Overview of simplicial sets

eCHT Kan seminar

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§1. The singular simplicial set.

$|\Delta^2| =$

$|\Delta^1| =$

$|\Delta^0| =$

Defn: Topological n-simplex

$$|\Delta^n| = \boxed{\left\{(e_0, \ldots, e_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_i e_i = 1, \quad e_i \geq 0\right\}}$$

The combinatorics of $|\Delta^n|$ is established by the maps

$$d^i : |\Delta^{n-1}| \longrightarrow |\Delta^n| \quad \forall\ 0 \leq i \leq n$$

$$(e_0, \ldots, e_{n-1}) \longmapsto \boxed{(e_0, \ldots, e_{i-1}, 0, e_i, \ldots, e_{n-1})}$$

&

$$\rho^i : |\Delta^{n+1}| \longrightarrow |\Delta^n| \quad \forall\ 0 \leq i \leq n$$

$$(e_0, \ldots, e_{n+1}) \longmapsto \boxed{(e_0, \ldots, e_{i-1}, e_i + e_{i+1}, e_{i+2}, \ldots, e_{n+1})}$$

Thus, the seq. of sets for a space Z

$$S_n(Z) := \mathrm{Map}(|\Delta^n|, Z) \quad \text{as a set}$$

admit maps

$\boxed{\text{Face maps}}$ $\partial_i : S_n(Z) \longrightarrow S_{n-1}(Z) \quad \forall\ 0 \leq i \leq n$

&

$$f \longmapsto \boxed{f \circ d^i}$$

$\boxed{\text{Degeneracy maps}}$ $s_i : S_n(Z) \longrightarrow S_{n+1}(Z) \quad \forall\ 0 \leq i \leq n$

$$f \longmapsto \boxed{f \circ \rho^i}$$

Furthermore, if $Z \xrightarrow{\varphi} W$ is a contns map, we get functions

$$\varphi_n : S_n(Z) \longrightarrow S_n(W)$$

which satisfies: $\quad f : |\Delta^n| \to Z \longmapsto \boxed{|\Delta^n| \xrightarrow{f} Z \xrightarrow{\varphi} W}$

$$\begin{tikzcd} S_n(Z) \arrow[r, "\varphi_n"] \arrow[d, "\partial_i"'] \arrow[dr, phantom, "\circlearrowleft"] & S_n(W) \arrow[d, "\partial_i"] \\ S_{n-1}(Z) \arrow[r, "\varphi_{n-1}"'] & S_{n-1}(W) \end{tikzcd} \quad \& \quad \begin{tikzcd} S_n(Z) \arrow[r, "\varphi_n"] \arrow[d, "s_i"'] \arrow[dr, phantom, "\circlearrowright"] & S_n(W) \arrow[d, "s_i"] \\ S_{n+1}(Z) \arrow[r, "\varphi_{n+1}"'] & S_{n+1}(W) \end{tikzcd}.$$

Moreover, these maps ∂_i & s_j satisfies:

(*) — The simplicial identities

$$\boxed{\begin{array}{lll} \partial_i \partial_j = \partial_{j-1} \partial_i & & \text{if } i<j \\ s_i s_j = s_{j+1} s_i & & \text{if } i \leq j \\ \partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{if } i<j \\ \mathrm{id} & \text{if } i=j, j+1 \\ s_j \partial_{i-1} & \text{if } i>j+1 \end{cases} \end{array}}$$

This motivates the following defn, in order to study spaces more algebraically.

Defn.: (Simplicial sets) A simplicial set X is a seq. of sets $\{X_n\}_{n\geq 0}$ together with maps

$$\boxed{\begin{array}{l} \partial_i : X_n \longrightarrow X_{n-1} \\ \& \quad s_i : X_n \longrightarrow X_{n+1} \\ \text{Satisfying } (*). \end{array}}$$

A simplicial map $f: X \to Y$ is a collection of functions

$$\{f_n : X_n \longrightarrow Y_n\}_{n \geq 0}$$

s.t. the following commutes:

$$\boxed{\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\partial_i"'] \arrow[dr, phantom, "\circlearrowright"] & Y_n \arrow[d, "\partial_i"] \\ X_{n-1} \arrow[r, "f_{n-1}"'] & Y_{n-1} \end{tikzcd}} \quad \& \quad \boxed{\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "s_i"'] \arrow[dr, phantom, "\circlearrowright"] & Y_n \arrow[d, "s_i"] \\ X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1} \end{tikzcd}}.$$

This gives the category of simplicial sets with simplicial maps, denoted

$$\boxed{sSet}.$$

There is an important property, satisfied by $S_*(Z)$ which gives an important class of simplicial sets.

Defn: ((n,k)-horn) For a simplicial set K, an (n,k)-horn for $0 \leq k \leq n$ is a collection of n many $(n-1)$-simplices

$$\{z_0, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n\} \subseteq K_{n-1}$$

satisfying

$$\boxed{\partial_i z_j = \partial_{j-1} z_i \quad \text{for } 0 \leq i < j \leq n, \; i \neq k \neq j}.$$

A Kan complex is a simplicial set K for which all (n,k)-horns for $0 \leq k \leq n$ $\{z_i\}$, has an n-simplex

$$\boxed{z \in K_n \quad \text{s.t.} \quad \partial_i z = z_i \;\; \forall \;\; i \neq k.}$$

"Kan filler condition"

This is also called an ∞-groupoid. If $0<k<n$, then it is an ∞-category.

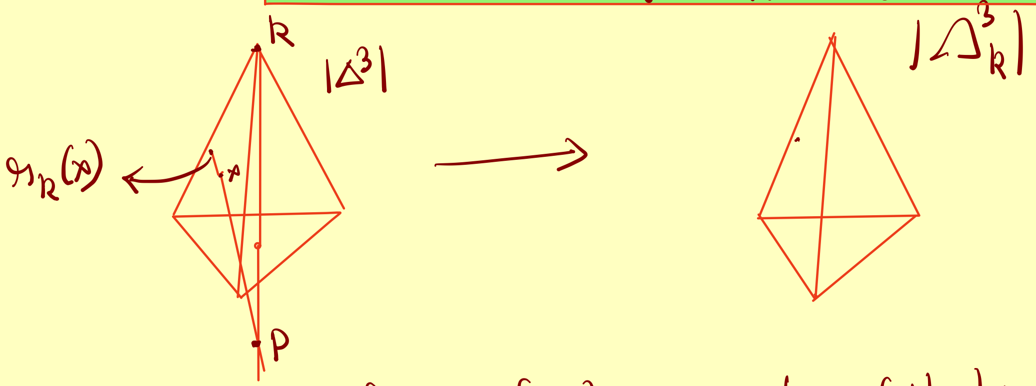
Lemma/Example: $S_*(Z)$ is a Kan complex

Proof: **Step 1:** An (n,k)-horn $\{z_i\}$ in $S_*(Z)$ gives a map $|\Lambda^n_k| \xrightarrow{\{z_i\}} Z$ where

$$|\Lambda^n_k| = \bigcup_{i \neq k} |\partial_i \Delta^n| \quad , \quad |\partial_i \Delta^n| = \{(e_0, \ldots, e_n) \mid c_i = 0\}$$

Step 2: We have a retraction

$|\Delta^n| \xrightarrow{r_k} |\Lambda^n_k|$ given by



Step 3: Any horn $\{z_i\}$ can be filled:

$$\begin{tikzcd}
|\Delta^n| \arrow[r, dashed, "z"] \arrow[d, bend left, "r_k"] & Z \\
|\Lambda^n_k| \arrow[u, hook] \arrow[ur, "\{z_i\}"'] &
\end{tikzcd}$$

$\square$

There's a more functorial way of constructing simplicial sets which is helpful.

__Defn__: (Δ) Consider a category Δ whose obj. are $\boxed{[n] = \{0<1< \cdots < n\}}$

& whose morphisms are

$$\boxed{\begin{array}{l}\text{non-decr. maps}\\ \quad f : [n] \longrightarrow [m].\end{array}}$$

There are two important classes of maps in Δ, given by

$$\boxed{\begin{array}{l} d^i : [n-1] \longrightarrow [n] \\ \quad k \longmapsto \begin{cases} k & \text{if } k<i \\ k+1 & \text{if } k \geq i \end{cases}\end{array}} \quad \forall\ 0 \leq i \leq n$$

&

$$\boxed{\begin{array}{l} s^i : [n+1] \longrightarrow [n] \\ \quad k \longmapsto \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i \end{cases}\end{array}} \quad \forall\ 0 \leq i \leq n$$

In-fact, any map $f : [n] \to [m]$ in Δ can be written as

$$\boxed{\begin{array}{l} \qquad f = d_f\, s_f \\ \text{where } d_f \text{ is comp. of } d^i\text{'s} \\ \quad \& \ s_f \text{ is comp. of } s^j\text{'s} \end{array}}$$

This gives us the following:

Lemma: TFAE!

$$\boxed{\begin{array}{l} \text{(1) } X \text{ is a sSet.} \\ \text{(2) } X \text{ is a presheaf} \\ \qquad X : \Delta^{op} \longrightarrow \text{Set} \end{array}}$$

$\square$

§2. Homotopy of simplices in a Kan complex.

We define when two simplices of a simpl. set are said to be homotopic.

Defn: (Homotopy of simplices) Let K be a simplicial set & $x, x' \in K_n$ be two n-simplices which are compatible, i.e. $\boxed{\partial_i x = \partial_i x' \ \forall\ 0 \leq i \leq n.}$

Then they are homotopic if $\exists\ y \in K_{n+1}$ s.t.

$$\boxed{\begin{array}{c} \partial_n y = x, \quad \partial_{n+1} y = x' \\ \& \\ \partial_i y = s_{n-1} \partial_i x = s_{n-1} \partial_i x' \ \forall\ 0 \leq i \leq n-1. \end{array}}$$

Similarly, for a triple (K, L, ϕ) where $L \subseteq K$ is a sub Kan complex & $\phi \in L_0$ is a 0-simplex, we define a notion of homotopy relative to L as follows:

If $x, x' \in K_n$ are two n-simplices satisfying

$$\boxed{\partial_i x = \partial_i x' \quad \forall\ 1 \leq i \leq n \ \&\ \partial_0 x, \partial_0 x' \in L_{n-1}}$$

Then they are said to be <u>homotopic rel L</u> if

$$\boxed{\begin{array}{l} \exists\ y \in K_{n+1} \text{ s.t. } \partial_n y = x, \quad \partial_{n+1} y = x' \\ \qquad \partial_i y = s_{n-1} \partial_i x = s_{n-1} \partial_i x' \\ \qquad\qquad\qquad \forall\ 1 \leq i \leq n-1 \\ \& \\ \qquad \partial_0 y : \partial_0 x \sim \partial_0 x' . \end{array}}$$

Just like in topology, the first thing we must do is to show that:

<u>Proposition</u>: If K is a Kan-complex then the homotopy of compatible simplices is an equivalence reln.

<u>Proof Sketch</u>: Reflexivity: $x \in K_n$, then $\boxed{y = s_n x}$ is the required homotopy.

Symmetry & transitivity: It suffices to show that

$$\begin{array}{l} x \sim x' \\ x \sim x'' \end{array} \implies x' \sim x''$$

Indeed, let $y' : x \sim x'$ & $y'' : x \sim x''$.

Then consider the collection of $n+2$ many $(n+1)$-simplices

$$z_i = \begin{cases} \partial_i s_n s_n x', & 0 \leq i \leq n-1 \\ y', & i = n \\ y'', & i = n+1 . \end{cases}$$

Claim: $\boxed{\{z_i\} \text{ forms an } (n+2, n+2)\text{-horn.}}$

$\Rightarrow$ Easy, do it case by case.

As K is a Kan complex, thus $\exists\ z \in K_{n+2}$ s.t. $\partial_i z = z_i \quad \forall\ i \neq k$.

Take $\boxed{h := \partial_{n+2} z}$.

Claim: $\boxed{h : x' \sim x''}$.

$\Rightarrow$ Easy, use simplicial ident. & bound. of z.

$\square$

We can now define homotopy groups of a Kan pair & triples.

Defn: (Simplicial homotopy groups) Let (K, L, ϕ) be a Kan triple.

Define $\boxed{\partial(K,\phi)_n = \{x \in K_n \mid \partial_i x = \phi,\ 1 \leq i \leq n\}}$

& Thus, $\pi_n(K,\phi) := \boxed{\dfrac{\partial(K,\phi)_n}{\sim_{\text{homotopy}}}}$.

Similarly, define

$$\boxed{\partial(K,L,\phi)_n = \left\{x \in K_n \;\middle|\; \begin{array}{l} \partial_i x = \phi \quad \forall\ 1 \leq i \leq n \\ \& \\ \partial_0 x \in L_{n-1} \end{array}\right\}}$$

&

$$\pi_n(K,L,\phi) := \boxed{\dfrac{\partial(K,L,\phi)_n}{\sim_{\text{rel } L}}}.$$

There is a group structure on $\pi_n(K,\phi)$ given by

$$\pi_n(K,\phi) \times \pi_n(K,\phi) \longrightarrow \pi_n(K,\phi)$$
$$[x]\ ,\ [x'] \longmapsto [\partial_n z]$$

where $z \in K_{n+1}$ is obtained by filling the $(n+1,n)$-horn

$$\boxed{z_i = \begin{cases} \phi, & \text{if } i \neq n-1, n+1 \\ x, & \text{if } i = n-1 \\ x', & \text{if } i = n+1. \end{cases}}$$

An application of these defns is the following classical l.e.s.

Theorem: [L.E.S. of a pair] Let (K,L,ϕ) be a Kan triple & $i:(L,\phi) \hookrightarrow (K,\phi)$ & $j:(K,\phi,\phi) \hookrightarrow (K,L,\phi)$. Then there is a l.e.s.

$$\cdots \to \pi_{n+1}(K,L,\phi) \xrightarrow{\partial} \pi_n(L,\phi) \xrightarrow{i} \pi_n(K,\phi) \xrightarrow{j} \pi_n(K,L,\phi) \to \cdots$$
$$[x] \longmapsto [\partial_0 x]$$

Proof Sketch: We show exactness at $\pi_n(L,\phi)$.

To Show: $i\partial = \phi$.

Pick $[x] \in \pi_{n+1}(K,L,\phi)$. To Show: $\partial_0 x \in K_n$ is null-homotopic.

Consider the $n+2$ many $(n+1)$-simplices

$$\{\underset{1}{\phi}, \underset{2}{\phi}, \ldots, \underset{n+1}{\phi}, \underset{n+2}{x}\}$$

Claim: This is an $(n+2,0)$-horn.

$\Rightarrow$ Easy.

Thus, there exists $z \in K_{n+2}$ extending the horn.

Claim: $\partial_0 z \in K_{n+1}$ is a homotopy $\partial_0 x \sim \phi$.

$\Rightarrow$ Easy by simplicial ident.

Similarly one can show $\ker \partial_i \subseteq \operatorname{Im} \partial$ & the others. $\square$

Remark: (On other notions of homotopy).

Recall from homotopy theory of spaces that

$$\boxed{\pi_n(X, x_0) = [(D^n, S^{n-1}), (X, x_0)]}.$$

Q: What are the simplicial sets which plays the role of (D^n, S^{n-1}) in topology?

Define $\Delta^n = h_{[n]} \in sSet$ &

$\partial\Delta^n \subseteq \Delta^n$ whose m-simplices are

$$\boxed{(\partial\Delta^n)_m = \{f : [m] \to [n] \mid f \text{ is NOT surjective}\}}$$

The pair $(\Delta^n, \partial\Delta^n)$ acts as the pair (D^n, S^{n-1}).

(But what's the notion of homotopy of two simplicial maps?)

<u>Defn:</u> Two simplicial maps $f, g : K \rightrightarrows L$ are said to be homotopic if there exists

$\boxed{\begin{array}{l}\text{a simplicial map} \\ \quad H : K \times \Delta^1 \longrightarrow L \quad \text{such that} \\ \text{on } q\text{-simplices, we have} \\ \quad H_q(x_q, \vec{1}) = f_q, \\ \quad H_q(x_q, \vec{0}) = g_q.\end{array}}$.

With this notion, it can be shown that for a Kan pair $(K, \emptyset)$, we have:

$$\boxed{\pi_n(K, \emptyset) \cong [(\Delta^n, \partial\Delta^n), (K, \emptyset)]}.$$

§3. Geometric Realization.

<u>Construction</u>: (Realizing a simplicial set)

Let K be a simplicial set. Define

$$\overline{K} = \coprod_{n \geq 0} K_n \times |\Delta^n|$$

&

$\boxed{\begin{array}{ll}TK \text{ or } |K| := \overline{K}/\sim & \text{where } \sim \text{ is} \\ \text{gen. by} & \\ (k_n, d^i u_{n-1}) \sim (\partial_i k_n, u_{n-1}) & \text{where } k_i \in K_i \\ (k_n, \rho^i u_{n+1}) \sim (s_i k_n, u_{n+1}) & u_i \in |\Delta^i|.\end{array}}$

Denote the pt. $(k_n, u_n) \in K_n \times |\Delta^n|$ induces in $|K|$ by $\boxed{[k_n, u_n]}$.

We wish to show that $|K|$ is a CW-complex.

Theorem: [Milnor] Let K be a simplicial set. Then $|K|$ is a CW-complex with one n-cell for each non-degenerate n-simplex of K.

Proof: Step 1: For all $[k_n, u_n]$, $\exists\, k'_m \in K_m$, $u'_m \in |\Delta^m|$ s.t.

a) $[k_n, u_n] = [k'_m, u'_m]$.

b) k'_m is non-deg.

c) u'_m is in the inter. of $|\Delta^m|$.

d) (k'_m, u'_m) is unique wrt a, b, c.

Step 2: There is a pushout square in sSet $\forall n \geq 1$

$$\begin{tikzcd}
sk_n K \arrow[dr, phantom, "\ulcorner"] & sk_{n-1} K \arrow[l] \\
\coprod_{NK_n} \Delta^n \arrow[u] & \coprod_{NK_n} \partial\Delta^n \arrow[l] \arrow[u]
\end{tikzcd}$$

where $NK_n \subseteq K_n$ is the set of non-deg. simplices.

& $sk_0 K = K_0$

where $sk_i K$ are defined inductively & are sub-simpl sets of K.

Step 3: There is an adjunction

$$\begin{tikzcd} sSet \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & Top \arrow[l, bend left, "S"] \end{tikzcd}.$$

Step 4: Conclude.

By step 3, T preserves colimits. Thus, we have a pushout in Top by step 2

$$\begin{tikzcd} {|sk_n K|} & {|sk_{n-1} K|} \arrow[l] \\ \coprod_{NK_n} |\Delta^n| \arrow[u] & \coprod_{NK_n} |\partial\Delta^n| \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

with $|sk_n K| \hookrightarrow |K|$.

By Step 1, $\bigcup_n |sk_n K| = |K|$ & the above diagram gives the required skeletal filtration. $\square$

Remark: 1) The natural isom.

$$\mathrm{Hom}_{Top}(TK, X) \cong \mathrm{Hom}_{sSet}(K, SX)$$

also induces an isom. at the level of homotopy classes.

2) The counit of the adjunction

$$\begin{tikzcd} sSet \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & Top \arrow[l, bend left, "S"] \end{tikzcd}$$

$\varepsilon : TS \longrightarrow \mathrm{id}_{Top}$ is a

$\boxed{\text{weak homotopy equivalence}}$.

It thus shows that any space is $\boxed{\text{weakly equivalent to a CW-complex}}$!!