Length

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In this note, we prove the main theorem on length of rings (finite length iff artinian, Theorem 6) and modules (finite length iff finite support, Corollary 9), apart from some other useful results. This is part of my master's thesis on intersection theory.

Definition 1 (Length of a module). Let R be a ring and M be an R-module. Then the length of M is given by the length of the longest ascending chain of submodules of M:

$$\operatorname{len}_R(M) := \sup\{r \in \mathbf{N} \mid M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r \text{ is a chain of submodules of } M\}.$$

A finite chain $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r$ is called a maximal length chain if it cannot be extended, that is, each factor M_i/M_{i-1} is a simple module. A maximal length chain is also called a composition series. Consequently, length of a module M is defined to be the length of the longest composition series.

An important result about length of modules is the fact that over a local ring R, any two composition series have the same length and composition factors.

Theorem 2 (Jordan-Hölder). *Let R be a local ring and M be an R-module which contains a composition series. Then any other composition series has the same length and composition factors. That is, length of M is equal to length of any composition series.*

The following are essential properties of length which one uses while dealing with maps.

Lemma 3. *Let $f : R \to S$ be a map of rings and M be an S-module. Then $\operatorname{len}_R(M) \geq \operatorname{len}_S(M)$ and equality holds if f is surjective.*

Proof. Follows from correspondence of submodules via a quotient map. □

The following is an easy exercise.

Lemma 4 (Additivity of length). *If M_i are finite length R-modules and the following is exact:*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to 0,$$

then

$$\sum_{i=0}^{n} (-1)^i \operatorname{len}_R(M_i) = 0.$$

We wish to characterize finite length modules over a noetherian ring. We begin with a lemma.

Lemma 5. *Any finite length R-module is finitely generated.*

Proof. If M is not finitely generated, then let $\{f_\alpha\}_{\alpha \in I}$ be a generating set of M and let $\{f_n\}_n$ be a subsequence. Then, the chain

$$0 \subsetneq \langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \dots$$

is a chain of submodules of M which doesn't stabilizes, a contradiction to finite length. □

Using results on artinian rings, we see an important characterization of artinian rings and finite length rings.

Theorem 6. *Let R be a ring. The following are equivalent:*
1. *R is artinian.*
2. *R has finite length.*

Proof. (1. $\Rightarrow$ 2.) By structure theorem of artinian rings, we reduce to assuming R is local artinian, $(R, \mathfrak{m})$. Recall that for an artinian ring, the Jacobson radical of R is nilpotent, which is just $\mathfrak{m}$. We construct a chain of ideals of R, where each subquotient has finite length. Indeed, consider the chain

$$0 = \mathfrak{m}^n \subsetneq \mathfrak{m}^{n-1} \subsetneq \dots \subsetneq \mathfrak{m}^2 \subsetneq \mathfrak{m} \subsetneq R.$$

Note that $\mathfrak{m}^{i-1}/\mathfrak{m}^i$ is an $\kappa = R/\mathfrak{m}$-module. If any one of $\mathfrak{m}^i/\mathfrak{m}^{i-1}$ is infinite dimensional as an κ-vector space, then the above chain of ideals can be refined to an infinite chain of strictly decreasing ideals, a contradiction to artinian condition. Hence each subquotient is a finite dimensional κ-module and hence its length as an R-module is equal to its dimension as a κ-module (Lemma 3).

(2. $\Rightarrow$ 1.) Take any descending chain of ideals $I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \dots$. If it doesn't stabilize, then we have an infinite length chain, so that $\operatorname{len}(R)$ is not finite, a contradiction. □

The following is an essential result which we'll use later.

Proposition 7. *Let R be a noetherian ring and M be a finitely generated R-module. If $\mathfrak{p} \in \operatorname{Supp}(M)$ is a minimal prime of M, then $M_\mathfrak{p}$ is a finite length $R_\mathfrak{p}$-module.*

Proof. As $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$, therefore a minimal prime $\mathfrak{p} \in \operatorname{Supp}(M)$ is an isolated/minimal prime of $\operatorname{Ann}(M)$. As $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module, therefore it suffices to construct a composition series of $M_\mathfrak{p}$. Let M be generated by $f_1, \dots, f_n \in M$, so that $M_\mathfrak{p}$ is also generated by their respective images. We thus get the following chain:

$$0 \subseteq \langle f_1 \rangle \subseteq \langle f_1, f_2 \rangle \subseteq \dots \subseteq \langle f_1, \dots, f_n \rangle = M_\mathfrak{p}.$$

It suffices to show that $\frac{\langle f_1, \dots, f_i \rangle}{\langle f_1, \dots, f_{i-1} \rangle}$ is a finite length $R_\mathfrak{p}$-module. Indeed, we have a surjection

$$\langle f_i \rangle \twoheadrightarrow \frac{\langle f_1, \dots, f_i \rangle}{\langle f_1, \dots, f_{i-1} \rangle},$$

hence it suffices to show that $\langle f_i \rangle$ is a finite length $R_\mathfrak{p}$-module. To this end, pick any $x \in M$. We'll show that $\langle x \rangle = xR_\mathfrak{p}$ is a finite length $R_\mathfrak{p}$-module. Observe that $\langle x \rangle = xR_\mathfrak{p}$ is isomorphic to $R_\mathfrak{p}/I$ where I is the annihilator of x in $R_\mathfrak{p}$. We may write $I = \mathfrak{a}R_\mathfrak{p}$ where $\mathfrak{a} \leq R$ is contained in $\mathfrak{p}$. Hence, we wish to show that $S = R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}$ is a finite length $R_\mathfrak{p}$-module, that is S is a finite length ring. Indeed, as $S = (R/\mathfrak{a})_\mathfrak{p}$ and $\mathfrak{p}$ is a minimal prime in $\operatorname{Supp}(M)$, that is, minimal prime containing $\operatorname{Ann}(M)$, and since $\operatorname{Ann}(M) \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, therefore $\mathfrak{p}$ is a minimal prime of $\mathfrak{a}$ as well. It follows that $S = (R/\mathfrak{a})_\mathfrak{p}$ is a dimension 0 ring. Since R is noetherian and noetherian property is inherited by quotients and localizations, therefore S is a noetherian ring of dimension 0, hence artinian. From Theorem 6, it follows that S is of finite length, as required. □

Theorem 8. *Let R be a noetherian ring and M be an R-module. Then the following are equivalent:*

1. *M has finite length.*
2. *M is finitely generated and $\dim R/\operatorname{Ann}(M) = 0$, i.e. $R/\operatorname{Ann}(M)$ is an artinian ring.*

Proof. (1. $\Rightarrow$ 2.) By Lemma 5, M is finitely generated. Let $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M$ be a composition series of M, which exists as $\operatorname{len}(M) < \infty$. We thus get that $M_i/M_{i-1} \cong R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$, as these subquotients are simple. Note that $\dim R/\operatorname{Ann}(M) = 0$ if and only if $\operatorname{Supp}(M)$ consists only of maximal ideals. So let $\mathfrak{p} \in \operatorname{Supp}(M)$. Thus $M_\mathfrak{p} \neq 0$. It follows that for some i, $(M_i/M_{i-1})_\mathfrak{p} \neq 0$. As $(M_i/M_{i-1})_\mathfrak{p} = (R/\mathfrak{m}_i)_\mathfrak{p}$, therefore this can only happen if $\mathfrak{m}_i \subseteq \mathfrak{p}$, i.e. $\mathfrak{m}_i = \mathfrak{p}$, as required. This also shows that $\operatorname{Supp}(M) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$.

(2. $\Rightarrow$ 1.) We need only construct a composition series of M. We have $\operatorname{Supp}(M)$ consists only of maximal ideals. Consider $\operatorname{Supp}(M) \subseteq \operatorname{Spec}(R)$. As M is finitely generated, say by $f_1, \dots, f_n$. Then we get a chain of submodules

$$0 \subsetneq \langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \cdots \subsetneq \langle f_1, \dots, f_n \rangle = M.$$

We need only show that each subquotient is a finite length R-module. Indeed, as we have a surjection

$$\langle f_i \rangle \twoheadrightarrow \frac{\langle f_1, \dots, f_i \rangle}{\langle f_1, \dots, f_{i-1} \rangle},$$

so it suffices to show that $\langle f_i \rangle$ is a finite length R-module. To this end, it suffices to show that for each $x \in M$, the submodule Rx is of finite length. Indeed, we have $Rx \cong R/I$ where $I = \operatorname{Ann}(x)$. As $I \supseteq \operatorname{Ann}(M)$, therefore

$$R/I \cong \frac{R/\operatorname{Ann}(M)}{I/\operatorname{Ann}(M)}.$$

As $R/\operatorname{Ann}(M)$ is an artinian ring and any quotient of artinian ring is an artinian ring, it follows at once that R/I is an artinian ring. By Theorem 6, $R/I \cong Rx$ is of finite length, as required. □

From the above proof, we can deduce the following.

Corollary 9. *Let R be a noetherian ring and M be a finitely generated R-module. Then the following are equivalent:*

1. *M is of finite length.*

2. *There exists a chain*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

where $M_i/M_{i-1} \cong R/\mathfrak{m}_i$ *and* $\mathrm{Supp}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$.

3. *Support of M consists of finitely many maximal ideals.*

Proof. (1. $\Leftrightarrow$ 2.) If M is a finite length module, then the maximal chain has each subquotient a simple module. Let $R/\mathfrak{m}_i$ be the subquotients. We wish to show that $\mathrm{Supp}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. This is what we showed in the proof of Theorem 8. Conversely, suppose M has the a chain as above with each subquotient a field. Then this is a maximal chain, as required. Note that (2. $\Rightarrow$ 3.) is immediate.

(3. $\Rightarrow$ 1.) Let $\mathrm{Supp}(M) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. As showed in the forward part of proof of Theorem 8, $R/\mathrm{Ann}(M)$ is of dimension 0 if and only if $\mathrm{Supp}(M)$ consists of finitely many maximals. The result follows from the other part of the theorem. □

The following lemma shows how one should generalize the valuation of a rational function at a codimension 1 subvariety.

Lemma 10. *Let R be a DVR with valuation $v : K \to \mathbf{Z}$ and fix $a \in R$. Then,*

$$v(a) = \mathrm{len}_R\, R/aR.$$

Proof. Let $t \in R$ be the local parameter of R. Then $a = ut^n$ where $u \in R^\times$ is a unit and $n \geq 0$. As $v(a) = n$, therefore we must show $\mathrm{len}_R\, R/aR = n$. Indeed, as R is a DVR, therefore every ideal of R is a principally generated by a power of t. Hence, $R/aR = R/t^nR$. We first show that $M = R/t^nR$ is a finite length R-module. Indeed, as $\mathrm{Ann}(R) = t^nR$, therefore by Theorem 8 we must show R/t^nR is a dimension 0 ring. Indeed, as R has only two primes, therefore R/t^nR has only one prime. It follows that $\dim R/t^nR = 0$, as required.

The only chain of ideals in R/t^nR is the image of the following chain in R:

$$t^nR \subsetneq t^{n-1}R \subsetneq \cdots \subsetneq t^2R \subsetneq tR = \mathfrak{m} \subsetneq R.$$

Thus, $\mathrm{len}_R\, R/t^nR = n$, as required. □

The following is an essential property of lengths.

Lemma 11. *Let R be a ring such that for $f, g \in R$, the R-modules $R/fR, R/gR$ and R/fgR has finite length. Then,*

$$\mathrm{len}_R\, R/fgR = \mathrm{len}_R\, R/fR + \mathrm{len}_R\, R/gR.$$

Proof. Follows from simple comparions of chains of ideals. □

Here's a more general result.

Lemma 12. *Let M be a finite length R-module. Then,*

$$\mathrm{len}_R(M) = \sum_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{len}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \sum_{\mathfrak{m} \in \mathrm{Supp}(M)} \mathrm{len}_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. We first have to show that the sum is finite. Indeed, observe that as M has finite length, therefore M is finitely generated and $R/\mathrm{Supp}\,(M)$ is an artinian ring by Theorem 8. Consequently, $\mathrm{Supp}\,(M) = V(\mathrm{Ann}(M))$ has only finitely many maximal ideals of R. This shows that the sum is finite.

To show the equality, it suffices to show that each maximal $\mathfrak{m} \in \mathrm{Supp}(M)$ occurs $\mathrm{len}_{R_\mathfrak{m}}(M_\mathfrak{m})$ many times as a composition factor in any composition series of M. Take a composition series $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ of M with $M_i/M_{i-1} \cong R/\mathfrak{m}_\mathfrak{i}$ where $\mathfrak{m}_i \in \mathrm{Supp}\,(M)$ and fix $\mathfrak{m} \in \mathrm{Supp}\,(M)$. Localizing at $\mathfrak{m}$, we get the chain $(M_0)_\mathfrak{m} \subseteq (M_1)_\mathfrak{m} \subseteq \cdots \subseteq (M_r)_\mathfrak{m}$ where the subquotient $(M_i/M_{i-1})_\mathfrak{m} \cong (R/\mathfrak{m}_i)_\mathfrak{m} \cong R/\mathfrak{m}_i \cong R_\mathfrak{m}/\mathfrak{m}_i R_\mathfrak{m}$. Hence, $\mathfrak{m}_i R_\mathfrak{m}$ is the maximal ideal of $R_\mathfrak{m}$, showing that $\mathfrak{m}_i = \mathfrak{m}$. Hence $\mathfrak{m}$ appears $\mathrm{len}_{R_\mathfrak{m}}(M_\mathfrak{m})$-many times as a composition factor of the chain of M, as required. ☐

We deal exclusively with length of modules over local rings. The following therefore shows the effect on length of a module under a map of local rings.

Proposition 13. *Let $\varphi : A \to B$ be a local homomorphism of local rings. We thus have a field extension of residue fields $\kappa(B)/\kappa(A)$. Then the following are equivalent for a B-module M:*

1. *As an A-module via φ, we have $\mathrm{len}_A(M) < \infty$.*
2. *We have $\mathrm{len}_B(M) < \infty$ and $[\kappa(B) : \kappa(A)] < \infty$.*

Moreover, if any of the above is satisfied, then

$$\mathrm{len}_A(M) = [\kappa(B) : \kappa(A)] \cdot \mathrm{len}_B(M).$$

Proof. Let $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$ be a composition series of M as a B-module. Hence $\mathrm{len}_B(M) = n$. As B is local, therefore each subquotient is isomorphic to $\kappa(B)$. Now for any $1 \le i \le n$, we have the following exact sequence of A-modules:

$$0 \to M_{i-1} \to M_i \to \kappa(B) \to 0.$$

By Lemma 4, we have the following equalities:

$$\begin{aligned}\mathrm{len}_A(M) &= \mathrm{len}_A(M_{n-1}) + \mathrm{len}_A(\kappa(B)) \\ &\;\vdots \\ &= \mathrm{len}_A(M_0) + n \cdot \mathrm{len}_A(\kappa(B)).\end{aligned}$$

Consequently, $\mathrm{len}_A(M) = \mathrm{len}_B(M) \cdot \mathrm{len}_A(\kappa(B))$. We need only show that $\mathrm{len}_A(\kappa(B)) = [\kappa(B) : \kappa(A)]$. As $\mathrm{len}_A(\kappa(B)) = \mathrm{len}_{A/\mathfrak{m}_A} \kappa(B)$ since $\mathfrak{m}_A \cdot \kappa(B) = 0$, therefore $\mathrm{len}_A(\kappa(B)) = \mathrm{len}_{\kappa(A)} \kappa(B) = [\kappa(B) : \kappa(A)]$, as required. ☐

Corollary 14. *Let $\varphi : A \to B$ be a local homomorphism of local rings. We thus have a field extension of residue fields $\kappa(B)/\kappa(A)$. Let $f \in A$ be a non zero-divisor. Let M be a B-module be a finite length modules and $\kappa(B)/\kappa(A)$ is a finite extension. Then*

$$\mathrm{len}_A\left(\frac{M}{fM}\right) = [\kappa(B) : \kappa(A)] \cdot \mathrm{len}_B\left(\frac{M}{fM}\right).$$

Proof. Immediate from Proposition 13. ☐

Corollary 15. *Let A be a ring, M be a finite length A-module and $f \in A$ be a non zero-divisor. Then,*

$$\operatorname{len}_A\left(\frac{M}{fM}\right) = \sum_{\mathfrak{p}\in \operatorname{Supp}(M/fM)} \operatorname{len}_{A_\mathfrak{p}}\left(\frac{M_\mathfrak{p}}{fM_\mathfrak{p}}\right).$$

Proof. Immediate from Lemma 12 □

Lemma 16. *Let $(A, \mathfrak{m})$ be a local ring, $\psi : A \to B$ be a finite A-algebra and M be a finite length B-module. If $\mathfrak{m}_i \in \operatorname{Spec}(B)$ are the finitely many maximal ideals of B such that $\psi^{-1}(\mathfrak{m}_i) = \mathfrak{m}$, then*

$$\sum_i \operatorname{len}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \cdot [\kappa(B_{\mathfrak{m}_i}) : \kappa(A)] = \operatorname{len}_A(M).$$

Proof. Follows from Proposition 13. □

The following lemma relates the length of the length of a flat local A-algebra with that of A.

Lemma 17. *Let A and B be local rings and $\varphi : A \to B$ be a flat map. Then*

1. *The induced map $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective,*
2. *If A and B are Artinian local rings, then*

$$\operatorname{len}_B(B) = \operatorname{len}_A(A) \cdot \operatorname{len}_B B/\mathfrak{m}_A B.$$

Proof. 1. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ be a prime. We wish to find $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Going modulo $\mathfrak{p}$, we get the map $\bar{\varphi} : A/\mathfrak{p} \to B/\mathfrak{p}B$. This map is further flat as base change of a flat map is flat. We thus reduce to assuming that A is a domain and $\mathfrak{p} = 0$. Observe that $\varphi(a)$ in B is a non zero-divisor for each non-zero $a \in A$ since $0 \to A \xrightarrow{\times a} A$ remains injective by flatness of B. Thus $\operatorname{Im}(\varphi)$ consists of non zero-divisors of B. Any prime corresponding to $B/\varphi(A) \cdot B$ will then work. If this quotient is zero, then B is a domain and hence zero ideal will work.

2. By Theorem 6, ring B has finite length, say r. Thus we have a maximal chain of ideals of A

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = A.$$

Consequently, I_i/I_{i-1} is a simple A-modules, that is,

$$I_i/I_{i-1} \cong A/\mathfrak{m}_A.$$

As B is a flat A-algebra, therefore by tensoring with B, we get a chain of ideals of B

$$0 = I_0B \subseteq I_1B \subseteq \cdots \subseteq I_rB = B.$$

Flatness further yields that $I_iB/I_{i-1}B \cong I_i/I_{i-1} \otimes_A B \cong A/\mathfrak{m}_\mathfrak{A} \otimes_A B \cong B/\mathfrak{m}_A B$. Since the following is exact

$$0 \to I_{i-1}B \to I_iB \to B/\mathfrak{m}_\mathfrak{A} B \to 0,$$

thus by Lemma 4 (additivity of length), we have the recurrence relation

$$\operatorname{len}_B I_iB = \operatorname{len}_B I_{i-1}B + \operatorname{len}_B B/\mathfrak{m}_A B.$$

From this, the result follows at once. □