_\mathfrak{p}}\right).$$

Proof. Immediate from Lemma A.4.12 □

Lemma A.4.16. *Let $(A, \mathfrak{m})$ be a local ring, $\psi : A \to B$ be a finite A-algebra and M be a finite length B-module. If $\mathfrak{m}_i \in \operatorname{Spec}(B)$ are the finitely many maximal ideals of B such that $\psi^{-1}(\mathfrak{m}_i) = \mathfrak{m}$, then*

$$\sum_i \operatorname{len}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \cdot [\kappa(B_{\mathfrak{m}_i}) : \kappa(A)] = \operatorname{len}_A(M).$$

Proof. Follows from Proposition A.4.13. □

The following result is used in proper pushforward.

Theorem A.4.17. *Let A be a one-dimensional domain, $K = Q(A)$, M be a finitely generated A-module and $f \in A$. Let $m_f : M \otimes_A K \to M \otimes_A K$ be the K-linear map induced by multiplication by f on M. Let $\det(m_f) \in K$ be the determinant of m_f. Then,*

$$\operatorname{len}_A\left(\frac{M}{fM}\right) = \operatorname{len}_A\left(\frac{M_K}{\det(m_f)M_K}\right).$$

Proof. Lemma A.3 of [Ful84]. □

The following lemma relates the length of the length of a flat local A-algebra with that of A.

Lemma A.4.18. *Let A and B be local rings and $\varphi : A \to B$ be a flat map. Then*

1. *The induced map $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective,*
2. *If A and B are Artinian local rings, then*

$$\operatorname{len}_B(B) = \operatorname{len}_A(A) \cdot \operatorname{len}_B B/\mathfrak{m}_A B.$$

Proof. 1. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ be a prime. We wish to find $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Going modulo $\mathfrak{p}$, we get the map $\bar{\varphi} : A/\mathfrak{p} \to B/\mathfrak{p}B$. This map is further flat as base change of a flat map is flat. We thus reduce to assuming that A is a domain and $\mathfrak{p} = 0$. Observe that $\varphi(a)$ in B is a non zero-divisor for each non-zero $a \in A$ since $0 \to A \stackrel{\times a}{\to} A$ remains injective by flatness of B. Thus $\operatorname{Im}(\varphi)$ consists of non zero-divisors of B. Any prime corresponding to $B/\varphi(A) \cdot B$ will then work. If this quotient is zero, then B is a domain and hence zero ideal will work.

2. By Theorem A.4.6, ring B has finite length, say r. Thus we have a maximal chain of ideals of A

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = A.$$

Consequently, I_i/I_{i-1} is a simple A-modules, that is,

$$I_i/I_{i-1} \cong A/\mathfrak{m}_A.$$

As B is a flat A-algebra, therefore by tensoring with B, we get a chain of ideals of B

$$0 = I_0 B \subseteq I_1 B \subseteq \cdots \subseteq I_r B = B.$$

Flatness further yields that $I_i B/I_{i-1}B \cong I_i/I_{i-1} \otimes_A B \cong A/\mathfrak{m}_{\mathfrak{A}} \otimes_A B \cong B/\mathfrak{m}_A B$. Since the following is exact

$$0 \to I_{i-1}B \to I_i B \to B/\mathfrak{m}_{\mathfrak{A}} B \to 0,$$

thus by Lemma A.4.4 (additivity of length), we have the recurrence relation

$$\operatorname{len}_B I_i B = \operatorname{len}_B I_{i-1} B + \operatorname{len}_B B/\mathfrak{m}_A B.$$

From this, the result follows at once. □

A.5 Normalization

We discuss some technical results about normalizations that we need in the main text, assuming all the basic facts. We begin with a basic fact about normalizations.

Lemma A.5.1. *Let X be a variety and $p : \tilde{X} \to X$ be its normalization so that $\tilde{X}$ is a normal variety. Let $Z \in \mathrm{PDiv}(X)$ is a prime divisor of X and R be the integral closure of $\mathcal{O}_{X,Z}$ in $K(X) = K(\tilde{X})$. Then,*

1. *R is a semi-local 1-dimensional domain with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$.*
2. *The local rings $R_{\mathfrak{m}_i}$ in $K(\tilde{X})$ corresponds to local rings $\mathcal{O}_{\tilde{X},\eta_i}$ of irreducible components W_i of $p^{-1}(Z) = \bigcup_{i=1}^{r} W_i$, $\eta_i \in \tilde{X}$ is the generic point of W_i.*
3. *The local rings $\mathcal{O}_{\tilde{X},\eta_i}$ are of dimension 1. Consequently, the subvarieties W_i are codimension 1 in $\tilde{X}$.*

Proof. We begin with the following observation. If $U = \mathrm{Spec}\,(A)$ is an open affine of X, then let $\eta \in U$ be the generic point of Z. Note that $K(X) = Q(A)$. By definition of normalization, we have $\tilde{U} = \mathrm{Spec}\left(\tilde{A}\right)$ is an open affine of $\tilde{X}$ where $A \hookrightarrow \tilde{A}$ is the normalization of A in $K(X)$. The following diagram then commutes where left vertical is induced by normalization

$$\begin{tikzcd} \tilde{U} \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ U \arrow[r, hook] & X \end{tikzcd} .$$

Thus, $\mathcal{O}_{X,Z} = A_\eta$. As localization and normalization commutes, therefore $R = \widetilde{\mathcal{O}_{X,Z}} = \widetilde{A_\eta} = \tilde{A}_\eta$ where $\tilde{A}$ is an A-module.

1. Note that $\mathcal{O}_{X,Z} \subseteq K(X) = K(\tilde{X})$ is 1-dimensional local domain. By Cohen-Seidenberg, $R \subseteq K(\tilde{X})$ is 1-dimensional domain as well. To see the semi-local property, pick the maximal $\mathfrak{m}$ of $\mathcal{O}_{X,Z}$. As R is the normalization of $\mathcal{O}_{X,Z}$, therefore every maximal of R lies over $\mathfrak{m}$. As $\dim R = 1$, therefore $\dim R/\mathfrak{m}R = 0$ and hence $R/\mathfrak{m}R$ is artinian. It follows that $R/\mathfrak{m}R$ has finitely many maximals and hence there are only finitely many maximals of R.

2. As in the remark above item 1, we have $p : \tilde{U} \to U$. Let $\eta_1, \dots, \eta_s$ be the generic points of irreducible components of $p^{-1}(Z)$. By replacing $\tilde{X}$ by $\tilde{U}$, X by U and $Z = V(\eta)$, we have that each $\eta_i \in \tilde{U}$ is a minimal prime of the ideal $\eta\tilde{A}$ since $p^{-1}(Z) = V(\eta\tilde{A})$. We first show that maximal ideals of $R = \tilde{A}_\eta$ and minimal primes of $\tilde{A}/\eta\tilde{A}$ are in bijection.

First, observe that as $\dim \tilde{A} = 1$, therefore $\tilde{A}/\eta\tilde{A}$ is dimension 0 and hence artinian so that every prime is maximal. Next, maximal ideals of $\tilde{A}_\eta$ are maximal ideals of $\tilde{A}$ which lie over η. On the other hand, maximal ideals of $\tilde{A}/\eta\tilde{A}$ are the maximals of $\tilde{A}$ which contains $\eta\tilde{A}$, i.e. lie over η. Hence we have a bijection and thus $r = s$. Finally, we have $R_{\mathfrak{m}_i} = (\tilde{A}_\eta)_{\mathfrak{m}_i} = \tilde{A}_{\mathfrak{m}_i} = \mathcal{O}_{\tilde{X},\eta_i}$, as required.

3. By item 2, we need only show that $\dim R_{\mathfrak{m}_i} = 1$. Since $\dim R_{\mathfrak{m}_i} = \mathrm{ht}\,\mathfrak{m}_i$ and $\dim R = 1$, so we are done. □

A.6 Smoothness, differentials & regularity

We discuss some more geometric aspects of schemes and varieties.

A.6.1 Tangent spaces

Recall that for a scheme X (recall our schemes are separated finite type over $\mathbf{k}$), we define two types of tangent spaces at a point $x \in X$. One is Zariski tangent space, given by

$$T_x^{\mathrm{zar}} X = \mathrm{Hom}_{\kappa(x)}\left(\frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^2}, \kappa(x)\right).$$

The other the usual $\mathbf{k}$-tangent space given by

$$T_x X = \mathrm{Hom}_{\mathbf{k}}\left(\frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^2}, \mathbf{k}\right).$$

If $x \in X$ is a rational point, then $\kappa(x) = \mathbf{k}$ and thus $T_x^{\mathrm{zar}} X = T_x X$. As our schemes are supposed to be over $\mathbf{k}$, thus we choose not to work with $T_x^{\mathrm{zar}} X$, primarily because there's an easy description of maps of tangent spaces.

Construction A.6.1 (Map on tangent spaces)**.** Let $f : X \to Y$ be a map of schemes. Then for each $x \in X$, we get a map of tangent spaces given by $f_x^\flat : T_x X \to T_{f(x)} Y$. Indeed, we first have a local map of local rings $f_x^\flat : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, which descends by local condition onto a map on the cotangent spaces:

$$f_x^\flat : \frac{\mathfrak{m}_{Y,f(x)}}{\mathfrak{m}_{Y,f(x)}^2} \longrightarrow \frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^2}.$$

Dualizing this with respect to $\mathbf{k}$, we get the required $\mathbf{k}$-linear map

$$f_x^\flat : T_x X \longrightarrow T_{f(x)} Y.$$

Remark A.6.2 (The "immersion" in a closed immersion)**.** Now suppose that $i : X \hookrightarrow Y$ is a closed immersion of a subscheme in Y. Then, the corresponding map on tangent spaces

$$i_x^\flat : T_x X \hookrightarrow T_x Y$$

is an injective $\mathbf{k}$-linear map since the map on local rings

$$i_x^\flat : \mathcal{O}_{Y,x} \longrightarrow \mathcal{O}_{X,x}$$

is surjective as $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}$ and thus the local ring $\mathcal{O}_{X,x}$ is the quotient of $\mathcal{O}_{Y,x}$ at some ideal and $i_x^\flat$ is that quotient map. Consequently, there is a surjection on cotangent spaces and by left-exactness of $\mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k})$, we ge that $T_x X \to T_x Y$ is an injective map. This is the reason we call a closed immersion an immersion, as it induces an injective map on tangent spaces, just like the terminology in differential geometry.

We next wish to see some simple results about tangent spaces which are helpful in discussion regarding transversality.

Lemma A.6.3. *Let X be a noetherian scheme and $x \in X$ be a reduced point. Then there exists an open affine $x \in U \subseteq X$ where U is a reduced affine scheme.*

Proof. Consider $V = \operatorname{Spec}(A)$ an open affine containing x where A is noetherian. Let $x = \mathfrak{p} \in V$ and $\mathfrak{n} \leq A$ be the nilradical of A. Note $\mathfrak{n}_\mathfrak{p}$ is the nilradical of $A_\mathfrak{p}$, hence is zero by hypothesis. By noetherian property, let $f_1, \dots, f_n \in \mathfrak{n}$ be a generating set of $\mathfrak{n}$. Hence by $\mathfrak{n}_\mathfrak{p} = 0$, we get that there exists $g_i \notin \mathfrak{p}$ such that $g_i f_i = 0$. Let $g = g_1 \dots g_n$. As nilradical of A_g is $\mathfrak{n}_g$, where $f_i/1 = g \cdot f_i/g = 0$ for each i, therefore $\mathfrak{n}_g = 0$. Hence A_g is reduced, so that we may take $U = \operatorname{Spec}(A_g)$, as required,. □

Lemma A.6.4. *Let X be a variety and $A, B \subseteq X$ be two subschemes. Let C is an irreducible component of $A \cap B$ and $p \in C$.*

1. *We have $T_pC = T_pA \cap T_pB$.*
2. *We have that A and B are transverse at $p \in C$.*

$$\operatorname{codim} T_pC = \operatorname{codim} T_pA + \operatorname{codim} T_pB.$$

Lemma A.6.5. *Let X be a variety and $A, B \subseteq X$ be two subvarieties. If A, B meet transversely at $p \in C \subseteq A \cap B$ where C is an irreducible component of $A \cap B$, then in T_pX we have*

$$T_pA \cap T_pB = T_pC.$$

Proof. This is true by above lemma but we give a different proof. Since we have inclusions $T_pC \subseteq T_pA, T_pB$, therefore we have $T_pC \subseteq T_pA \cap T_pB$ in T_pX. We need only show that T_pC and $T_pA \cap T_pB$ have same dimension. In particular, it is sufficient to show that $\dim T_pC \geq \dim T_pA \cap T_pB$. Note that as p is a smooth point for A, B and C, therefore, $\dim C = \dim \mathcal{O}_{C,p} = \dim T_pC$ and similarly $\dim A = \dim T_pA$, $\dim B = \dim T_pB$. By Corollary A.1.10, we have

$$\dim T_pC \geq \dim T_pA + \dim T_pB - \dim X.$$

By transversality, we further have (Remark 2.1.2)

$$\dim T_pA \cap T_pB = \dim T_pA + \dim T_pB - \dim X.$$

This completes the proof. □

A.6.2 Tangent cones

Definition A.6.6 (Tangent cones at a point). Let X be a scheme and $p \in X$. Denote $\mathfrak{m}$ to be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. Note that we have a filtration of the local ringConstruct the following graded $\mathbf{k}$-algebra

$$A = \bigoplus_{\alpha \geq 0} \frac{\mathfrak{m}^\alpha}{\mathfrak{m}^{\alpha+1}} = \mathbf{k} \oplus \frac{\mathfrak{m}}{\mathfrak{m}^2} \oplus \frac{\mathfrak{m}^2}{\mathfrak{m}^3} \oplus \dots.$$

As A is generated as an algebra on $\mathfrak{m}/\mathfrak{m}^2$, therefore we have a surjection (induced by universal property of Sym)

$$\varphi_p : \operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow A.$$

Denote the affine space Spec $(\mathrm{Sym}(V^*))$ over a vector space V as $\overline{V}$ and the projective space over V as $\mathbb{P}V = \mathrm{Proj}(\mathrm{Sym}(V^*))$. The affine and projective tangent cones of X at p are defined respectively as the following schemes in the affine and projective tangent space $\overline{T_pX}$ and $\mathbb{P}T_pX$:

$$TC_pX = \mathrm{Spec}\,(A) \hookrightarrow \overline{T_pX}$$
$$\mathbb{P}TC_pX = \mathrm{Proj}(A) \hookrightarrow \mathbb{P}T_pX.$$

Lemma A.6.7. *Let $X \hookrightarrow \mathbb{A}^n$ be a closed subscheme of affine space with ideal $I \leq \mathbf{k}[x_1, \dots, x_n]$. Then the affine tangent cone to X at origin p*

$$TC_pX \subseteq \overline{T_pX} \subseteq \overline{T_p\mathbb{A}^n} = \mathbb{A}^n$$

is given by the ideal generated by smallest degree homogeneous terms of each $f \in I$.

Proof. Let $A = \bigoplus_{\alpha \geq 0} \frac{\mathfrak{m}^\alpha}{\mathfrak{m}^{\alpha+1}}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$. Denoting $B = \mathbf{k}[x_1, \dots, x_n]/I$ the coordinate ring of X and $\mathfrak{m}_0 = \langle x_1, \dots, x_n \rangle$, we get that $\mathcal{O}_{X,p} = B_{\mathfrak{m}_0}$ and $\mathfrak{m} = \mathfrak{m}_0 B_{\mathfrak{m}_0}$. We have surjections

$$\mathbf{k}[x_1, \dots, x_n] = \mathrm{Sym}(\mathfrak{m}_0/\mathfrak{m}_0^2) \overset{\psi_0}{\twoheadrightarrow} \mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2) \overset{\varphi_0}{\twoheadrightarrow} A.$$

The map $\psi_0 : \mathbf{k}[x_1, \dots, x_n] \twoheadrightarrow \mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2)$ is given by $x_i \mapsto \bar{x}_i$. The map φ_0 on the other hand is given as follows. Pick a polynomial $p \in \mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2)$ and write $p = p_m + p_{m+1} + \dots$ where each p_k denotes the k-degree homogeneous part of p. Then, $\varphi_0(p) = \sum_k \bar{p}_k$ where each $\bar{p}_k \in \frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}}$. Thus the composition

$$\pi : \mathbf{k}[x_1, \dots, x_n] \twoheadrightarrow A$$

maps $p(x_1, \dots, x_n)$ to $\sum_k \bar{p}_k$ where $\bar{p}_k \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$. We claim that

$$\mathrm{Ker}\,(\pi) = \langle f_m \mid f \in I,\ f_m \text{ is the least degree homogeneous term of } f \rangle =: J.$$

To this end, we first calculate $\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$. As

$$\frac{\mathfrak{m}^\alpha}{\mathfrak{m}^{\alpha+1}} = \frac{\langle \mathfrak{m}_0^\alpha, I \rangle}{\langle \mathfrak{m}_0^{\alpha+1}, I \rangle}.$$

Pick f_m a generator of J corresponding to some $f \in I$. We wish to show that $\pi(f_m) = 0$. Indeed, we have

$$\pi(f_m) = \bar{f}_m \in \frac{\mathfrak{m}^m}{\mathfrak{m}^{m+1}} = \frac{\langle \mathfrak{m}_0^m, I \rangle}{\langle \mathfrak{m}_0^{m+1}, I \rangle}.$$

Note that $f_k \in \mathfrak{m}_0^{m+1}$ for all $k \geq m+1$, therefore $f_m = f - \sum_{k \geq m+1} f_k$ is in $\langle \mathfrak{m}_0^{k+1}, I \rangle$ as required. Conversely, pick any $g \in \mathrm{Ker}\,(\pi)$. We wish to show $g \in J$. As $\bar{g}_k = 0$ in $\mathfrak{m}^k/\mathfrak{m}^{k+1}$, therefore $g_k \in \langle \mathfrak{m}_0^{k+1}, I \rangle$. Consequently, we can write

$$g_k = \sum_i h_{i,k+1} \cdot l_i + f$$

where $h_{i,k+1}$ is a degree $k+1$-monomial, l_i some polynomial and $f \in I$. As $\deg(h_{i,k+1} \cdot l_i) \geq k+1$, therefore after writing

$$f = g_k - \sum_i h_{i,k+1} \cdot l_i,$$

we see that g_k is the least degree term of f. As $g = \sum_k g_k$ and each g_k is a generator of J, hence $g \in \mathrm{Ker}\,(\pi)$, as required. $\square$

Here are some examples of tangent cones to affine schemes.

Example A.6.8. Consider the curve $X : y^2 - x^2 - x^3$ in $\mathbb{A}^2$ and $p = (0,0)$ the origin. Then TC_pX in $\mathbb{A}^2$ is the scheme given by the ideal $y^2 - x^2$.

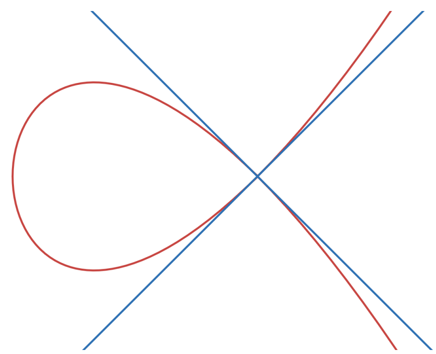

Figure 2: The curve in red is $y^2 - x^2 - x^3$ and the scheme in blue is the tangent cone at origin.

A.7 Tor & flatness

Recall that $\mathrm{Tor}_i^R(M, -)$ is the i^{th}-left derived functor of $N \mapsto M \otimes_R N$. A module M is said to be flat if the tensor functor $N \mapsto M \otimes_R N$ is exact. Here are equivalent notions of flatness:

Theorem A.7.1. *Let R be a ring and M be an R-module. Then the following are equivalent:*

1. *M is a flat R-module.*
2. *$\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$ and R-modules N.*
3. *$\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N.*
4. *$\mathrm{Tor}_1^R(M, N) = 0$ for all finitely generated R-modules N.*
5. *$\mathrm{Tor}_1^R(M, R/I) = 0$ for every ideal $I \leq R$.*
6. *$I \otimes_R M \to M$ is injective for every ideal $I \leq R$.*
7. *$I \otimes_R M \to IM$ is an isomorphism for every ideal $I \leq R$.*
8. *$M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*

Some more properties of flat modules are as follows.

Theorem A.7.2. *Let R be a ring and M be an R-module.*

1. *If M is projective, then M is flat.*
2. *If R is local and M is flat, then M is free.*
3. *If M is finitely generated, then M is projective if and only if M is flat.*

4. *If $M = \oplus_i M_i$, then M is flat if and only if M_i are flat.*
5. *If $S \subseteq R$ is a multiplicative set, then $S^{-1}A$ is flat.*
6. *If $0 \to M' \to M \to M'' \to 0$ is exact and M'' is flat, then M is flat if and only if M' is flat.*
7. *(Extension of scalars) If $f : R \to S$ is a ring homomorphism and M is flat, then $M \otimes_R S$ is a flat S-module.*
8. *(Restriction of scalars for flat maps) If $f : R \to S$ is a flat ring homomorphism and N is a flat S-module, then N is a flat R-module.*
9. *Rings $R[x_1, \dots, x_n]$ and $R[[x_1, \dots, x_n]]$ are flat R-modules.*
10. *If R is a PID, then M is flat if and only if M is torsion free.*

A.7.1 Flat maps

Recall that a map $f : X \to Y$ of schemes is said to be flat if for all $x \in X$ the comorphism $f_x^\flat : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat map of rings. The following hypothesis is usually used with the flatness of a map.

Definition A.7.3 (**Relative dimension** n)**.** A map of schemes $f : X \to Y$ is said to have relative dimension n if for all subvarieties $V \subseteq Y$, the subscheme $f^{-1}(V)$ has pure dimension $\dim V + n$, that is, every irreducible component of $f^{-1}(V)$ is of dimension $\dim V + n$.

We mention here some nice properties of flat maps of schemes.

Theorem A.7.4 (Proposition III.9.5, [Har77])**.** *Let $f : X \to Y$ be a flat map of finite type schemes over a field k. If Y is irreducible, then the following are equivalent:*

1. *For any irreducible component $Z \subseteq X$, we have $\dim Z = \dim Y + n$.*
2. *For every point $y \in Y$, every irreducible component X_y has dimension n.*

Theorem A.7.5 (Proposition III.9.7, [Har77])**.** *Let $f : X \to Y$ be a map of schemes where Y is a regular curve and X is reduced. Then the following are equivalent:*

1. *The map f is flat.*
2. *Every component of X maps dominantly to Y.*

Proposition A.7.6 (EGA.IV.14.2)**.** *If $f : X \to Y$ is a flat map of finite type schemes where Y is irreducible and every irreducible component of X has dimension $\dim Y + n$, then f is relative of dimension n and all base extensions of f are flat of dimension n.*

Finite type flat maps are open.

Proposition A.7.7. *Let $f : X \to Y$ be a finite type flat map of noetherian schemes. Then f is an open map.*

From the above result, one would expect that any open immersion is flat. It is indeed the case.

Lemma A.7.8. *Let X be a scheme and $i : U \hookrightarrow X$ be an open immersion. Then i is flat.*

Proof. Observe that the map on stalks is $i_x^\flat : \mathcal{O}_{X,x} \to \mathcal{O}_{X|U,x}$ for $x \in U$ is identity, which is flat. $\square$

A.8 Euler characteristic of modules

Serre in his book [Ser00] introduced the notion of Euler characteristic of two finitely generated modules over a regular local ring of finite dimension. The main theorem for our purposes is the following.

Theorem A.8.1 (Serre)**.** *Let X be an affine scheme and A be a regular local ring of dimension n obtained by localizing X at a regular point of X. Let M, N be two finitely generated A-modules such that $\operatorname{len}_A(M \otimes_A N)$ is finite. Let*

$$\chi(M, N) = \sum_{i \geq 0} (-1)^i \operatorname{len}_A(\operatorname{Tor}_i^A(M, N))$$

be the Euler characteristic of M and N. Then,

1. *The length of each tor module $\operatorname{len}_A(\operatorname{Tor}_i^A(M, N))$ is finite.*
2. *We have $\dim M + \dim N \leq n$.*
3. *The Euler characteristic $\chi(M, N) \geq 0$.*
4. *We have that $\chi(M, N) = 0$ if and only if $\dim M + \dim N < n$.*

A.9 Cohen-Macaulay rings

Recall a noetherian local ring R is Cohen-Macaulay (or simply, CM), if $\operatorname{depth}(R) = \dim(R)$, where $\operatorname{depth}(R)$ is the $\mathfrak{m}$-depth. These rings are generalizations of regular local rings.

Theorem A.9.1 (Grothendieck)**.** *Let R be a regular local ring of dimension n and M, N be two non-zero finitely generated R-modules. Denote*

$$d_{\mathrm{Tor}} = \max\{i \in \mathbf{N} \mid \operatorname{Tor}_i^R(M, N) \neq 0\}.$$

If $\operatorname{len}_R(M \otimes_R N) < \infty$, then

$$d_{\mathrm{Tor}} = \operatorname{pd}(M) + \operatorname{pd}(N) - n.$$

This is used to prove the following important result.

Corollary A.9.2. *Let R be a regular local ring of dimension n and M, N be two non-zero finitely generated R-modules such that $\operatorname{len}_R(M \otimes_R N) < \infty$. Then the following are equivalent:*

1. *$\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*
2. *M and N are Cohen-Macaulay and $\dim M + \dim N = n$.*

A.10 Bundles

We study geometric vector bundles over varieties.

Definition A.10.1 (**Geometric vector bundles**)**.** Let X be a scheme. A geometric vector bundle of rank n over X is a map $p : E \to X$ such that there is a cover U_i of X and isomorphisms $\varphi_i : p^{-1}(U_i) \to \mathbb{A}^n_{\mathbf{k}} \times U_i$ such that

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "\varphi_i"] \arrow[d, "p"'] & U_i \times \mathbb{A}^n_{\mathbf{k}} \arrow[dl, "\pi_1"] \\ U_i & \end{tikzcd}$$

commutes (i.e. φ_i is an U_i-morphism) and for any open affine $V = \mathrm{Spec}\,(A) \subseteq U_i \cap U_j$, the composite

$$\mathbb{A}^n_A \cong V \times \mathbb{A}^n_{\mathbf{k}} \xleftarrow{\varphi_i} p^{-1}(V) \xrightarrow{\varphi_j} V \times \mathbb{A}^n_{\mathbf{k}} \cong \mathbb{A}^n_A$$

is a linear isomorphism of $\mathbb{A}^n_A$, i.e. $\varphi_j \circ \varphi_i^{-1} : \mathbb{A}^n_A \to \mathbb{A}^n_A$ is given by $\theta : A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$ which is A-linear and $\theta(x_i) = \sum_j a_{ij} x_j$ for some $a_{ij} \in A$.

If $p : E \to X$ and $p' : E' \to X$ are two vector bundles of rank n and m over X, then a map of vector bundles is an X-morphism $f : E \to E'$ such that if $\varphi : p^{-1}(U) \to U \times \mathbb{A}^n_{\mathbf{k}}$ and $\psi : p'^{-1}(U') \to U' \times \mathbb{A}^m_{\mathbf{k}}$ are local trivializations of E and E', then the horizontal composite

$$(U \cap U') \times \mathbb{A}^n_{\mathbf{k}} \xleftarrow[\cong]{\varphi} p^{-1}(U \cap U') \xrightarrow{f} p'^{-1}(U \cap U') \xrightarrow[\cong]{\psi} (U \cap U') \times \mathbb{A}^m_{\mathbf{k}}$$

is a linear map of affine spaces $\mathbb{A}^n_{U \cap U'} \to \mathbb{A}^m_{U \cap U'}$.

A.11 Algebraic operations on vector bundles

Let $\mathbf{k}$ be a fixed base field, $\mathcal{V}$ be the category whose objects are *geometric* vector spaces, i.e. objects are $\hat{V} := \mathrm{Spec}\,(\mathrm{Sym}(V^*))$ for every finite dimensional $\mathbf{k}$-vector space V, and maps $\hat{V} \to \hat{W}$ are linear isomorphisms on the underlying vector spaces $f : V \to W$ (if exists) but seen as the map induced from the linear map on coordinate rings $f^* : \mathrm{Sym}(W^*) \to \mathrm{Sym}(V^*)$. Thus $\mathcal{V}$ is a groupoid where each homset is either $\mathrm{GL}_n(\mathbf{k})$ for some n or $\emptyset$. As $\mathrm{GL}_n(\mathbf{k})$ is an open subscheme of $\mathbb{A}^{n^2}$, thus $\mathcal{V}$ is in particular a groupoid enriched over $\mathcal{S}ch_{/\mathbf{k}}$.

Definition A.11.1 (Gluable functor). Let $T : \mathcal{V}^{\times k} \to \mathcal{V}$ be a functor. We call T gluable if the map induced on homsets

$$T : \mathrm{Hom}_{\mathcal{V}}(V_1, W_1) \times \cdots \times \mathrm{Hom}_{\mathcal{V}}(V_k, W_k) \longrightarrow \mathrm{Hom}_{\mathcal{V}}(T(V_1, \ldots, V_k), T(W_1, \ldots, W_k))$$

is a map of schemes, where we view $\mathrm{Hom}_{\mathcal{V}}(V_i, W_i)$ as the scheme $\mathrm{GL}_{n_i}(\mathbf{k})$. This is equivalent to saying that the functor $T : \mathcal{V}^{\times k} \to \mathcal{V}$ is an enriched functor over $\mathcal{S}ch_{/\mathbf{k}}$.

The goal of this note is to prove the following result.

Theorem A.11.2. *Let $T : \mathcal{V}^{\times k} \to \mathcal{V}$ be a gluable functor and X be a $\mathbf{k}$-scheme. If $p_\alpha : E_\alpha \to X$ are k-many vector bundles over X where* $\mathrm{rank} E_\alpha = n_\alpha$*, then there exists a vector bundle $p : E \to X$ which is unique with respect to the property that for any common local trivialization $U_i \subseteq X$ of all bundles E_α, there is a trivialization*

$$\begin{tikzcd} p^{-1}(U_i) \arrow[d, "p"'] & U_i \times T(\mathbb{A}^{n_1}, \ldots, \mathbb{A}^{n_k}) \arrow[l, "\phi_i"', "\cong"] \arrow[dl, "\pi_1"] \\ U_i & \end{tikzcd}$$

of E such that for $U_i \cap U_j$, the composite

$$U_i \cap U_j \times T(\mathbb{A}^{n_1}, \ldots, \mathbb{A}^{n_k}) \xrightarrow[\cong]{\phi_i} p^{-1}(U_i \cap U_j) \xleftarrow[\cong]{\phi_j} U_i \cap U_j \times T(\mathbb{A}^{n_1}, \ldots, \mathbb{A}^{n_k})$$

given by $\phi_j^{-1} \circ \phi_i$ is given as follows: for each $\alpha = 1, \dots, k$ and i, consider the local trivialization $h_i^\alpha : U_i \times \mathbb{A}^{n_\alpha} \to p_i^{-1}(U_i)$ of the bundle $p_\alpha : E_\alpha \to X$, then the map $\phi_j^{-1} \circ \phi_i$ is given by the following map

$$\begin{aligned} \varphi_{ij} : U_i \cap U_j \times T(\mathbb{A}^{n_1}, \dots, \mathbb{A}^{n_k}) &\longrightarrow U_i \cap U_j \times T(\mathbb{A}^{n_1}, \dots, \mathbb{A}^{n_k}) \\ (x, v) &\longmapsto \left(x, T\left((h_{j,x}^1)^{-1} \circ h_{i,x}^1, \dots, (h_{j,x}^k)^{-1} \circ h_{i,x}^k\right)(v)\right). \end{aligned}$$

The proof is essentially gluing of all of $U_i \times T(\mathbb{A}^{n_1}, \dots, \mathbb{A}^{n_k})$.

Proof. Define the following quantities where $\{U_i\}_{i \in I}$ is the cover of X by all common local trivializations of each p_α:

$$\begin{aligned} X_i &:= U_i \times T(\mathbb{A}^{n_1}, \dots, \mathbb{A}^{n_k}) \\ X_{ij} &:= U_i \cap U_j \times T(\mathbb{A}^{n_1}, \dots, \mathbb{A}^{n_k}) \subseteq X_i \end{aligned}$$

and define the gluing map

$$\begin{aligned} \varphi_{ij} : X_{ij} &\longrightarrow X_{ji} \\ (x, v) &\longmapsto \left(x, T\left((h_{j,x}^1)^{-1} \circ h_{i,x}^1, \dots, (h_{j,x}^k)^{-1} \circ h_{i,x}^k\right)(v)\right). \end{aligned}$$

We claim that the map φ_{ij} is an isomorphism of schemes. Indeed, by functoriality of T, we need only show that φ_{ij} is a map of schemes as then φ_{ji} is its inverse. To this end, observe that φ_{ij} is obtained from the gluing maps of each bundle p_α

$$g_{ij}^\alpha : U_i \cap U_j \to \mathrm{GL}_{n_\alpha}(\mathbf{k})$$

by composing it with T as follows:

$$g_{ij} : U_i \cap U_j \xrightarrow{(g_{ij}^\alpha)_\alpha} \mathrm{GL}_{n_1}(\mathbf{k}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbf{k}) \xrightarrow{T} \mathrm{GL}_n(\mathbf{k})$$

where $n = \mathrm{rank} T(\mathbb{A}^{n_1}, \dots, \mathbb{A}^{n_k})$. As by definition of gluable functors, the map T above is a map of schemes, hence the map φ_{ij} is a map of schemes, as required.

Next, we wish to show that φ_{ij} satisfies the cocycle condition, that is,

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik} \text{ on } X_{ij} \cap X_{ik}.$$

Indeed, pick $(x, v) \in X_{ij} \cap X_{ik}$. Then observe that

$$\begin{aligned} \varphi_{jk} \circ \varphi_{ij}(x, v) &= \varphi_{jk}\left(x, T\left((h_{j,x}^1)^{-1} \circ h_{i,x}^1, \dots, (h_{j,x}^k)^{-1} \circ h_{i,x}^k\right)(v)\right) \\ &= \left(x, T\left((h_{k,x}^1)^{-1} \circ h_{j,x}^1 \circ (h_{j,x}^1)^{-1} \circ h_{i,x}^1, \dots, (h_{k,x}^k)^{-1} \circ h_{j,x}^k \circ (h_{j,x}^k)^{-1} \circ h_{i,x}^k\right)(v)\right) \\ &= \left(x, T\left((h_{k,x}^1)^{-1} \circ h_{i,x}^1, \dots, (h_{k,x}^k)^{-1} \circ h_{i,x}^k\right)(v)\right) \\ &= \varphi_{ik}(x, v), \end{aligned}$$

as required. Thus by gluing of schemes, we get a scheme E together with open embeddings $\phi_i : X_i \to E$ and E is unique w.r.t. the property that $\{\phi_i(X_i)\}$ covers E, $\phi_i(X_i) \cap \phi_j(X_j) = \phi_i(X_{ij}) = \phi_j(X_{ji})$ and the following triangle commutes

$$\begin{tikzcd} & E & \\ X_{ij} \arrow[ur, hook, "\phi_i"] \arrow[rr, "\varphi_{ij}"'] & & X_{ji} \arrow[ul, hook', "\phi_j"'] \end{tikzcd}.$$

We now wish to show that E is a vector bundle over X. To this end, we first show that there is a canonical map $p : E \to X$. Indeed, we have maps $f_i : X_i \to X$ given by $(x, v) \mapsto x$. We claim that f_i can be glued to X. For this, we would need to show that $f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ \varphi_{ij}$ for all $i, j \in I$. Indeed, for $(x, v) \in X_{ij}$, we have $f_i(x, v) = x = f_j(\varphi_{ij}(x, v))$. Thus we get a unique map

$$f : E \to X$$

such that the following commutes for each $i \in I$:

$$\begin{tikzcd} X_i = U_i \times T(\mathbb{A}^{n_1}, \ldots, \mathbb{A}^{n_k}) \arrow[r, "\phi_i", "\cong"'] \arrow[dr, "\pi_1"'] & \phi_i(X_i) \arrow[d, "f"] \\ & U_i \end{tikzcd}.$$

That is, $f : E \to X$ is locally trivial. Finally, the transitions of $f : E \to X$ are linear as it is just the map φ_{ij} which is linear by hypothesis on T. This completes the proof. $\square$

Remark A.11.3. As an application of Theorem A.11.2, we have the following operations on vector bundles $\mathcal{E}, \mathcal{F}$ of ranks n and m over X:
1. $\mathcal{E} \oplus \mathcal{F}$ is a bundle of rank $n + m$ over X whose fibers are isomorphic to $\mathbb{A}^n \oplus \mathbb{A}^m$.
2. $\mathcal{E} \otimes \mathcal{F}$ is a bundle of rank nm over X whose fibers are isomorphic to $\mathbb{A}^n \otimes \mathbb{A}^m$.
3. $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is a bundle of rank nm over X whose fibers are isomorphic to $\mathrm{Hom}_{\mathbf{k}}(\mathbb{A}^n, \mathbb{A}^m)$ of all $\mathbf{k}$-linear maps $\mathbb{A}^n \to \mathbb{A}^m$.
4. $\mathrm{Sym}^k \mathcal{E}$ is a bundle of rank ${}^{n+k-1}C_k$ whose fibers are isomorphic to $\mathrm{Sym}^k \mathbb{A}^n$.
5. $\wedge^k \mathcal{E}$ is a bundle of rank ${}^nC_k$ whose fibers are isomorphic to $\wedge^k \mathbb{A}^n$.

$\vdots$

Thus, we know now how to construct new vector bundles out of old.

A.11.1 Maps of vector bundles

We next wish to lay out a general procedure to construct maps between geometric vector bundles. We first begin by the following criterion to detect isomorphism of bundles.

Lemma A.11.4. *Let $\mathcal{E} = (E, p, X)$ and $\mathcal{E}' = (E', p', X)$ be two bundles of rank n and m respectively on a scheme X. Let $f : \mathcal{E} \to \mathcal{E}'$ be a map of bundles. Then the following are equivalent:*
1. *f is an isomorphism of bundles.*
2. *$f_x : E_x \to E'_x$ is an isomorphism on fibers for all $x \in X$.*

Proof. We need only show (2. $\Rightarrow$ 1.). Let $U \subseteq X$ be a common local trivialization of both $\mathcal{E}$ and $\mathcal{E}'$. It is sufficient to show that the composite

$$U \times \mathbb{A}^n \xrightarrow{\cong} p^{-1}(U) \xrightarrow{f} p'^{-1}(U) \xleftarrow{\cong} U \times \mathbb{A}^n$$

is an isomorphism as then f will be an isomorphism on an open cover of E'.

We may thus assume that $\mathcal{E}$ and $\mathcal{E}'$ are trivial and hence obtain a fiberwise linear isomorphism $f : X \times \mathbb{A}^n \to X \times \mathbb{A}^n$. This map is, however, obtained from the gluing map $X \to \mathrm{GL}_n(\mathbf{k})$. Thus composing with the inverse map $\mathrm{GL}_n(\mathbf{k}) \to \mathrm{GL}_n(\mathbf{k})$, we get the gluing map for the inverse of f, which shows that f is an isomorphism. $\square$

The following shows when a vector bundle is trivial.

Lemma A.11.5. *Let X be a scheme and $p : E \to X$ be a rank n vector bundle over X. Then the following are equivalent:*

1. *$p : E \to X$ is trivial.*
2. *The corresponding $\mathcal{O}_X$-module of sections Γ_p is free of rank n.*

Proof. Follows immediately from the equivalence of locally free sheaves and vector bundles. $\square$

Remark A.11.6. Next, suppose we have two bundles $\mathcal{E}$ and $\mathcal{E}'$ of rank n and m on X. One can construct a map $\mathcal{E} \to \mathcal{F}$ by first defining a map of bundles $\mathcal{E}_{|U_i} \to \mathcal{E}'$ where $\{U_i\}$ is a cover of X by common local trivializations and then gluing these maps up.

Bundles over varieties are varieties again.

Lemma A.11.7. *Let X be a $\mathbf{k}$-variety. If $p : Y \to X$ is a vector bundle over X, then Y is a $\mathbf{k}$-variety.*

Proof. As X is quasi-compact and if $U = \mathrm{Spec}\,(A)$ is a trivializing open affine of X, then $p^{-1}(U)$ is isomorphic to $U \times \mathbb{A}^n_{\mathbf{k}}$, which is again quasi-compact, thus, Y is quasi-compact. Consequently, Y is irreducible and of finite type over $\mathbf{k}$. Clearly, fiber of Y at $x \in X$ is $\mathbb{A}^n_{\mathbf{k}}$ which is reduced again. Separatedness of Y is exhibited by the following fiber square

$$\begin{tikzcd} Y \arrow[r, "\Delta_Y"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times_{\mathbf{k}} Y \arrow[d, "p \times p"] \\ X \arrow[r, "\Delta_X"'] & X \times_{\mathbf{k}} X \end{tikzcd}$$

since Δ_X is a closed immersion and thus Δ_Y is. $\square$

Theorem A.11.8. *Any finite rank vector bundle $p : E \to \mathbb{A}^n_{\mathbf{k}}$ is trivial.*

Proof. It is sufficient to show that any locally free $\mathcal{O}_X$-module $\mathcal{F}$ over $\mathbb{A}^n_{\mathbf{k}}$ is free. Note that $\mathcal{F} = \widetilde{M}$ where M is a $R = \mathbf{k}[x_1, \dots, x_n]$-module. As $\mathcal{F}$ is locally free of finite rank, therefore M is finitely generated projective module over R by faithfully flat descent. By Quillen-Suslin theorem, M is free, as required. $\square$

This result is more elementary for vector bundles over $\mathbb{A}^1_{\mathbf{k}}$.

Lemma A.11.9. *Any vector bundle $p : E \to \mathbb{A}^1_{\mathbf{k}}$ is trivial.*

Proof. Indeed, any projective module over $\mathbf{k}[x]$ is free since $\mathbf{k}[x]$ is a PID. $\square$

A.12 Chern classes

A higher degree polynomial is usually difficult to understand. However we understand linear polynomials very well. The topic of vector bundles is an implementation of the idea of linearization, where we try to reduce the data in a high degree polynomial into a family of linear equations, family being indexed by some base space. Characteristic classes can then be thought of as the numerical invariants associated to vector bundles in order to distinguish them from one another.

We will work axiomatically, via the help of the following theorem.

Theorem A.12.1. *Let X be a smooth quasi-projective variety. There exists a unique assignment for each $1 \leq i$*

$$c_i : \mathcal{V}B(X) \longmapsto A^i(X)$$

which maps $\mathcal{E} \mapsto c_i(\mathcal{E})$ satisfying the following properties (we denote $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \dots$ in the ring $A^{\Pi}(X)$ which is a power series ring):

1. *(Line bundles) If $\mathcal{L}$ is a line bundle, then*

$$c(\mathcal{L}) = 1 + c_1(\mathcal{L})$$

where $c_1(\mathcal{L}) \in A^1(X)$ is obtained via the isomorphisms

$$\mathrm{Pic}(X) \to \mathrm{CaCl}(X) \to A^1(X).$$

2. *(Degeneracy locus) Let $s_0, \dots, s_{r-i}$ be global sections of $\mathcal{E}$ where $r = \mathrm{rank}\mathcal{E}$. If the degeneracy locus of their dependency*

$$D = V(s_0, \dots, s_{r-i})$$

has codimension i, then

$$c_i(\mathcal{E}) = [D].$$

3. *(Whitney's formula) If*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is a short exact sequence of vector bundles over X, then

$$c(\mathcal{F}) = c(\mathcal{E}) \cdot c(\mathcal{G}).$$

4. *(Functoriality) If $\varphi : Y \to X$ is a morphism of smooth quasi-projective varieties, then*

$$\varphi^*(c(\mathcal{E})) = c(\varphi^*\mathcal{E}).$$

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