Strongly Local Constructions & Normalization

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1 Strongly local constructions on rings

A lot of times we have the situation that a certain construction on a ring A leads to a map $\varphi : A \to \tilde{A}$. Consequently, we obtain maps $f : \operatorname{Spec}\left(\tilde{A}\right) \to \operatorname{Spec}(A)$. If X is a scheme, then for each open affine $V_i = \operatorname{Spec}(A_i)$, we thus get a map $\operatorname{Spec}\left(\tilde{A}_i\right) \to V_i$. Consequently, we are interested in the conditions that the construction $\varphi : A \to \tilde{A}$ must satisfy so that $X_i = \operatorname{Spec}\left(\tilde{A}_i\right)$ glue together to give a scheme $\tilde{X} \to X$, which would thus represent the local construction globally.

The main theorem is Theorem 1.9.

Definition 1.1 (**Construction on rings**)**.** A construction on rings is a collection of maps $\{\varphi_A : A \to \tilde{A}\}$ one for each ring A such that for any isomorphism $\eta_{AB} : A \overset{\cong}{\to} B$, we have an isomorphism $\tilde{\eta}_{AB} : \tilde{A} \to \tilde{B}$ which is id if η is id, the diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi_A"] \arrow[d, "\eta_{AB}"', "\cong"] & \tilde{A} \arrow[d, "\tilde{\eta}_{AB}"', "\cong"] \\
B \arrow[r, "\varphi_B"'] & \tilde{B}
\end{tikzcd}$$

commutes and if $\eta_{BC} \circ \eta_{AB} = \eta_{AC}$, then $\tilde{\eta}_{BC} \circ \tilde{\eta}_{AB} = \tilde{\eta}_{AC}$. That is, we demand constructions to be functorial on isomorphisms.

Definition 1.2 (**Strongly local constructions**)**.** A construction on rings $\{\varphi_A : A \to \tilde{A}\}$ is said to be strongly local if it naturally commutes with localization. That is, for each $g \in A$ not in nilradical, there exists an isomorphism $\widetilde{A_g} \cong \tilde{A}_g$ such that

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi_A"'] & A_g \arrow[d, "(\varphi_A)_g"'] \arrow[dr, "\varphi_{A_g}"] & \\
\tilde{A} \arrow[r] & \tilde{A}_g \arrow[r, "\cong"'] & \widetilde{A_g}
\end{tikzcd}$$

commutes where $(\varphi_A)_g : A_g \to \tilde{A}_g$ is the localization of map $\varphi_A : A \to \tilde{A}$ at the element $g \in A$ and the horizontal arrows of the square are localization maps.

A canonical example of strongly local construction is normalization of domains.

Remark 1.3 (*Normalization is a strongly local construction*)**.** Let A be an arbitrary domain. Then we get an inclusion $\varphi_A : A \hookrightarrow \tilde{A}$ where $\tilde{A}$ is the normalization of A in its fraction field. We claim that the collection of maps $\{\varphi_A : A \hookrightarrow \tilde{A}\}$ one for each domain is a construction which is strongly local on domains (see Definitions 1.1 & 1.2).

Indeed, first $\{\varphi_A : A \hookrightarrow \tilde{A}\}$ is a construction on domains as if $\eta : A \to B$ is an isomorphism, then we have an isomorphism $\tilde{\eta} : \tilde{A} \to \tilde{B}$ given as follows: we have an isomorphism $\bar{\eta} : K_A \to K_B$ between their fraction fields, given by $a/a' \mapsto \eta(a)/\eta(b)$. Now $a/a' \in K_A$ is integral over A if and only if $\eta(a)/\eta(a') \in K_B$ is integral over B. This shows that $\bar{\eta} : K_A \to K_B$ restricts to an isomorphism $\tilde{\eta} : \tilde{A} \to \tilde{B}$. Moreover, if $\eta : A \to A$ is id, then so is $\tilde{\eta}$ and it satisfies the square and cocycle condition as well of Definition 1.1. We now claim that normalization is strongly local.

Indeed, pick $g \in A$ non-zero. Then, the localization of the inclusion $\varphi_A : A \hookrightarrow \tilde{A}$ at element g yields $(\varphi_A)_g : A_g \hookrightarrow \tilde{A}_g = \widetilde{A_g}$ which is equal to the normalization of the domain $\varphi_{A_g} : A_g \hookrightarrow \widetilde{A_g}$ as normalization commutes with localization.

Remark 1.4. Let $\eta : A_f \cong B_g$ be an isomorhism where $f \in A$ and $g \in B$. Then we get an isomorphism $\hat{\eta} : \tilde{A}_f \cong \tilde{B}_g$ as in the following commutative diagram:

$$\begin{tikzcd}
\tilde{A}_f \arrow[r, dashed, "\hat{\eta}"] & \tilde{B}_g \\
\widetilde{A_f} \arrow[u, "\cong"] \arrow[r, "\cong", "\tilde{\eta}"'] & \widetilde{B_g} \arrow[u, "\cong"'] \\
A_f \arrow[u, "\varphi_{A_f}"'] \arrow[uu, bend left=60, "(\varphi_A)_f"] \arrow[r, "\cong", "\eta"'] & B_g \arrow[u, "\varphi_{B_g}"] \arrow[uu, bend right=60, "(\varphi_B)_g"']
\end{tikzcd}.$$

Let X be a scheme. Our main goal is to show that strongly local constructions done on each affine open subset of X can be glued to give a scheme $\tilde{X}$ admitting a map $\tilde{X} \to X$.

We will achieve this in steps. We first translate strongly local property more geometrically.

Lemma 1.5. *Let $\{\varphi_A : A \to \tilde{A}\}$ be a strongly local construction on rings. For any ring A denote $\phi_A : \operatorname{Spec}\left(\tilde{A}\right) \to \operatorname{Spec}(A)$ to be the map corresponding to φ_A. Then, for any $f \in A$ not in nilradical, the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Spec}(A_f) & & \operatorname{Spec}\left(\tilde{A}_f\right) \arrow[ll, "\phi_A|_{\operatorname{Spec}\left(\tilde{A}_f\right)}"'] \arrow[dl, "\cong"] \\
& \operatorname{Spec}\left(\widetilde{A_f}\right) \arrow[ul, "\phi_{A_f}"] &
\end{tikzcd}.$$

Proof. This is the translation of Definition 1.1 in $\operatorname{Spec}(-)$ where localization amounts to restricting to the corresponding open subscheme. $\square$

The following is an important observation which will help in checking the cocycle condition.

Lemma 1.6. *Let* $\{\varphi_A : A \to \tilde{A}\}$ *be a strongly local construction on rings and the following be a commutative triangle of isomorphisms*

$$\begin{tikzcd}
R_f \arrow[r] \arrow[dr] & S_g \arrow[d] \\
 & T_h
\end{tikzcd}$$

for $f \in R$, $g \in S$ *and* $h \in T$. *Then, the following triangle of isomorphisms as constructed in Remark 1.4 also commutes*

$$\begin{tikzcd}
\tilde{R}_f \arrow[r] \arrow[dr] & \tilde{S}_g \arrow[d] \\
 & \tilde{T}_h
\end{tikzcd} .$$

Proof. By definition of a construction, we get that the following triangle commutes

$$\begin{tikzcd}
\widetilde{R_f} \arrow[r] \arrow[dr] & \widetilde{S_g} \arrow[d] \\
 & \widetilde{T_h}
\end{tikzcd} .$$

By the construction of isomorphism $\tilde{R}_f \to \tilde{S}_g$ and others as in Remark 1.4, we immediately get that the required triangle commutes. ☐

Lemma 1.7. *Let* $X = \operatorname{Spec}(A)$ *and* $Y = \operatorname{Spec}(B)$ *be two affine schemes. Let* R *be a ring with isomorphisms* $A_f \cong R \cong B_g$ *for some* $f \in A$ *and* $g \in B$. *Let* $\{\varphi_S : S \to \tilde{S}\}$ *be a strongly local construction on rings. Then there are open immersions* $\operatorname{Spec}\left(\tilde{R}\right) \hookrightarrow \operatorname{Spec}\left(\tilde{A}\right)$ *and* $\operatorname{Spec}\left(\tilde{R}\right) \hookrightarrow \operatorname{Spec}\left(\tilde{B}\right)$ *so that the following commutes*

$$\begin{tikzcd}
\operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\phi_A"'] & \operatorname{Spec}\left(\tilde{R}\right) \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\phi_R"'] & \operatorname{Spec}\left(\tilde{B}\right) \arrow[d, "\phi_B"] \\
\operatorname{Spec}(A) & \operatorname{Spec}(R) \arrow[l, hook'] \arrow[r, hook] & \operatorname{Spec}(B)
\end{tikzcd} .$$

Proof. This follows from the following diagram

$$\begin{tikzcd}
\operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\phi_A"'] & \operatorname{Spec}\left(\tilde{A}_f\right) \cong \operatorname{Spec}\left(\widetilde{A_f}\right) \arrow[l, hook'] \arrow[d, "\phi_A|_{\operatorname{Spec}\left(\tilde{A}_f\right)}"] & \operatorname{Spec}\left(\tilde{R}\right) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\phi_R"] & \operatorname{Spec}\left(\widetilde{B_g}\right) \cong \operatorname{Spec}\left(\tilde{B}_g\right) \arrow[r, hook] \arrow[d, "\phi_B|_{\operatorname{Spec}(\tilde{B}_g)}"] & \operatorname{Spec}\left(\tilde{B}\right) \arrow[d, "\phi_B"] \\
\operatorname{Spec}(A) & \operatorname{Spec}(A_f) \arrow[l, hook'] & \operatorname{Spec}(R) \arrow[l, "\cong"] \arrow[r, "\cong"'] & \operatorname{Spec}(B_g) \arrow[r, hook] & \operatorname{Spec}(B)
\end{tikzcd}$$

the commutativity of which follows from Lemma 1.5 and the definition of a construction. ☐

Let X be a scheme and $U = \mathrm{Spec}\,(A)$ and $V = \mathrm{Spec}\,(B)$ be two open affines. We can now glue $\mathrm{Spec}\left(\tilde{A}\right)$ and $\mathrm{Spec}\left(\tilde{B}\right)$ along the intersection $U \cap V$ as follows.

Proposition 1.8. *Let X be a scheme and $U = \mathrm{Spec}\,(A)$ and $V = \mathrm{Spec}\,(B)$ be two open affines. Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Let $\phi_A : \tilde{U} = \mathrm{Spec}\left(\tilde{A}\right) \to \mathrm{Spec}\,(A)$ and $\phi_B : \tilde{V} = \mathrm{Spec}\left(\tilde{B}\right) \to \mathrm{Spec}\,(B)$ be the maps corresponding to φ_A and φ_B. Then, there exists an isomorphism of schemes*

$$\Theta : \phi_A^{-1}(U \cap V) \xrightarrow{\cong} \phi_B^{-1}(U \cap V)$$

such that the following commutes for any affine open $\mathrm{Spec}\,(R) \subseteq U \cap V$ which is basic in both U and V by the isomorphisms $A_f \cong R \cong B_g$

$$\begin{tikzcd}
\phi_A^{-1}(U \cap V) \arrow[r, "\Theta", "\cong"'] & \phi_B^{-1}(U \cap V) \\
\mathrm{Spec}\left(\tilde{A}_f\right) \arrow[u, hook] \arrow[r, "\cong", "\Theta_f"'] & \mathrm{Spec}\left(\tilde{B}_g\right) \arrow[u, hook]
\end{tikzcd}$$

where Θ_f is obtained from $\theta : A_f \cong B_g$ via˜ construction (Remark 1.4).

Proof. Cover $U \cap V$ by open affines which are basic in both U and V and write $U \cap V = \bigcup_{i \in I} \mathrm{Spec}\,(A_{f_i}) = \bigcup_{i \in I} \mathrm{Spec}\,(B_{g_i})$ where $f_i \in A$ and $g_i \in B$. Consequently we may write

$$\phi_A^{-1}(U \cap V) = \bigcup_{i \in I} \phi_A^{-1}(\mathrm{Spec}\,(A_{f_i})) = \bigcup_{i \in I} \mathrm{Spec}\left(\tilde{A}_{f_i}\right)$$

and thus similarly,

$$\phi_B^{-1}(U \cap V) = \bigcup_{i \in I} \mathrm{Spec}\left(\tilde{B}_{g_i}\right).$$

For each $i \in I$, Lemma 1.7 provides us with an isomorphism

$$\Theta_i : \mathrm{Spec}\left(\tilde{A}_{f_i}\right) \xrightarrow{\cong} \mathrm{Spec}\left(\tilde{B}_{g_i}\right) \hookrightarrow \tilde{V}.$$

We claim that Θ_i can be glued. Indeed, for $i \neq j$, we have $\mathrm{Spec}\left(\tilde{A}_{f_i}\right) \cap \mathrm{Spec}\left(\tilde{A}_{f_j}\right) = \mathrm{Spec}\left(\tilde{A}_{f_i f_j}\right)$, therefore we reduce to showing that Θ_i and Θ_j are equal when restricted to $\mathrm{Spec}\left(\tilde{A}_{f_i f_j}\right)$. We know that the isomorphism $A_{f_i} \cong B_{g_i}$ takes $f_i \mapsto g_i$. The above is now equivalent to showing that the isomorphisms $\theta_i : \tilde{A}_{f_i} \cong \tilde{B}_{g_i}$ and $\theta_j : \tilde{A}_{f_j} \cong \tilde{B}_{g_j}$ obtained from $A_{f_i} \cong B_{g_i}$ and $A_{f_j} \cong B_{g_j}$ fit in the following commutative diagram

$$\begin{tikzcd}
\tilde{A}_{f_i f_j} \arrow[r, "(\theta_i)_{f_j}"] \arrow[d, equal, "\mathrm{id}"'] & \tilde{B}_{g_i f_j} \arrow[d, equal, "\mathrm{id}"] \\
\tilde{A}_{f_j f_i} \arrow[r, "(\theta_j)_{f_i}"'] & \tilde{B}_{g_j f_i}
\end{tikzcd} .$$

But $\theta_i(f_j) = g_j$ and $\theta_j(f_i) = g_i$, as mentioned above. Therefore $\tilde{B}_{g_i f_j} = \tilde{B}_{g_i g_j} = \tilde{B}_{g_j g_i} = \tilde{B}_{g_j f_i}$ and the above square commutes, showing that Θ_i glues to give a map $\Theta : \phi_A^{-1}(U \cap V) \to \phi_B^{-1}(U \cap V)$, which is an isomorphism as locally it is an isomorphism. □

Using Proposition 1.8, we can now globalize a strongly local construction.

Theorem 1.9. *Let X be a scheme and $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction on rings. Then there exists a scheme $\alpha : \tilde{X} \to X$ such that for any affine open* $\operatorname{Spec}(A) \hookrightarrow X$*, the following square commutes*

$$\begin{tikzcd}
\operatorname{Spec}\left(\tilde{A}\right) \arrow[r, hook] \arrow[d, "\phi_A"'] & \tilde{X} \arrow[d, "\alpha"] \\
\operatorname{Spec}(A) \arrow[r, hook] & X
\end{tikzcd}.$$

Proof. We first construct $\tilde{X}$ by gluing each $\operatorname{Spec}\left(\tilde{A}\right)$. Indeed, let $\{V_i = \operatorname{Spec}(A_i)\}_{i\in I}$ be the collection of affine opens in X and let $\{\tilde{X}_i = \operatorname{Spec}\left(\tilde{A}_i\right)\}$ be the collection of corresponding $\tilde{}$-constructions. Let $\phi_i : \tilde{X}_i \to V_i$ be the maps corresponding to φ_{A_i}.

For each $i \neq j \in I$ we wish to construct open subschemes $U_{ij} \subseteq \tilde{X}_i$ and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ satisfying the gluing conditions. We let

$$U_{ij} = \phi_i^{-1}(V_i \cap V_j).$$

Then Proposition 1.8 provides us with an isomorphism

$$\varphi_{ij} : U_{ij} \xrightarrow{\cong} U_{ji}.$$

It is immediate that $U_{ii} = \tilde{X}_i$ and $\varphi_{ii} = \mathrm{id}_{U_{ii}}$. Moreover, $\varphi_{ji} = \varphi_{ij}^{-1}$ by construction. We now check the cocycle condition. Indeed, pick $i, j, k \in I$ and pick an open affine $\operatorname{Spec}(R) \subseteq V_i \cap V_j \cap V_k$ in X which is basic open in V_i, V_j and V_k such that we have isomorphisms $A_{i,f_i} \cong A_{j,f_j} \cong A_{k,f_k} \cong R$ so that the following triangle commutes

$$\begin{tikzcd}
A_{i,f_i} \arrow[r, "\cong"] \arrow[dr, "\cong"'] & A_{j,f_j} \arrow[d, "\cong"] \\
& A_{k,f_k}
\end{tikzcd}. \tag{$*$}$$

By taking inverse images under ϕ_i, it follows that $\operatorname{Spec}\left(\tilde{A}_{i,f_i}\right) \subseteq U_{ij} \cap U_{ik}$ is basic open in both $\tilde{X}_i$ and $\tilde{X}_j$. We wish to show that φ_{ik} restricted to $\operatorname{Spec}\left(\tilde{A}_{i,f_i}\right)$ is the composition $\varphi_{jk} \circ \varphi_{ij}$. By Proposition 1.8, we get that φ_{ik} on this open affine is an isomorphism to $\operatorname{Spec}\left(\tilde{A}_{k,f_k}\right)$ and φ_{ij} is an isomorphism to $\operatorname{Spec}\left(\tilde{A}_{j,f_j}\right)$. Consequently, we wish to show that the following triangle of isomorphisms commute

$$\begin{tikzcd}
\operatorname{Spec}\left(\tilde{A}_{i,f_i}\right) \arrow[r, "\varphi_{ij}"] \arrow[dr, "\varphi_{ik}"'] & \operatorname{Spec}\left(\tilde{A}_{j,f_j}\right) \arrow[d, "\varphi_{jk}"] \\
& \operatorname{Spec}\left(\tilde{A}_{k,f_k}\right)
\end{tikzcd}.$$

But these isomorphisms are obtained by the following isomorphisms on the localizations (Proposition 1.8):

$$\begin{tikzcd}
\tilde{A}_{i,f_i} \arrow[r, "\cong"] \arrow[dr, "\cong"'] & \tilde{A}_{j,f_j} \arrow[d, "\cong"] \\
& \tilde{A}_{k,f_k}
\end{tikzcd} .$$

Hence it suffices to show that the above triangle commutes. The Lemma 1.6 applied on $(*)$ yields the required commutativity. $\square$

Definition 1.10 ($\tilde{}$**-fication**)**.** Let $\{\varphi_S : S \to \tilde{S}\}$ be a strongly local construction of rings and let X be a scheme. The scheme $\tilde{X} \to X$ obtained in Theorem 1.9 is called the$\tilde{}$-fication of X.

2 Normalization

Using Theorem 1.9 and Remark 1.3, we can immediately obtain a normal integral scheme out of an integral scheme. However, to prove universal property of normalization and the fact that it glues requires some work.

The following is immediate from local nature of normal domains.

Lemma 2.1. *Let X be an integral scheme. Then the following are equivalent:*

1. *X is a normal scheme.*
2. *For all open affine* $\operatorname{Spec}(A) \subseteq X$*, the ring A is a normal domain.*

Proof. As X is integral, therefore for every open affine $\operatorname{Spec}(A)$ of X, A is a domain. As X is normal iff $\mathscr{O}_{X,x}$ is a normal domain for all $x \in X$, the result follows from local nature of normal domains (R is normal iff $R_{\mathfrak{p}}$ is normal for each $\mathfrak{p} \in \operatorname{Spec}(R)$). $\square$

We have a universal property for normalization of domains, which we will later globalize to an arbitrary integral scheme.

Proposition 2.2. *Let A be a domain and $\tilde{A}$ be the normalization of A in its fraction field. Then for any normal domain B and an injective map $A \hookrightarrow B$, there exists a unique map $\tilde{A} \to B$ such that following commutes:*

$$\begin{tikzcd}
\tilde{A} \arrow[r, dashed] & B \\
A \arrow[u, hook] \arrow[ur, hook]
\end{tikzcd} .$$

Proof. Let $f : A \hookrightarrow B$. This, by universal property of fraction fields, induces a unique injective map $\varphi : K \hookrightarrow L$ from fraction field of A to that of B such that $\varphi|_A = f$ (this is where injectivity of f is used). Let $x \in \tilde{A}$. Then

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

holds in K where $a_i \in A$. Applying φ on the above equation yields

$$\varphi(x)^n + f(a_{n-1})\varphi(x)^{n-1} + \cdots + f(a_1)\varphi(x) + f(a_0) = 0$$

in L. It follows that $\varphi(x)$ is an integral element of L over B. As B is normal it follows that $\varphi(x) \in B$. Consequently, we have a unique map

$$\varphi|_{\tilde{A}} : \tilde{A} \to B$$

such that the triangle commutes, as required. ☐

The main result in normal schemes is that any integral scheme induces a unique normal scheme obtained by normalizing each open affine.

Theorem 2.3. [1] *Let X be an integral scheme. Then there exists a scheme $\tilde{X} \to X$ over X where $\tilde{X}$ is a normal integral scheme such that for any normal integral scheme Z and a dominant map $f : Z \to X$, there exists a unique map $\tilde{f} : Z \to \tilde{X}$ such that the following commutes*

$$\begin{tikzcd}
\tilde{X} \arrow[d] & Z \arrow[l, dashed, "\tilde{f}"'] \arrow[dl, "f"] \\
X &
\end{tikzcd}.$$

The scheme $\tilde{X} \to X$ is called the normalization of X and is unique upto isomorphism.

We first see this for affine domains.

Lemma 2.4. *Let $X = \operatorname{Spec}(A)$ be an integral affine scheme and $Z = \operatorname{Spec}(B)$ be a normal integral affine scheme. Let $\tilde{X} = \operatorname{Spec}\left(\tilde{A}\right)$ be the normalization of X and denote the natural map $\pi : \tilde{X} \to X$. If $f : Z \to X$ is any dominant map, then there exists a map $\tilde{f} : Z \to \tilde{X}$ such that $\pi \circ \tilde{f} = f$.*

$$\begin{tikzcd}
\operatorname{Spec}\left(\tilde{A}\right) \arrow[d, "\pi"'] & \operatorname{Spec}(B) \arrow[l, dashed, "\tilde{f}"'] \arrow[dl, "f"] \\
\operatorname{Spec}(A) &
\end{tikzcd}.$$

Proof. Indeed, by applying $\operatorname{Spec}(-)$ on Proposition 2.2, this follows immediately where we know that injective map of rings gives a dominant map on affine schemes. ☐

Remark 2.5. By Remark 1.3, it follows that normalization is a strongly local property. Thus Theorem 2.3 holds for normalization.

Proof of Theorem 2.3. By Remark 1.3, it follows that normalization is a strongly local construction for domains. Let $A \hookrightarrow \tilde{A}$ be the normalization map for any domain A. Therefore

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[1] Exercise II.3.8 of Hartshorne.

by Theorem 1.9, we have a scheme $\alpha : \tilde{X} \to X$ such that for any open affine $\mathrm{Spec}\,(A) \hookrightarrow X$, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Spec}\left(\tilde{A}\right) \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d, "\alpha"] \\
\mathrm{Spec}\,(A) \arrow[r, hook] & X
\end{tikzcd}$$

where the left vertical map is the map corresponding to normalization $A \hookrightarrow \tilde{A}$. This shows the construction of $\alpha : \tilde{X} \to X$.

We now show the required universal property of normalization. Let Z be an arbitrary normal integral scheme and $f : Z \to X$ be a dominant map. Pick any open affine $\mathrm{Spec}\,(A) \subseteq X$ and consider the non-empty (f is dominant) open subset $f^{-1}(\mathrm{Spec}\,(A))$. Write

$$f^{-1}(\mathrm{Spec}\,(A)) = \bigcup_{i \in I} \mathrm{Spec}\,(B_i)$$

where $\mathrm{Spec}\,(B_i) \subseteq Z$ are open affine. As Z is normal integral, therefore B_i are normal domains from Lemma 2.1. By restriction we thus have the map

$$f|_{\mathrm{Spec}(B_i)} : \mathrm{Spec}\,(B_i) \to \mathrm{Spec}\,(A)$$

for each $i \in I$. Observe that $\alpha^{-1}(\mathrm{Spec}\,(A)) \supseteq \mathrm{Spec}\left(\tilde{A}\right)$. By Lemma 2.4, it follows that we have a unique map $\tilde{f_i} : \mathrm{Spec}\,(B_i) \to \mathrm{Spec}\left(\tilde{A}\right)$ such that the following commutes

$$\begin{tikzcd}
\mathrm{Spec}\left(\tilde{A}\right) \arrow[d, "\alpha|_{\mathrm{Spec}(\tilde{A})}"'] & \mathrm{Spec}\,(B_i) \arrow[l, dashed, "\tilde{f_i}"'] \arrow[dl, "f|_{\mathrm{Spec}(B_i)}"] \\
\mathrm{Spec}\,(A) &
\end{tikzcd}.$$

It thus follows that for every open affine $\mathrm{Spec}\,(B_{ij}) \subseteq \mathrm{Spec}\,(B_i)$, we have a map $\tilde{f_i} : \mathrm{Spec}\,(B_i) \to \mathrm{Spec}\left(\tilde{A}\right)$ by restriction. Hence by Lemma 2.4, we have that this is unique. As $\mathrm{Spec}\,(A) \subseteq X$ is arbitrary open affine, therefore we have an open affine covering $\{\mathrm{Spec}\,(A_i)\}_{i \in I}$ of X which by inverse image gives an open affine covering $\{\mathrm{Spec}\,(B_{ij})\}$ of Z and a collection of open affines $\{\mathrm{Spec}\left(\tilde{A_i}\right)\}$ of $\tilde{X}$ such that for each i, we have a unique map $\tilde{f_{ij}} : \mathrm{Spec}\,(B_{ij}) \to \tilde{X}$ such that

$$\begin{tikzcd}
\tilde{X} \arrow[d, "\alpha"'] & \mathrm{Spec}\left(\tilde{A_i}\right) \arrow[l, hook'] \arrow[d, "\alpha"'] & \mathrm{Spec}\,(B_{ij}) \arrow[l, dashed, "\tilde{f_{ij}}"'] \arrow[dl, "f"] \\
X & \mathrm{Spec}\,(A_i) \arrow[l, hook'] &
\end{tikzcd}$$

commutes. We claim that $\tilde{f_{ij}}$ can be glued to a unique map $\tilde{f} : Z \to \tilde{X}$, which would complete the proof. First, for a fixed i, we glue $\tilde{f_{ij}}$ and $\tilde{f_{il}}$. Indeed, covering the intersection

$\operatorname{Spec}(B_{ij}) \cap \operatorname{Spec}(B)_{il}$ by open affines $\operatorname{Spec}(C_p)$, we immediately by restriction get maps $\tilde{f}_{ij} : \operatorname{Spec}(C_p) \to \operatorname{Spec}\left(\tilde{A}_i\right)$ and $\tilde{f}_{il} : \operatorname{Spec}(C_p) \to \operatorname{Spec}\left(\tilde{A}_i\right)$ which are thus equal by uniqueness. Hence, for each i, we may glue the maps $\{\tilde{f}_{ij}\}_j$ to obtain a unique map $\tilde{f}_i : Z_i = f^{-1}(\operatorname{Spec}(A_i)) \to \operatorname{Spec}\left(\tilde{A}_i\right)$ as in

$$\begin{tikzcd}
\operatorname{Spec}\left(\tilde{A}_i\right) \arrow[d, "\alpha"'] & Z_i \arrow[l, dashed, "\tilde{f}_i"'] \arrow[dl, "f"] \\
\operatorname{Spec}(A_i) &
\end{tikzcd}.$$

We now wish to glue these $\tilde{f}_i$. To this end, pick an affine open $\operatorname{Spec}(C) \subseteq Z_i \cap Z_k = f^{-1}(\operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k))$ and observe $\alpha^{-1}(\operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k)) \supseteq \operatorname{Spec}\left(\tilde{A}_i\right) \cap \operatorname{Spec}\left(\tilde{A}_k\right)$. We thus have the following diagram

$$\begin{tikzcd}
\operatorname{Spec}\left(\tilde{A}_i\right) \arrow[d, "\alpha"'] & \operatorname{Spec}(C) \arrow[l, "\tilde{f}_i"'] \arrow[r, "\tilde{f}_k"] \arrow[d, "f"] & \operatorname{Spec}\left(\tilde{A}_k\right) \arrow[d, "\alpha"] \\
\operatorname{Spec}(A_i) & \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k) \arrow[l, hook'] \arrow[r, hook] & \operatorname{Spec}(A_k)
\end{tikzcd}.$$

By Lemma 2.4, it then suffices to show that $\tilde{f}_i(\operatorname{Spec}(C)), \tilde{f}_k(\operatorname{Spec}(C)) \subseteq \operatorname{Spec}\left(\tilde{A}_i\right) \cap \operatorname{Spec}\left(\tilde{A}_k\right)$, as then uniqueness would imply $\tilde{f}_i$ and $\tilde{f}_k$ are equal over $\operatorname{Spec}(C)$. By symmetry, it suffices to show this for $\tilde{f}_i$. Since $\alpha \circ \tilde{f}_i(\operatorname{Spec}(C)) \subseteq \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k)$, therefore $\tilde{f}_i(\operatorname{Spec}(C)) \subseteq \alpha^{-1}(\operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k)) \cap \operatorname{Spec}\left(\tilde{A}_i\right) \subseteq \operatorname{Spec}\left(\tilde{A}_i\right) \cap \operatorname{Spec}\left(\tilde{A}_k\right)$, as required. Hence $\tilde{f}_i$ can be glued to a unique map $\tilde{f} : Z \to \tilde{X}$, thus completing the proof. □