SWAN'S THEOREM FOR COMPLETELY REGULAR SPACES

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Abstract. A well known theorem of Swan establishes an equivalence between vector bundles over compact-Hausdorff space and finitely generated projective modules. We generalize this equivalence to spaces which are only completely regular. Our method of proof avoids partitions of unity and is motivated from Atiyah's proof of the same theorem in the compact-Hausdorff case.

1 The statement

Let B be a compact-Hausdorff space, $C(B)$ be the commutative unital ring of all $\mathbf{R}$-valued continuous maps on B, $\mathcal{V}B(B)$ the category of all real vector bundles over B and $\mathcal{P}roj(C(B))$ the category of finitely generated projective $C(B)$-modules. The classical Swan's theorem states that the global sections functor

$$\Gamma : \mathcal{V}B(B) \longrightarrow \mathcal{P}roj(C(B))$$

is an equivalence of categories, that is, a fully-faithful and essentially surjective functor.

The goal of this note is to prove the following generalization of Swan's theorem.

Theorem 1.1. *Let B be a completely regular space and let $C(B)$ be the ring of all $\mathbf{R}$-valued continuous functions on B. Then the global sections functor*

$$\Gamma : \mathcal{V}B^{\oplus}(B) \longrightarrow \mathcal{P}roj(C(B))$$

from the category of finite rank vector bundles which are direct summands of finite rank trivial bundles to finitely generated projective modules over $C(B)$ is an equivalence of categories.

2010 *Mathematics Subject Classification.* 13C10, 57R22.

Key words and phrases. Swan's theorem, projective modules, vector bundles.

Remark 1.2. The hypothesis that all our bundles be direct summands of trivial bundles is not too restrictive:

(1) Every bundle over B is a direct summand of a trivial bundle if B is compact-Haudorff. Indeed, then any bundle ξ has a finite dimensional ample subspace in $\Gamma(\xi)$ (see Lemma 1.4.12 of [1]). Consequently, the statement of Theorem 1.1 reduces to classical Swan's theorem when B is compact-Hausdorff.

(2) Any soft[1] bundle is a direct summand of a trivial bundle. Indeed, same proof as Lemma 1.4.12 of [1] will work.

Our method to prove Theorem 1.1 is as follows. First, we show that Γ establishes an equivalence between trivial vector bundles of finite rank over B and free modules of finite rank over $C(B)$ (Proposition 2.1). Second, we show that the bundles which are direct summand of a trivial bundle have global sections module which is always finitely generated projective (Lemma 2.2). Third, we introduce the notion of projectors in a category which allows us to relate free modules and projective modules on one hand (Construction 3.2), and trivial bundles and vector bundles on the other (Construction 3.5). Having constructed these functors, we relate them via a commutative diagram (Proposition 3.6), which then allows us to give a proof of Swan's theorem in the penultimate section. This idea is loosely related to the last step of the proof of classical Swan's theorem by Atiyah (pp. 31 of [1]).

2 Trivial bundles & free $C(B)$-modules

We prove the first equivalence as discussed above.

Proposition 2.1. *Let B be any topological space. Then the global sections functor establishes an equivalence between trivial finite rank vector bundles on B and finite rank free $C(B)$-modules:*

$$\Gamma : \mathcal{V}B^{triv}(B) \longrightarrow \mathfrak{F}ree(C(B)).$$

Proof. Note that the global sections functor is given by

$$\Gamma : \mathcal{V}B^{\mathrm{triv}}(B) \longrightarrow \mathfrak{F}ree(C(B))$$

[1] a bundle is soft if there is a finite trivializing cover of base and it admits a partition of unity.

which on a map of bundles $f:\xi\to\eta$, $\xi=(E,p,B)$ and $\eta=(E',p',B)$ gives $\Gamma(f):\Gamma(\xi)\to\Gamma(\eta)$, mapping a section $s:B\to E$ of ξ to $f\circ s:B\to E'$. We first show that Γ is fully-faithful. To this end, we have to show that the following is a bijection:

$$\Gamma:\mathrm{Hom}_{\mathcal{VB}^{\mathsf{triv}}(B)}(\xi,\eta)\longrightarrow \mathrm{Hom}_{\mathcal{F}ree(C(B))}(\Gamma(\xi),\Gamma(\eta)).$$

We construct an inverse as follows. Pick any $C(B)$-linear map $\varphi:\Gamma(\xi)\to\Gamma(\eta)$, where $\Gamma(\xi)$ and $\Gamma(\eta)$ are free of rank n and m respectively. We define a map $f:E\to E'$ which on fiber at b maps as $f(\sum_i c_i s_i(b))\mapsto\sum_i c_i\varphi(s_i)(b)$ where $\{s_1,\dots,s_n\}$ is a free basis of $\Gamma(\xi)$. We need only show continuity of f. It is sufficient to show that the composite is continuous:

$$B\times\mathbf{R}^n\xrightarrow{\cong}E\xrightarrow{f}E'\xleftarrow{\cong}B\times\mathbf{R}^m\ .$$

Indeed, this composite maps as follows:

$$(b,c_1,\dots,c_n)\mapsto\sum_i c_i s_i(b)\mapsto\sum_i c_i\varphi(s_i)(b)\mapsto\left(b,\sum_i c_i d_{i1},\dots,\sum_i c_i d_{im}\right)$$

where $\varphi(s_i)=\sum_j d_{ij}t_j$ where $\{t_1,\dots,t_m\}$ is the free basis of $\Gamma(\eta)$. It is thus clear that the composite is continuous and hence so is f. This defines a map

$$\theta:\mathrm{Hom}_{\mathcal{F}ree(C(B))}(\Gamma(\xi),\Gamma(\eta))\longrightarrow\mathrm{Hom}_{\mathcal{VB}^{\mathsf{triv}}(B)}(\xi,\eta)$$

which is inverse of Γ, as required. Next we show that Γ is essentially surjective. Indeed, for any free $C(B)$-module of rank n, say M, there is an isomorphism to $C(B)^n$. As $C(B)^n=\Gamma(\epsilon^n)$, hence Γ is essentially surjective. This shows that Γ is an equivalence of categories. $\square$

Next, we show that $\Gamma(\xi)$ for a vector bundle ξ is a finitely generated projective $C(B)$-module. This is also Problem 3-F of [3].

Lemma 2.2. *Let B be a topological space and Γ be the global sections functor for bundles on B.*

(1) We have $\Gamma(\xi\oplus\eta)\cong\Gamma(\xi)\oplus\Gamma(\eta)$.
(2) A vector bundle ξ is trivial of rank n if and only if $\Gamma(\xi)$ is a free $C(B)$-module of rank n.
(3) If $\xi\oplus\eta$ is trivial of finite rank, then $\Gamma(\xi)$ is a finitely generated projective module $C(B)$-module.

Proof. 1. Recall that there are bundle maps $\pi_1 : \xi \oplus \eta \to \xi$ and $\pi_2 : \xi \oplus \eta \to \eta$, which are corresponding projections on fibers. Consider the map

$$\begin{aligned} \theta : \Gamma(\xi \oplus \eta) &\longrightarrow \Gamma(\xi) \oplus \Gamma(\eta) \\ s &\longmapsto (\pi_1 \circ s, \pi_2 \circ s). \end{aligned}$$

This is a well-defined map. We claim that this is $C(B)$-linear. Indeed, we see that for $f \in C(B)$, we have

$$\begin{aligned} \theta(s + ft) &= (\pi_1(s + ft), \pi_2(s + ft)) \\ &= (\pi_1 \circ s + f\pi_1 \circ t, \pi_2 \circ s + f\pi_2 \circ t) \\ &= \theta(s) + f\theta(t), \end{aligned}$$

as required. We now show that this is a bijection. Indeed, consider the map

$$\begin{aligned} \kappa : \Gamma(\xi) \oplus \Gamma(\eta) &\longrightarrow \Gamma(\xi \oplus \eta) \\ (s, t) &\longmapsto s \oplus t \end{aligned}$$

where $s \oplus t : B \to E_1 \oplus E_2$ is the section given by $b \mapsto (s(b), t(b)) \in E_{1,b} \oplus E_{2,b}$. We first show that this is indeed a continuous section of $\xi \oplus \eta$. Let $U \subseteq B$ be a common local trivialization of ξ and η. We then have the following commutative triangle:

$$\begin{tikzcd} p^{-1}(U) \arrow[d, "p"] & U \times \mathbf{R}^n \oplus \mathbf{R}^m \arrow[l, "h_1 \oplus h_2"'] \arrow[dl, "\pi_1"] \\ U \arrow[u, dashed, bend left, "s \oplus t"] & \end{tikzcd}$$

where $h_1 : U \times \mathbf{R}^n \to p_1^{-1}(U)$ and $h_2 : U \times \mathbf{R}^m \to p_2^{-1}(U)$ are trivializations of ξ and η respectively and thus by the theory of continuous functors (Chapter 3, [3]), $h_1 \oplus h_2$ forms local trivialization of $\xi \oplus \eta$. As $h_1 \oplus h_2$ is an isomorphism, therefore to show continuity of $s \oplus t$, it is sufficient to show that $(h_1 \oplus h_2)^{-1} \circ (s \oplus t) : U \to U \times (\mathbf{R}^n \oplus \mathbf{R}^m)$ is continuous. Indeed, this map is continuous as it is given by $b \mapsto (b, h_{1,b}^{-1}(s(b)), h_{2,b}^{-1}(t(b)))$ where are all components are continuous maps. Hence $s \oplus t$ is a continuous section of $\xi \oplus \eta$ and thus κ is well-defined. It is clear that $\kappa \circ \theta = \mathrm{id}$ and $\theta \circ \kappa = \mathrm{id}$, hence θ is an isomorphism of $C(B)$-modules, as required.

2. ($\Rightarrow$) Suppose $\xi = (E, p, B)$ is trivial of rank n. We wish to show that $\Gamma(\xi)$ is free of rank n. Indeed, we have n-sections $s_1, \ldots, s_n \in \Gamma(\xi)$ which

are nowhere dependent. We claim that these form a free $C(B)$-basis of $\Gamma(\xi)$. To this end, take $c_i \in C(B)$ such that $c_1 s_1 + \cdots + c_n s_n = 0$ in $\Gamma(\xi)$. Hence for any $b \in B$, we get $\sum_i c_i(b)s_i(b) = 0$ and by linear independence of $s_i(b)$, it follows that all $c_i(b) = 0$. Hence $c_i = 0$, as required, therefore s_i are linearly independent. Next we have to show that $\Gamma(\xi)$ is spanned by $s_1, \ldots, s_n$. Indeed, if $s \in \Gamma(\xi)$, then for any $b \in B$, we have $s(b) = \sum_i c_i(b)s_i(b)$. We need only show that each $c_i : B \to \mathbf{R}$ is continuous. As $E_b = \mathbf{R}^n$ is spanned by $\{s_i(b)\}$, $s : B \to B \times \mathbf{R}^n$ maps $b \mapsto (b, c_1(b), \ldots, c_n(b))$ and hence each c_i is continuous by continuity of s, as required.

($\Leftarrow$) Suppose $\Gamma(\xi)$ is free of rank n. Then there are sections $s_1, \ldots, s_n \in \Gamma(\xi)$ which forms a free basis. Consequently, they are linearly independent and hence for each $b \in B$, the vectors $s_1(b), \ldots, s_n(b)$ are linearly independent. We have thus found n-sections of ξ which are nowhere dependent.

3. By item 1, $\Gamma(\xi \oplus \eta) \cong \Gamma(\xi) \oplus \Gamma(\eta)$. By item 2, $\Gamma(\xi \oplus \eta)$ is free. Hence, $\Gamma(\xi)$ is a direct summand of a free $C(B)$-module, hence projective. The module $\Gamma(\xi)$ is finitely generated since there is a surjection from a finitely generated free $C(B)$-module: $\Gamma(\xi \oplus \eta) \twoheadrightarrow \Gamma(\xi)$. $\square$

Remark 2.3. It follows from Lemma 2.2 that the global sections functor Γ on $\mathcal{V}B^{\oplus}(B)$ maps any vector bundle to a finitely generated projective $C(B)$-module. Thus, we have a functor

$$\Gamma : \mathcal{V}B^{\oplus}(B) \longrightarrow \mathcal{P}roj(C(B))$$

as required in the statement of Theorem 1.1.

3 Projectors & image functor

Let $\mathcal{C}$ be a category. We define a *projector* in $\mathcal{C}$ to be an idempotent endomorphism $f : C \to C$ of an object $C \in \mathcal{C}$, i.e. $f \circ f = f$. A map of projectors is a commutative square in $\mathcal{C}$ where horizontal maps are same:

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "f"'] & C' \arrow[d, "f'"] \\ C \arrow[r, "h"'] & C' \end{tikzcd}.$$

We denote the category of projectors of $\mathcal{C}$ as $\mathrm{Pro}(\mathcal{C})$. The following is a lemma of interest to us.

Lemma 3.1. *Let $\Gamma : \mathcal{C} \to \mathcal{D}$ be an equivalence of catgories. Then the functor*

$$\tilde{\Gamma} : \mathrm{Pro}(\mathcal{C}) \longrightarrow \mathrm{Pro}(\mathcal{D})$$

which maps a projector $f : C \to C$ to a projector $\Gamma(f) : \Gamma(C) \to \Gamma(C)$ is also an equivalence of categories.

Proof. Let $f : C \to C$ and $g : C' \to C'$ be two projectors and suppose $h, k : f \to g$ are two maps such that $\Gamma(h) = \Gamma(k) : \Gamma(f) \to \Gamma(g)$. It follows from faithfulness of Γ that $h = k$. This shows that $\tilde{\Gamma}$ is faithfull. If $\varphi : \Gamma(f) \to \Gamma(g)$ is a map of projectors in $\mathcal{D}$, then φ is a map $\Gamma(C) \to \Gamma(C')$. It follows by fullness of Γ that there exists $h : C \to C'$ such that $\Gamma(h) = \varphi$, as required. $\square$

Construction 3.2 (Image functor-1)**.** Let R be a commutative ring with 1. We construct a functor

$$\mathrm{Im}\,(-) : \mathrm{Pro}(\mathcal{F}ree(R)) \longrightarrow \mathcal{P}roj(R)$$

which is full and essentially surjective. This functor on a projector $\varphi : M \to M$ gives $\mathrm{Im}\,(\varphi)$, which is clearly a projective module. On a map of projectors

$$\begin{array}{ccc} M & \xrightarrow{\beta} & N \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ M & \xrightarrow[\beta]{} & N \end{array}$$

$\mathrm{Im}\,(\beta) : \mathrm{Im}\,(\varphi) \to \mathrm{Im}\,(\psi)$ is the restriction of β on $\mathrm{Im}\,(\varphi)$ since $M = \mathrm{Im}\,(\varphi) \oplus \mathrm{Ker}\,(\varphi)$ and $N = \mathrm{Im}\,(\psi) \oplus \mathrm{Ker}\,(\psi)$. We first show that this is essentially surjective. Indeed, if P is a f.g. projective R-module, then $M \cong P \oplus Q$ where M is a free R-module of finite rank. Thus $\varphi : M \twoheadrightarrow P \hookrightarrow M$ is the required projection such that $\mathrm{Im}\,(\varphi) \cong P$. To see that this is full, take any R-linear map $\beta : P \to Q$ of projective modules. Let $\varphi : M \to M$ and $\psi : N \to N$ be projectors corresponding to P and Q respectively. Then $M = P \oplus \mathrm{Ker}\,(\varphi)$ and $N = Q \oplus \mathrm{Ker}\,(\psi)$. Define a map of projectors given by the following square:

$$\begin{array}{ccc} P \oplus \mathrm{Ker}\,(\varphi) & \xrightarrow{\beta\oplus 0} & Q \oplus \mathrm{Ker}\,(\psi) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ P \oplus \mathrm{Ker}\,(\varphi) & \xrightarrow[\beta\oplus 0]{} & Q \oplus \mathrm{Ker}\,(\psi) \end{array} .$$

Clearly, the functor $\mathrm{Im}\,(-)$ on $\beta \oplus 0$ gives the map $\beta : P \to Q$, as required. This shows that $\mathrm{Im}\,(-)$ is full, as required.

We next construct a functor from a projector of trivial bundle to vector bundles. To this, we first need to observe that a projector of trivial bundles is of locally constant rank, so that its image is a vector bundle.

Lemma 3.3. *A projector $f : \xi \to \xi$ of any bundle $\xi = (E, p, B)$ of rank n is of locally constant rank.*

Proof. As $f \circ f = f$, therefore $f \circ (\mathrm{id} - f) = 0$ where id is the identity map of E. Thus, for each $b \in B$, we have $\dim(\mathrm{Im}\,(f_b)) + \dim(\mathrm{Im}\,(\mathrm{id} - f_b)) = \dim E_b = n$. Note that $\mathrm{Im}\,(\mathrm{id} - f_b) = \mathrm{Ker}\,(f_b)$. By upper semi-continuity of rank function, it follows that $n - \dim(\mathrm{Im}\,(\mathrm{id} - f_b))$ is lower-semicontinous. It follows that $\dim \mathrm{Im}\,(f_b)$ is continuous and hence f is of locally constant rank. □

Remark 3.4. By Theorem 8.2 of [2], it follows that if $f : \xi \to \xi$ is a projector, then $\mathrm{Im}\,(f)$ is a vector bundle.

Construction 3.5 (Image functor-2)**.** Consider the mapping

$$\mathrm{Im}\,(-) : \mathrm{Pro}(\mathcal{V}B^{\mathrm{triv}}(B)) \to \mathcal{V}B^{\oplus}(B)$$

mapping a projector on trivial bundle $f : \xi \to \xi$ to the image bundle $\mathrm{Im}\,(f)$. On a map of projectors

$$\begin{tikzcd} \xi \arrow[r, "h"] \arrow[d, "f"'] & \eta \arrow[d, "g"] \\ \xi \arrow[r, "h"'] & \eta \end{tikzcd},$$

we get an induced map $h : \mathrm{Im}\,(f) \to \mathrm{Im}\,(g)$, which is a bundle map. Note that we have the following split exact sequence of vector bundles:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}\,(f) \arrow[r] & \xi \arrow[r, "f"] & \mathrm{Im}\,(f) \arrow[r] \arrow[l, bend left, dashed, "f"] & 0 \end{tikzcd}.$$

Consequently, $\xi \cong \mathrm{Ker}\,(f) \oplus \mathrm{Im}\,(f)$. This shows that $\mathrm{Im}\,(f)$ is a direct summand of a trivial bundle and hence lies in $\mathcal{V}B^{\oplus}(B)$, as required. This defines the desired functor $\mathrm{Im}\,(-)$.

Our next claim is that the following diagram commutes:

Proposition 3.6. *Let B be any topological space. Then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Pro}(\mathcal{V}B^{triv}(B)) \arrow[r, "\tilde{\Gamma}"] \arrow[d, "\mathrm{Im}(-)"'] & \mathrm{Pro}(\mathcal{F}ree(C(B))) \arrow[d, "\mathrm{Im}(-)"] \\ \mathcal{V}B^{\oplus}(B) \arrow[r, "\Gamma"'] & \mathcal{P}roj(C(B)) \end{tikzcd}$$

where $\tilde{\Gamma}$ is the functor induced on projectors from the functor $\Gamma : \mathcal{V}B^{triv}(B) \to \mathcal{F}ree(C(B))$.

Proof. Let $f : \xi \to \xi$ be a projector for a trivial bundle ξ of rank n. We wish to show that $\mathrm{Im}\,(\Gamma(f)) = \Gamma(\mathrm{Im}\,(f))$, where $\mathrm{Im}\,(f)$ is the image bundle of f. Pick $s \in \Gamma(\mathrm{Im}\,(f))$. Then, $f \circ s$ is a section in $\Gamma(\xi)$ such that $\Gamma(f)(f \circ s) = f \circ f \circ s = f \circ s \in \mathrm{Im}\,(f)$. But $f|_{\mathrm{Im}(f)} = \mathrm{id}$ by above splitting in Construction 3.5, thus $f \circ s = s$. It follows that $\Gamma(\mathrm{Im}\,(f)) \subseteq \mathrm{Im}\,(\Gamma(f))$. Conversely, if $f \circ t \in \mathrm{Im}\,(\Gamma(f))$ for some $t \in \Gamma(\xi)$, then clearly $f(t(b)) \in \mathrm{Im}\,(f_b)$ for each $b \in B$, as required. This completes the proof. □

4 Proof of Swan's theorem

We now bring all the ideas together. We first show that global sections is already a faithful functor on vector bundles. This is the only place where we will use the completely regular hypothesis.

Lemma 4.1. *Let B be a completely regular space. Then the global sections functor is a faithful functor between finite rank vector bundles on B which are direct summands of a finite rank trivial bundle and finitely generated projective $C(B)$-modules:*

$$\Gamma : \mathcal{V}B^{\oplus}(B) \longrightarrow \mathcal{P}roj(C(B)).$$

Proof. By Remark 2.3, this functor is well-defined. Let $\xi = (E, p, B)$ and $\eta = (E', p', B)$ be rank n and m bundles respectively. We wish to show that the following is an injection:

$$\Gamma : \mathrm{Hom}_{\mathcal{V}B(B)}(\xi, \eta) \longrightarrow \mathrm{Hom}_{\mathcal{P}roj(C(B))}(\Gamma(\xi), \Gamma(\eta)).$$

Indeed, suppose $\Gamma(f) = \Gamma(g)$ for $f, g : \xi \to \eta$. Pick any common local trivialization $U \subseteq B$ for both the bundles. We claim that

$$\Gamma(f|_{p^{-1}(U)}) = \Gamma(g|_{p^{-1}(U)})$$

as homomorphisms $\Gamma(\xi|_U) \to \Gamma(\eta|_U)$. Indeed, let $s : U \to p^{-1}(U)$ be a section in $\Gamma(\xi|_U)$. We wish to show that $f|_{p^{-1}(U)} \circ s = g|_{p^{-1}(U)} \circ s$. Pick any $u_0 \in U$. As B is completely regular, therefore there exists a continuous map $\rho : B \to [0,1]$ such that $\rho(B - U) = 0$ and $\rho(u_0) = 1$. It follows that $\rho \cdot s : B \to E$ is a section in $\Gamma(\xi)$. Thus $f \circ (\rho \cdot s) = g \circ (\rho \cdot s)$. It follows that $f \circ (\rho(u_0)s(u_0)) = g \circ (\rho(u_0)s(u_0))$ from which it follows that $f \circ s(u_0) = g \circ s(u_0)$. As $u_0 \in U$ is arbitrary, thus $f|_{p^{-1}(U)} \circ s = g|_{p^{-1}(U)} \circ s$, as required.

Note that $f|_{p^{-1}(U)}, g|_{p^{-1}(U)} : p^{-1}(U) \to p'^{-1}(U)$ is a map of trivial bundles hence by the equivalence in Proposition 2.1, it follows that $f|_{p^{-1}(U)} = g|_{p^{-1}(U)}$. As U is arbitrary trivializing neighborhood, thus $f = g$, as required. □

To complete the proof of Theorem 1.1, we need only show that Γ is full and it is essentially surjective. This will now follow from discussions in past few sections.

Proof of Theorem 1.1. From Lemma 4.1, the global sections functor is already faithful. By Proposition 3.6, $\Gamma : \mathcal{V}B^{\oplus}(B) \to \mathcal{P}roj(C(B))$ is essentially surjective since $\mathrm{Im}\,(-) : \mathrm{Pro}(\mathcal{F}ree(C(B))) \to \mathcal{P}roj(C(B))$ is essentially surjective, $\tilde{\Gamma}$ is equivalence and the diagram commutes. Finally to show fullness, pick any homomorphism of modules $h : \Gamma(\xi) \to \Gamma(\eta)$ for two bundles $\xi, \eta \in \mathcal{V}B^{\oplus}(B)$. As $\mathrm{Im}\,(-) : \mathrm{Pro}(\mathcal{F}ree(C(B))) \to \mathcal{P}roj(C(B))$ is full, therefore there exists a map of projectors

$$\begin{tikzcd} M \arrow[r, "\tilde{h}"] \arrow[d, "\varphi"'] & N \arrow[d, "\psi"] \\ M \arrow[r, "\tilde{h}"'] & N \end{tikzcd}$$

where M, N are free of finite rank, $\mathrm{Im}\,(\varphi) = \Gamma(\xi)$ and $\mathrm{Im}\,(\psi) = \Gamma(\eta)$. By equivalence of of $\tilde{\Gamma}$ (Lemma 3.1), there is a unique projector map

$$\begin{tikzcd} \mu \arrow[r, "h'"] \arrow[d, "f"'] & \nu \arrow[d, "g"] \\ \mu \arrow[r, "h'"'] & \nu \end{tikzcd}$$

where μ and ν are trivial bundles of finite rank such that $\Gamma(\mu) = M$, $\Gamma(\nu) = N$ and similarly for maps. By commutativity of Proposition 3.6, it follows that the bundle map $h' : \mathrm{Im}\,(f) \to \mathrm{Im}\,(g)$ maps to $h : \Gamma(\xi) \to \Gamma(\eta)$ under the functor Γ, as required. $\square$

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