ALGEBRAIC K-THEORY OF RINGS

Contents

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1. Introduction

Algebraic K-theory is an invariant for rings which dually becomes an invariant for schemes. The lower K-theory of rings was the first view of algebraic K-theory that people found. Indeed, $K_0(R)$ for a ring R with unity was defined and used by Grothendieck whereas later $K_1(R), K_2(R)$ was defined later by Hymann Bass and John Milnor. Existence of a relative ideal sequence then raised a question whether one can define higher K-groups, which would extend this sequence akin to relative homology sequence. Indeed, this was done by D.G. Quillen in early 70s. The general procedure in which he defined $K_n(R)$ was to first obtain a topological space from a ring R in such a manner so to make its first three homotopy groups identical to K_0, K_1, K_2 and then defined higher K-groups by the higher homotopy groups of that space.

We will discuss below some basic examples of Grothendieck groups, notably the connection between class groups in algebraic number theory and reduced Grothendieck group for Dedekind domains. The functor K_0 behaves well with respect to finite direct sums and direct limits, which further facilitates computations for rings which admits decomposition results, like Artinian rings.

We will see that K-theory behaves a lot like a generalized homology theory, thus further facilitating computations. In fact, given a two-sided ideal I of a ring R, there is an exact sequence relating the K-groups of the rings R, R/I and a relative K-group of the pair (R, I). Moreover, the relative K-group $K_0(R, I)$ only depends on the difference between R and R/I, which is the ideal I thought of as a ring without unity. This is an analogue of the excision theorem in homology.

All rings will be associative with 1, but may not be commutative, unless stated otherwise. We denote $\mathcal{P}roj(R)$ to be the category of finitely generated projective left R-modules. Below are some easy to prove equivalent characterizations of projective modules and some of their properties.

Proposition 1.0.1. *Let R be a ring and P be a left R-module. Then the following are equivalent:*

1. *P is finitely generated projective.*
2. *Any short exact sequence $0 \to M \to N \to P \to 0$ is split exact.*
3. *There exists a module Q such that $P \oplus Q \cong R^n$.*
4. *There exists a surjection $\pi : R^n \twoheadrightarrow P$ which splits.*
5. *The functor $\operatorname{Hom}_R(P, -) : \mathcal{M}od(R) \to \mathcal{A}b$ is an exact functor, where $\mathcal{M}od(R)$ is the category of left R-modules.*

Proposition 1.0.2. *Let $P, Q \in \mathcal{P}roj(R)$ be two finitely generated projective modules. Then[1],*

1. *$P \oplus Q$ is a finitely generated projective module,*
2. *Any direct summand of P is a finitely generated projective module.*
3. *If R is commutative, then $P \otimes_R Q$ is a finitely generated projective R-module.*
4. *If R is commutative, then P is flat.*
5. *♣We have that $\check{P} = \operatorname{Hom}_R(P, R)$ is a projective R^{op}-module. If R is commutative, then $\check{P}$ is a projective R-module.*
6. *♣If R is commutative, then $\operatorname{rank}(\check{P}) = \operatorname{rank}(P)$.*
7. *♣If R is commutative, then trace of P, that is $\tau_P := \operatorname{Im}\left(\mathrm{ev} : \check{P} \otimes_R P \to R\right)$, is an idempotent ideal of R.*

Proof. † Item 1. and 2. are immediate from Proposition 1.0.1. For item 3, observe that if $P \oplus P' = R^{\oplus n}$, then $(P \otimes_R Q) \oplus (P' \otimes_R Q) = (P \oplus P') \otimes_R Q = R^{\oplus n} \otimes_R Q = Q^{\oplus n}$. As Q is projective, therefore

[1] We put ♣ wherever finite generation of P and Q are not needed, i.e. if only projectivity of P and Q are needed.

$Q^{\oplus n}$ is projective by item 1. We conclude by item 2.

For item 4, we need only show that for an injective map $f : M' \to M$, the map $f \otimes \mathrm{id} : M' \otimes_R P \to M \otimes_R P$ is also injective. As P is projective, so there exists Q f.g. projective module such that $P \oplus Q = R^n$. Consequently, we get the commutative diagram as below:

$$\begin{tikzcd} (M' \otimes_R P) \oplus (M' \otimes_R Q) \arrow[r, phantom, "\cong"] \arrow[d, "(f\otimes\mathrm{id})\oplus(f\otimes\mathrm{id})"] & M' \otimes_R (P \oplus Q) \arrow[r, phantom, "\cong"] \arrow[d, "f\otimes\mathrm{id}"] & (M')^{\oplus n} \arrow[d, "f^{\oplus n}"] \\ (M \otimes_R P) \oplus (M \otimes_R Q) \arrow[r, phantom, "\cong"] & M \otimes_R (P \oplus Q) \arrow[r, phantom, "\cong"] & M^{\oplus n} \end{tikzcd}.$$

The right vertical map is injective by hypothesis. By commutativity of the diagram above, the rest of the two vertical maps are also injective. Hence, $f \otimes \mathrm{id} : M' \otimes_R P \to M \otimes_R P$ is injective as well, as required.

Item 5 follows from existence of Q such that $P \oplus Q \cong R^n$ and that direct sum in first variable commutes with hom.

For item 6, first observe that $P \otimes_R \kappa(\mathfrak{p}) \cong P_\mathfrak{p}/\mathfrak{p}P_\mathfrak{p}$. Since $\mathrm{Hom}_R(P,R)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(P_\mathfrak{p}, R_\mathfrak{p}) = \check{P}_\mathfrak{p}$ as one of the modules in the hom is finitely presented (see Proposition 23.1.2.13 of [FoG]), therefore we need only show that $P_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(P_\mathfrak{p}, R_\mathfrak{p})$. To this end, as localization of projective modules is projective since localization is exact, we deduce that $P_\mathfrak{p}$ is projective $R_\mathfrak{p}$-module. Consequently, $P_\mathfrak{p}$ is free as $R_\mathfrak{p}$ is local (see Theorem 23.23.0.9 of [FoG]). Hence the required isomorphism $P_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(P_\mathfrak{p}, R_\mathfrak{p})$ is immediate.

For item 7, the fact that τ_P is an ideal is immediate from definition of ev as $\varphi \otimes x \mapsto \varphi(x)$. We now show that $\tau_P^2 = \tau_P$. To this end, we need only show that $\tau_P \subseteq \tau_P^2$. It can be seen that it is sufficient to show that any element $x \in P$ can be written as $x = \sum_{i=1}^n \psi_i(x)x_i$ for $x_i \in P$ and $\psi_i \in \check{P}$. Indeed, as there exists Q such that $P \oplus Q = R^F$, therefore for any $x \in P$, we may write $x = \sum_{i=1}^n r_i x_i$ where $r_i = f_i(x)$ where $\{f_i\}_{i\in F}$ is the dual basis of $(R^F)\check{}$. This completes the proof. ∎

Recall that an R-module M is locally free if for all $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists a basic open $\mathfrak{p} \in D(f) \subseteq \mathrm{Spec}(R)$ such that M_f is a free R_f-module[2]. An important local characterization of projective modules is the following.

Theorem 1.0.3. *Let R be a commutative ring and M be an R-module. Then the following are equivalent:*

(1) M is finitely generated projective.

(2) M is locally free of finite rank.

Proof. (1. $\Rightarrow$ 2.) Pick $\mathfrak{p} \in \mathrm{Spec}(R)$. Then, $M_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module which is also projective as localization is exact. It follows from Theorem 2.2.3 that $M_\mathfrak{p} = (R_\mathfrak{p})^{\oplus n}$. Let $\{m_i/s_i\}_{i=1,\dots,n}$ be an $R_\mathfrak{p}$-basis of $M_\mathfrak{p}$. It follows by multipliying by $s_1 \dots s_n$ that we have a map $f : R^n \to M$ which may not be surjective, however, $f_\mathfrak{p} : R_\mathfrak{p}^n \to M_\mathfrak{p}$ is surjective. Denoting $N = \mathrm{CoKer}(f)$, we deduce that $N_\mathfrak{p} = 0$. As N is finitely generated, it follows that there exists $s \in R$ such that $N_s = 0$. But since $N_s = \mathrm{CoKer}(f_s)$, where $f_s : R_s^n \to M_s$, thus, we deduce that f_s is surjective. Since M_s is a projective R_s-module, therefore $M_s \oplus P = R_s^n$ where P is a finitely generated projective R_s-module. Localizing at $\mathfrak{p}$ again, we see that $M_\mathfrak{p} \oplus P_\mathfrak{p} = R_\mathfrak{p}^n$, but since $M_\mathfrak{p} = R_\mathfrak{p}^n$, thus, $P_\mathfrak{p} = 0$. It follows

[2] That is, $\tilde{M}$ is locally free, i.e. a vector bundle over $\mathrm{Spec}(R)$.

by finite generation that there exists $t \in R$ such that $t \cdot P = 0$ and thus $P_t = 0$. It follows that $M_{st} \oplus P_t = R^n_{st}$ and thus $M_{st} = R^n_{st}$ so that $f = st$ will do the job.

(2. $\Rightarrow$ 1.) The proof is in two steps. In step 1, one shows that a locally free module of finite rank is finitely presented with free stalks. This follows from faithfully flat descent. In step 2, one shows that finitely presented modules with free stalks are projective. Indeed, let M be such a module. Then, we have an exact sequence

$$R^m \to R^n \xrightarrow{\pi} M \to 0.$$

By Proposition 1.0.1, it suffices to show that π splits. To this end, it is sufficient to show that $\pi_* : \mathrm{Hom}_R(M, R^n) \to \mathrm{Hom}_R(M, M)$ is surjective, as then id_M will have a section, as required. Indeed, as surjectivity of maps of modules is a local property ($f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is surjective for all $\mathfrak{p} \in \mathrm{Spec}(R)$ if and only if $f : M \to N$ is surjective), thus we reduce to showing that $(\pi_*)_\mathfrak{p}$ is surjective. As Hom and localization commutes if one of the modules is finitely presented (see Proposition 23.1.2.13 of [FoG]), therefore we wish to show that $\pi_{\mathfrak{p}*} : \mathrm{Hom}_{R_\mathfrak{p}}\left(M_\mathfrak{p}, R^n_\mathfrak{p}\right) \to \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, M_\mathfrak{p})$ is surjective. This is true as the map $\pi_\mathfrak{p} : R^n_\mathfrak{p} \to M_\mathfrak{p}$ is surjective by exactness of localization and since $M_\mathfrak{p}$ is a projective $R_\mathfrak{p}$-module as it is free by our hypothesis. This concludes the proof. □

Remark 1.0.4. By Theorem 1.0.3, it follows that vector bundles over $\mathrm{Spec}(R)$ are in one-to-one bijection with projective modules over R.

Using the above result, we can show that rank of a projective module is a continuous function from $\mathrm{Spec}(R)$ to $\mathbb{Z}$.

Proposition 1.0.5. *Let R be a commutative ring and M be a projective R-module. Then* rank : $\mathrm{Spec}(R) \to \mathbb{Z}$ *is a continuous map.*

Proof. † By discreteness of $\mathbb{Z}$, it suffices to show that each fibre of rank is an open set. Indeed,

$$\begin{aligned}\mathrm{rank}^{-1}(n) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p}) = n\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim_{\kappa(\mathfrak{p})} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = n\}.\end{aligned}$$

By Theorem 1.0.3, M is locally free, hence $M_\mathfrak{p} \cong R^k_\mathfrak{p}$ for all $\mathfrak{p}$ in some largest open set $U \subseteq \mathrm{Spec}(R)$. Consequently, $\dim_{\kappa(\mathfrak{p})} M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = \dim_{\kappa(\mathfrak{p})}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})^k = \dim_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})^k = k$ for all $\mathfrak{p} \in U$. Thus the above fibre is either empty or non-empty open set, as required. □

A simple example shows that $\mathcal{P}roj(R)$ cannot be abelian.

Example 1.0.6. Let $\mathbb{Z}$ be free $\mathbb{Z}$-module of rank 1. Observe that $2\mathbb{Z} \subseteq \mathbb{Z}$ is also a free module of rank 1. Hence both $\mathbb{Z}$ and $2\mathbb{Z}$ are projective $\mathbb{Z}$-modules. However, $\mathbb{Z}/2\mathbb{Z}$ is not a projective $\mathbb{Z}$-module as it cannot be a direct summand of $\mathbb{Z}^{\oplus n}$ for any $n \in \mathbb{N}$ since $\mathbb{Z}^{\oplus n}$ doesn't have any 2-torsion element. Consequently, $\mathcal{P}roj(R)$ is not abelian.

One observes that rank of a constant rank projective module remains same under extension of scalars.

Proposition 1.0.7. *Let $f : R \to S$ be a ring homomorphism between commutative rings. If P is a finitely generated projective R-module, then*

$$\operatorname{rank}(P \otimes_R S) = \operatorname{rank}(P) \circ f^*.$$

Hence, if P is constant rank n, then so is $P \otimes_R S$.

Proof. Let $\mathfrak{q} \in \operatorname{Spec}(S)$ and $f^*(\mathfrak{q}) = f^{-1}(\mathfrak{q}) = \mathfrak{p} \in \operatorname{Spec}(R)$. We need only show that if $P \otimes_R \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})^n$, then $(P \otimes_R S) \otimes_S \kappa(\mathfrak{q}) \cong \kappa(\mathfrak{q})^n$. Indeed, as

$$\begin{aligned}
(P \otimes_R S) \otimes_S \kappa(\mathfrak{q}) &\cong P \otimes_R \kappa(\mathfrak{q}) \\
&\cong P \otimes_R S_\mathfrak{q} \otimes_S S/\mathfrak{q} \\
&\cong P \otimes_R R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q} \otimes_S S/\mathfrak{q} \\
&\cong P_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q} \otimes_S S/\mathfrak{q} \\
&\cong R_\mathfrak{p}^n \otimes_{R_\mathfrak{p}} S_\mathfrak{q} \otimes_S S/\mathfrak{q} \\
&\cong S_\mathfrak{q}^n \otimes_S S/\mathfrak{q} \\
&\cong (S_\mathfrak{q} \otimes_S S/\mathfrak{q})^n \\
&\cong \kappa(\mathfrak{q})^n,
\end{aligned}$$

as required. □

It is quite intuitive to claim that finite rank projective modules ought to be finitely generated. Indeed it is true.

Proposition 1.0.8. *Let R be a commutative ring and M be a finite rank projective module. Then M is finitely generated.*

Proof. † A result of Kaplansky states that a module over commutative ring R is projective if and only if it is locally free (we have done the finite case above in Theorem 1.0.3). Since by Theorem 1.0.3, it is sufficient to show that M is locally free of finite rank, where by above we already know it is locally free, we need only show that M is also finitely locally free. Let $f \in R$ be such that $M_f \cong R_f^F$. We wish to show that $|F| < \infty$. As M is finite rank, therefore for each $\mathfrak{p} \in \operatorname{Spec}(R)$, $\dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p}) < \infty$. If $f \notin \mathfrak{p}$, then since $M_\mathfrak{p} = (M_f)_\mathfrak{p} \cong (R_f^F)_\mathfrak{p} \cong R_\mathfrak{p}^F$, we deduce that $M \otimes_R \kappa(\mathfrak{p}) \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})^F = \kappa(\mathfrak{p})^F$. Thus $|F| < \infty$, as required. □

An important conceptual result which will guide us in defining higher K-groups is the cofinality of free modules in projective modules.

Lemma 1.0.9. *Let R be a ring and let $\mathcal{F}ree(R)^{\cong}$ be the isomorphism classes of finitely generated free R-modules. This is a monoid under direct sum with identity 0. Then $\mathcal{F}ree(R)^{\cong}$ is cofinal in $\mathcal{P}roj(R)^{\cong}$.* □

There is also a characterization of finitely generated projective modules in terms of flatness.

Proposition 1.0.10. *Let R be a commutative ring and M be an R-module. Then the following are equivalent:*

(1) M is a finitely presented flat R-module.

(2) M is a finitely generated projective R-module.

Proof. (1. $\Rightarrow$ 2.) As M is finitely generated, thus to show that it is flat, it suffices to show that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for each $\mathfrak{p} \in \operatorname{Spec}(R)$. As localization is exact, we reduce to assuming that R is a local ring and M is a finitely presented flat R-module. By Corollary 6.6 of [Eis95], it follows that M is projective R-module. As projective modules over local rings are free (Proposition 23.23.0.9 of [FoG]), thus M is free, as required.

(2. $\Rightarrow$ 1.) As M is finitely generated projective, then it is finitely presented as if $M \oplus N \cong R^n$ where N is thus also finitely generated projective, then we get a presentation $N \to R^n \to M \to 0$, as required. Clearly, M is flat by Proposition 1.0.2, 4. $\square$

2. K_0

We study the first K-group, which is also the easiest to construct and understand. We begin by studying this for rings, before looking at it more geometrically via schemes.

2.1. K_0 of a ring & basic properties.

Definition 2.1.1 ($K_0(R)$)**.** Let R be a ring and consider $F(R)$ to be the free abelian group generated by objects of skeleton of $\mathcal{P}roj(R)$, denoted $\mathcal{P}roj(R)^{\cong}$. Consider the subgroup of $F(R)$

$$E = \langle \{[P \oplus Q] - [P] - [Q] \mid P, Q \in \mathcal{P}roj(R)\} \rangle$$

Then we define

$$K_0(R) := F(R)/E.$$

That is,

$$\mathcal{P}roj(R)^{\cong} \hookrightarrow F(R) \twoheadrightarrow K_0(R).$$

Consequently, $K_0(R)$ is an abelian group as it is quotient of the free abelian group $F(R)$. In particular, the addition in $K_0(R)$ of $P, Q \in \mathcal{P}roj(R)$ is $[P] + [Q] = [P \oplus Q]$.

Some observations from the definitions are as follows.

Lemma 2.1.2. *Let R be a ring and $P, Q \in \mathcal{P}roj(R)$ be two finitely generated projective left R-modules. Then, the following are equivalent:*

(1) $[P] = [Q]$ in $K_0(R)$.
(2) There exists $P' \in \mathcal{P}roj(R)$ such that $P \oplus P' \cong Q \oplus P'$[3].
(3) There exists $n \geq 0$ such that $P \oplus R^n \cong Q \oplus R^n$.

Proof. (1. $\Rightarrow$ 2.) Unravelling the definition, we deduce that there exists $P_i, Q_i, P'_j, Q'_j \in \mathcal{P}roj(R)$ such that

$$P - Q = \sum_{i=1}^{n} P_i \oplus Q_i - P_i - Q_i - \left(\sum_{j=1}^{m} P'_j \oplus Q'_j - P'_j - Q'_j \right)$$

in $F(R)$. By rearrangement, we deduce that P is either isomorphic to $P_i \oplus Q_i$ or P_i or Q_i and Q is either isomorphic to $P'_j \oplus Q'_j$ or P'_j or Q'_j. Conseuqently, the summand may be taken as P' can

[3]This is at times also called stable isomorphism of two modules.

be taken to be the direct sum of all P_i, Q_i, P'_j, Q'_j which will be projective.

(2. $\Rightarrow$ 3.) As $P' \in \mathcal{P}roj(R)$, thus, there exists $Q' \in \mathcal{P}roj(R)$ such that $P' \oplus Q' = R^n$ for some $n \in \mathbb{N}$. Hence taking direct sum with Q' in the given isomorphism will give us the required isomorphism.

(3. $\Rightarrow$ 1.) As $[P] = [P \oplus R^n] - [R^n] = [Q \oplus R^n] - [R^n] = [Q]$ in $K_0(R)$, hence we get the desired result. □

A simple corollary yields the precise meaning of $[P] = [R^n]$ in $K_0(R)$.

Corollary 2.1.3. *Let R be a ring and P be a finitely generated projective module. If $[P] = [R^n]$, then P is stably free.*

Proof. By Lemma 2.1.2, we deduce that $P \oplus R^k \cong R^{n+k}$, as required. □

We now establish that K_0 is a functor on $\mathcal{R}ing$ to $\mathcal{A}b$.

Construction 2.1.4 (Functor K_0)**.** Let $f : A \to B$ be a map of rings. We define $f_* : K_0(A) \to K_0(B)$ by extension of scalars:

$$\begin{aligned} f_* : K_0(A) &\longrightarrow K_0(B) \\ [P] &\longmapsto [P \otimes_A B]. \end{aligned}$$

As $(f \circ g)_* = f_* \circ g_*$, therefore K_0 is a functor.

The following shows that if R is a commutative ring then $K_0(R)$ is a commutative ring.

Lemma 2.1.5. *Let R be a commutative ring. Then the operation $([P], [Q]) \mapsto [P \otimes_R Q]$ for $P, Q \in \mathcal{P}roj(R)$ defines a commuative ring structure on $K_0(R)$.*

Proof. Indeed, this is immediate by commutativity of $\otimes$ upto isomorphism for commutative rings and distributivity of $\otimes$ over $\oplus$. □

The following states that K_0 preserves products.

Lemma 2.1.6. *Let R, R_1, R_2 be rings. If $R = R_1 \times R_2$, then*

$$K_0(R) \cong K_0(R_1) \times K_0(R_2).$$

Proof. We need only show a bijection $\mathcal{P}roj(R)^{\cong} \cong \mathcal{P}roj(R_1)^{\cong} \times \mathcal{P}roj(R_2)^{\cong}$. Indeed, consider the function $P \mapsto (P \otimes_R R_1, P \otimes_R R_2)$. We claim that the map $P_1 \oplus P_2 \leftarrow\!\shortmid (P_1, P_2)$ is an inverse of above. Indeed, we see $(P \otimes_R R_1) \oplus (P \otimes_R R_2) = P \otimes_R (R_1 \oplus R_2) = P \otimes_R R \cong P$. Similarly, $(P_1 \oplus P_2) \otimes_R R_1 = (P_1 \oplus P_2) \otimes_R \frac{R}{0 \times R_2} \cong \frac{P_1 \oplus P_2}{(0 \times R_2) \cdot (P_1 \oplus P_2)} \cong \frac{P_1 \oplus P_2}{P_2} \cong P_1$, as required. □

The following result says that K_0 is invariant of reducing the structure.

Proposition 2.1.7. *Let R be a ring and $I \leq R$ be a nilpotent ideal. Then $K_0(R) \cong K_0(R/I)$. In particular, $K_0(R) \cong K_0(R_{\text{red}})$.*

Proof. It is sufficient to show that $\mathcal{P}roj(R)^{\cong} \cong \mathcal{P}roj(R/I)^{\cong}$. Indeed, this is true by idempotent lifting (see Exercise I.2.2 of [Wei13]). □

The following is a simple characterization of units of the commutative ring $K_0(R)$.

Proposition 2.1.8. *Let R be a commutative ring. Then*

$$\mathrm{Pic}(R) \hookrightarrow K_0(R)^\times$$

and every element of the form $[P] \in K_0(R)^\times$ is in $\mathrm{Pic}(R)$.

Proof. † Consider the map

$$\begin{aligned}\varphi : \mathrm{Pic}(R) &\longrightarrow K_0(R)^\times \\ [P] &\longmapsto [P]\end{aligned}$$

which takes the isomorphism class of a line bundle to its K_0-class in $K_0(R)$. This is a group homomorphism as $\varphi([P]\cdot[Q]) = \varphi([P\otimes_R Q]) = [P\otimes_R Q] = [P]\cdot[Q]$. Now take any $[P] \in K_0(R)^\times$, then there is $[Q] \in K_0(R)^\times$ such that $[P\otimes_R Q] = [R]$. It follows by Lemma 2.1.2 that $P\otimes_R Q$ is stably free, i.e. $(P\otimes_R Q)\oplus R^n \cong R^{n+1}$. Comparing the rank, we see that $P\otimes_R Q$ is constant rank 1. Thus, $P\otimes_R Q$ is a line bundle which is stably free, so by an argument involving top exterior power, we deduce that $P\otimes_R Q$ is free of rank 1. By another rank argument, we deduce that P and Q are line bundles. Thus $\varphi([P]) = [P]$.

This is injective as if P is a line bundle such that $[P] = [R]$ in $K_0(R)$, then P is stably free by Lemma 2.1.2, and thus is free of rank 1, i.e. $P \cong R$ and is thus the identity of $\mathrm{Pic}(R)$, as required. □

2.2. **Computations.** There are few main computations for K_0 of a ring; PIDs, local rings and more generally, Dedekind domains. We recall that rings may not be commutative.

Theorem 2.2.1 (K_0 of a PID)**.** *Let R be a PID. Then the map*

$$\begin{aligned}\varphi : \mathbb{Z} &\longmapsto K_0(R) \\ 1 &\longmapsto [R]\end{aligned}$$

is an isomorphism.

Proof. Observe that $\varphi(n) = [R^n] = [R]+\dots+[R]$ n-times. To see injectivity, observe that if $\varphi(n) = [0]$, then $[R^n] = [0]$. It follows by Lemma 2.1.2, 3, that $R^{n+m} \cong R^m$. Going modulo any maximal ideal of R, we deduce that we have an $R/\mathfrak{m}$-vector space isomorphism $(R/\mathfrak{m})^{\oplus m+n} \cong (R/\mathfrak{m})^{\oplus m}$. It follows at once that $n = 0$, as required.

To see surjectivity, we need only show that $\mathrm{Im}\,(\varphi)$ contains the image of $\mathcal{P}roj(R)^{\cong}$ in $K_0(R)$. To this end, take any $P \in \mathcal{P}roj(R)^{\cong}$. We need only show that for some $n \in \mathbb{N}$, $[P] = [R^n]$ in $K_0(R)$. It suffices to show that every projective module over a PID is free. Indeed, this is true (see Proposition 23.23.0.8 of [FoG]). □

As a field is a PID, we deduce the following.

Corollary 2.2.2 (K_0 of a field)**.** *Let F be a field. Then $K_0(F) \cong \mathbb{Z}$.* □

Theorem 2.2.3 (K_0 of a local ring)**.** *Let $(R,\mathfrak{m})$ be a local ring. Then the group homomorphism*

$$\begin{aligned}\varphi : \mathbb{Z} &\longrightarrow K_0(R)\\ 1 &\longmapsto [R]\end{aligned}$$

is an isomorphism.

Proof. Observe that the map $\rho' : F(R) \to F(R/\mathfrak{m})$ given by $P \mapsto P \otimes_R R/\mathfrak{m} = P/\mathfrak{m}P$ defines a group homomorphism $\rho : K_0(R) \to K_0(R/\mathfrak{m}) = \mathbb{Z}$. Note that $\rho([R^n]) = [R^n/\mathfrak{m}R^n] = [(R/\mathfrak{m}R)^{\oplus n}]$ which yields $n \in \mathbb{Z}$ under ρ. Consequently, ρ is surjective. We need only show that ρ is injective. To this end, observe that it is sufficient to show that any finitely generated projective R-module is free, as ρ is injective over this subset of $K_0(R)$. Indeed, this is true (see Proposition 23.23.0.9 of [FoG]). ☐

The next computation is for Dedekind domains.

Theorem 2.2.4 (K_0 of a Dedekind domain)**.** *Let R be a Dedekind domain. Then*

$$K_0(R) \cong \mathbb{Z} \oplus \mathrm{Cl}(R)$$

is an isomorphism of groups where $\mathrm{Cl}(R)$ *is the ideal class group of R.*

However, we will show something general, which will showcase to us the use of geometric viewpoint. Recall that $H_0(X) = \mathbb{Z}^{\oplus r}$ where r is the number of path-components of X. Note that we may interpret $H_0(X) = C(X,\mathbb{Z})$, the set of all continuous functions from X to the discrete space $\mathbb{Z}$. We denote $H_0(R) = H_0(\mathrm{Spec}\,(R))$. Let R be a commutative ring. Then, any finitely generated projective module P gives a continuous map $\mathrm{rank}(P) : \mathrm{Spec}\,(R) \to \mathbb{Z}$ given by $\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} P \otimes_R \kappa(\mathfrak{p})$ (see Exercise I.2.11 of [Wei13]). Thus, we get the following result.

Lemma 2.2.5 (The rank map)**.** *Let R be a commutative ring. Then, the map*

$$\begin{aligned}\mathrm{rank} : K_0(R) &\longrightarrow H_0(R)\\ [P] &\longmapsto \mathrm{rank}(P)\end{aligned}$$

is a ring homomorphism.

Proof. We need only show that $\mathrm{rank}(P \oplus Q) = \mathrm{rank}(P) + \mathrm{rank}(Q)$ and $\mathrm{rank}(P \otimes_R Q) = \mathrm{rank}(P) \cdot \mathrm{rank}(Q)$. The former is immediate. For the latter, we need only observe that $P \otimes_R Q \otimes_R \kappa(\mathfrak{p}) \cong (P \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes_R Q)$, then the result follows from the basic fact of dimension of tensor product of vector spaces. ☐

Lemma 2.2.6. *Let R be a commutative ring. If R is noetherian, then $H_0(R)$ is a direct summand of $K_0(R)$. We write*

$$K_0(R) \cong H_0(R) \oplus \tilde{K}_0(R),$$

where $\tilde{K}_0(R) = \mathrm{Ker}\,(\mathrm{rank})$.

Proof. By Exercise I.2.4 of [Wei13], we have that for any continuous map $f : \mathrm{Spec}\,(R) \to \mathbb{Z}$, there is a decomposition $R = R_1 \times \dots R_k$ where on each $\mathrm{Spec}\,(R_i)$, f is constant n_i, say. Thus, for each such f, we construct the R-module

$$R^f = R_1^{n_1} \times \dots \times R_k^{n_k}$$

which we claim is finitely generated projective R-module. Indeed this can be seen by observing that if P_1 is projective R_1 and P_2 is projective R_2 modules, then $P_1 \oplus P_2$ is projective $R_1 \times R_2$-module, by making $P_1 \oplus P_2$ a direct summand of an $R_1 \times R_2$-free module. Thus, we define a map

$$\begin{aligned} H_0(R) &\longrightarrow K_0(R) \\ f &\longmapsto [R^f]. \end{aligned}$$

Now observe that the composite

$$H_0(R) \longrightarrow K_0(R) \xrightarrow{\text{rank}} H_0(R)$$

is such that $f \mapsto [R^f] \mapsto \mathrm{rank}(R^f)$. Since $\mathrm{rank}(R^f) = f$ by definition of R^f, thus the composite is id. It follows that we have a decomposition $K_0(R) \cong H_0(R) \oplus \mathrm{Ker}\,(\mathrm{rank})$, as required. □

As a Dedekind domain is noetherian, this gives us a hint towards the above result (Theorem 2.2.4).

Construction 2.2.7 (Picard group and determinant bundle)**.** Next map that we wish to discuss is the determinant map for a commutative ring, which will be a map from $K_0(R)$ to $\mathrm{Pic}(R)$, the Picard group of scheme $\mathrm{Spec}\,(R)$, which may be described as the commutative group of isomorphism classes of all finitely generated projective modules[4] of constant rank 1, where the group operation is $\otimes$ and inverse is taking dual module.

For a projective module P, we may define a rank 1 projective module given as follows. If P has constant rank, then $\det(P) = \wedge^{\mathrm{rank}(P)} P$, the top exterior power of P, which has rank 1 and is projective as it is locally free (since exterior powers commute with tensoring). Whereas if P doesn't have constant rank, then writing $R = R_1 \times \dots \times R_n$ such that $\mathrm{rank}(P)$ is constant on each $\mathrm{Spec}\,(R_i)$. We thus get a decomposition $P = P_1 \times \dots \times P_n$ where each P_i is a projective R_i-module. We then define $\det(P) = \wedge^{\mathrm{rank}(P_1)} P_1 \times \dots \times \wedge^{\mathrm{rank}(P_n)} P_n$. As an R-module, this has rank 1 by a simple tensor calculation.

The main observation here is the following.

Lemma 2.2.8 (The det map)**.** *Let R be a commutative ring. Then the map*

$$\det : K_0(R) \to \mathrm{Pic}(R)$$

is a surjective group homomorphism.

Proof. We need only show that det is a group homomorphism on the generators of $F(R)$. That is, we wish to show that $\det([P \oplus Q]) = \det([P]) \otimes_R \det([Q])$. To this end, by above discussion, we may reduce to assuming P has constant rank n and Q has constant rank m. Now, by binomial sum formula, we get, $\det(P \oplus Q) = \wedge^{n+m}(P \oplus Q) = \bigoplus_{i=0}^{n+m} \wedge^i P \otimes \wedge^{n+m-i} Q$. All of the terms except $\wedge^n P \otimes_R \wedge^m Q$ are zero above. Consequently, we get the required result. To see surjectivity, observe that any rank 1 projective module P is such that $\det([P]) = \wedge^1 P \cong P$. □

[4] also called algebraic line bundles over $\mathrm{Spec}\,(R)$.

We come to the main theorem.

Theorem 2.2.9. *Let R be a commutative ring. Then the map*

$$\mathrm{rank} \oplus \det : K_0(R) \longrightarrow H_0(R) \oplus \mathrm{Pic}(R)$$

is surjective with kernel being the ideal

$$SK_0(R) = \langle [P] - [R^m] \mid P \in \mathcal{P}roj(R)^{\cong} \text{ of constant rank } m \ \& \ \wedge^m P \cong R \ \rangle.$$

Proof. Indeed, $SK_0(R)$ is in the $\mathrm{Ker}\,(\mathrm{rank} \oplus \det)$. Conversely, as $\mathrm{Ker}\,(\mathrm{rank} \oplus \det) \subseteq \tilde{K}_0(R) = \mathrm{Ker}\,(\mathrm{rank})$ and $\tilde{K}_0(R)$ is the filtered limit of the set $F_n(R) = \{[P] \mid P \in \mathcal{P}roj(R)^{\cong}$ of constant rank $n\}$ via the map $F_n(R) \to \tilde{K}_0(R)$ mapping as $[Q] \mapsto [Q] - [R^n]$ (see Lemma 2.3.1 of [Wei13]), thus we deduce that any $[P] \in \mathrm{Ker}\,(\mathrm{rank} \oplus \det)$ is of form $[P] = [Q] - [R^m]$ where Q is projective of constant rank m.

As $[P] \in \mathrm{Ker}\,(\mathrm{rank} \oplus \det)$, therefore $\wedge^m Q = R$, as can be seen easily. ☐

Corollary 2.2.10. *Let R be a commutative noetherian ring of dimension one. Then*

$$\mathrm{rank} \oplus \det : K_0(R) \longrightarrow H_0(R) \oplus \mathrm{Pic}(R)$$

is an isomorphism.

The proof of Theorem 2.2.4 is immediate as $\mathrm{Pic}(R)$ of a Dedekind domain is the ideal class group.

Proof. By classification of finitely generated projective modules over a commutative noetherian ring of dimension one, it follows that if $P \in \mathcal{P}roj(R)^{\cong}$ is of constant rank, then it is isomorphic to $\det(P) \oplus R^{\mathrm{rank}(P)-1}$. Consequently, the ideal $SK_0(R) = 0$ and we conclude by Theorem 2.2.9. ☐

Another simple calculation is of Artin rings.

Proposition 2.2.11. *Let R be an Artinian ring with $|\mathrm{mSpec}\,(R)| = n$. Then*

$$K_0(R) \cong \mathbb{Z}^{\oplus n}.$$

Proof. † By structure theorem for Artinian rings, we know that $A \cong \prod_{i=1}^n A/\mathfrak{m}_i^k$ for some $k > 0$ where $\mathrm{mSpec}\,(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ and each $A/\mathfrak{m}_i^k$ is an Artin local ring (see Theorem 8.7, pp 90, [AM69]). By Lemma 2.1.6, we deduce that

$$K_0(R) \cong \prod_{i=1}^n K_0(A/\mathfrak{m}_i^k).$$

By Theorem 2.2.3, we have that $K_0(A/\mathfrak{m}_i^k) \cong \mathbb{Z}$, giving the required result. ☐

There is a famous important calculation of K_0 done by Quillen and Suslin around the same time.

Theorem 2.2.12 (Quillen-Suslin)**.** *Let R be a PID (for example, a field). Then all projective modules over $R[x_1, \dots, x_n]$ is free. In particular,*

$$K_0(R[x_1, \dots, x_n]) \cong \mathbb{Z}.$$

2.3. **Homological properties of** K_0**.** We have already seen the reduced K_0, denoted $\tilde{K}_0$ in Lemma 2.2.6. We thus discuss some other homology-type results for K_0.

2.3.1. *Relative exact sequence for* K_0*.* To discuss excision and Mayer-Vietoris, we first need to understand what will be the analogue of union of two subspaces in this context. It is perhaps not that surprising that the right answer is geometrically motivated.

Lemma 2.3.1 (Milnor squares)**.** *Let* $f : R \to S$ *be a ring homomorphism and* $I \leq R$ *be an ideal such that* $f|_I : I \to f(I)$ *is a bijection onto an ideal* $f(I) \leq S$ *which we also denote by* I*. Then the following is a pullback square of rings:*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, two heads] \\ R/I \arrow[r, "\bar{f}"'] & S/I \end{tikzcd}.$$

We call them Milnor squares.

Proof. Recall that

$$R/I \times_{\bar{f}} S := \frac{R/I \times S}{\langle (r+I, s) \in R/I \times S \mid f(r) - s \in I \rangle}.$$

Consider the map

$$\begin{aligned} R &\longrightarrow R/I \times_{\bar{f}} S \\ r &\longmapsto (r+I, f(r)). \end{aligned}$$

This is injective as if $r \in I$ and $f(r) = 0 \in I$, then since $f|_I$ is bijective, then $r = 0$. This is surjective as for any $(r+I, s) \in R/I \times_{\bar{f}} S$, we have that $f(r) - s = f(i)$, $i \in I$ and thus $s = f(r-i)$. Consequently, $r - i \mapsto (r - i + I, f(r-i)) = (r+I, s)$, as required. $\square$

Remark 2.3.2. A Milnor square equivalently yields the following pushout diagram of affine schemes:

$$\begin{tikzcd} \operatorname{Spec}(R) \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec}(S) \arrow[l, "f^*"'] \\ \operatorname{Spec}(R/I) \arrow[u, hook] & \operatorname{Spec}(S/I) \arrow[l, "\bar{f}^*"] \arrow[u, hook] \end{tikzcd}.$$

Consequently, we get that $\operatorname{Spec}(R)$ is obtained by gluing $\operatorname{Spec}(R/I)$ to $\operatorname{Spec}(S)$ along the closed subspace $\operatorname{Spec}(S/I) \hookrightarrow \operatorname{Spec}(S)$ via the map $\bar{f}^*$, that is,

$$\operatorname{Spec}(R) = \operatorname{Spec}(S) \amalg_{\bar{f}^*} \operatorname{Spec}(R/I).$$

Remark 2.3.3 (Milnor squares and excisive triples)**.** By the above remark, it is clear that Milnor squares behave as excisive triples as seen in topology. That is a Milnor square

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, two heads] \\ R/I \arrow[r, "\bar{f}"'] & S/I \end{tikzcd}$$

can be seen as the excisive triple $(\mathrm{Spec}\,(R), \mathrm{Spec}\,(S), \mathrm{Spec}\,(R/I))$ for the space $\mathrm{Spec}\,(R)$.

Using the above idea, we can now define K-groups relative to an ideal I as follows.

Definition 2.3.4 (**Relative** K_0)**.** Let R be a commutative ring and $I \leq R$ be an ideal. Consider the commutative ring $R \oplus I$ with 1 whose operation is

$$(r, x) \cdot (s, y) := (rs, ry + sx + xy),$$

where identity is $(1, 0)$. We call this the augmented ring. Consider the projection homomorphism $p : R \oplus I \to R$. This induces the map $p_* : K_0(R \oplus I) \to K_0(R)$. We thus define

$$K_0(R, I) := \mathrm{Ker}\,(p_* : K_0(R \oplus I) \to K_0(R)).$$

We call $K_0(R, I)$ the relative K_0-group w.r.t ideal I. As the composite of the ring homomorphisms $R \to R \oplus I \to R$ is identity, therefore after applying K_0, we get $K_0(R) \to K_0(R \oplus I) \to K_0(R)$ is identity. It follows by splitting lemma that we have

$$K_0(R \oplus I) \cong K_0(R) \oplus K_0(R, I).$$

We define now a group which we will meet consistently.

Definition 2.3.5 ($\mathrm{GL}(R)$)**.** Let R be a ring and $\mathrm{GL}_n(R) = \mathrm{Aut}\,(R^n)$, the group of R-linear automorphisms of the free module R^n where the group operation is composition. We may think of $\mathrm{GL}_n(R)$ as $n \times n$ invertible matrix over R. Observe that we have injective maps

$$\begin{aligned} \mathrm{GL}_n(R) &\overset{\iota_n}{\hookrightarrow} \mathrm{GL}_{n+1}(R) \\ g_n &\longmapsto \begin{bmatrix} g_n & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Consequently we have a directed system $\{\mathrm{GL}_n(R), \iota_n\}_n$. We define

$$\mathrm{GL}(R) := \varinjlim_n \mathrm{GL}_n(R)$$

An element $[g_n] \in \mathrm{GL}(R)$ for some $g_n \in \mathrm{GL}_n(R)$ is an equivalence class where $g_n \sim h_m$ for some $h_m \in \mathrm{GL}_m(R)$ if and only if $\iota_{n,k}(g_n) = \iota_{m,k}(h_m)$ in $\mathrm{GL}_k(R)$ for some $k \geq n, m$. Observe that the group operation of $\mathrm{GL}(R)$ is as follows: for $g_n, h_m \in \mathrm{GL}(R)$, $[g_n] \cdot [h_m] = [\iota_{n,p}(g_n) \cdot \iota_{m,p}(h_m)]$ for some $p \geq n, m$. It is clear that $[g_n]$ only consists of all the elements $\iota_{n,k}(g_n)$ for all $k \geq n$, i.e. $[g_n]$ is the infinite invertible matrix obtained by padding by 1 on diagonals.

The main theorem here is the following:

Theorem 2.3.6. *Let R, S be commutative rings and let $I \leq R$ be an ideal. Then we have a natural exact sequence*

$$\mathrm{GL}(R) \to \mathrm{GL}(R/I) \xrightarrow{\partial} K_0(R, I) \to K_0(R) \to K_0(R/I).$$

2.3.2. *Excision for K_0.* A main observation here yields the independence of $K_0(R, I)$ on R.

Theorem 2.3.7. *Let R be a ring and $I \leq R$ be an ideal. If $f : R \to S$ is a ring homomorphism which maps ideal I isomorphically to an ideal $f(I)$ of S (which we denote by I again), then*

$$K_0(R, I) \cong K_0(S, I).$$

Proof. See Exercise II.2.3 of [Wei13]. □

2.3.3. *Mayer-Vietoris for K_0.* We begin by constructing a Mayer-Vietoris sequence for K_0, which will be later extended to a long-exact sequence while discussing higher K-groups, just like in homology theory. We do this essentially by using excision, as is usually done in singular homology.

Theorem 2.3.8 (Mayer-Vietoris)**.** *Consider a Milnor square*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, two heads] \\ R/I \arrow[r, "\bar{f}"'] & S/I \end{tikzcd} .$$

Then there is a long exact sequence

$$\mathrm{GL}(S/I) \to K_0(R) \to K_0(S) \oplus K_0(R/I) \to K_0(S/I).$$

Proof. Observe that we have maps relative sequences of Theorem 2.3.6 as follows:

$$\begin{tikzcd} \mathrm{GL}(R) \arrow[r] \arrow[d] & \mathrm{GL}(R/I) \arrow[r, "\partial"] \arrow[d] & K_0(R,I) \arrow[r] \arrow[d, "\cong"'] & K_0(R) \arrow[r] \arrow[d] & K_0(R/I) \arrow[d] \\ \mathrm{GL}(S) \arrow[r] & \mathrm{GL}(S/I) \arrow[r, "\partial"'] & K_0(S,I) \arrow[r] & K_0(S) \arrow[r] & K_0(S/I) \end{tikzcd}$$

where the middle vertical map is the excision isomorphism of Theorem 2.3.7. By Barrat-Whitehead lemma (Lemma 14.6, [GH19]), we get an exact sequence

$$\mathrm{GL}(S/I) \to K_0(R) \to K_0(S) \oplus K_0(R/I) \to K_0(S/I),$$

as required. □

2.4. **Applications.** We present some applications of the K_0 and its calculations.

2.4.1. *Wall's finiteness obstruction.*

Construction 2.4.1 (0^{th}-Whitehead group of a group G)**.** Let G be a group and $\mathbb{Z}[G]$ be the group ring of G[5]. Then we have the following commutative diagram of rings:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\iota"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{Z}[G] \arrow[r, "\epsilon"] & \mathbb{Z} \end{tikzcd} .$$

Applying K_0, we get the following commutative diagram of groups:

$$\begin{tikzcd} K_0(\mathbb{Z}) \arrow[r, "\iota_*"] \arrow[rr, bend right, "\mathrm{id}"'] & K_0(\mathbb{Z}[G]) \arrow[r, "\epsilon_*"] & K_0(\mathbb{Z}) \end{tikzcd} .$$

It hence follows that $\iota_* : K_0(\mathbb{Z}) \hookrightarrow K_0(\mathbb{Z}[G])$ is an injective map. We define the 0^{th}-Whitehead group of G to be the cokernel of ι_*:

$$\mathrm{Wh}_0(G) := \mathrm{CoKer}\,(\iota_*) = K_0(\mathbb{Z}[G])/K_0(\mathbb{Z}).$$

[5] Note that if G is infinite cyclic group with generator x, then $\mathbb{Z}[G] = \mathbb{Z}[x]$, the polynomial ring.

Moreover, as $\epsilon_* \circ \iota_* = \mathrm{id}_*$, therefore the following s.e.s. is split on the left:

$$\begin{tikzcd} 0 \arrow[r] & K_0(\mathbb{Z}) \arrow[r, "\iota_*"] & K_0(\mathbb{Z}[G]) \arrow[l, bend left, "\epsilon_*"] \arrow[r] & \mathrm{Wh}_0(G) \arrow[r] & 0 \end{tikzcd}.$$

Consequently, we get the following decomposition of $K_0(\mathbb{Z}[G])$:

$$K_0(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}) \oplus \mathrm{Wh}_0(G).$$

Remark 2.4.2 (Information in $\mathrm{Wh}_0(G)$.)**.** Observe that the map $\iota_* : K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[G])$ maps the generator $[\mathbb{Z}]$ to $[\mathbb{Z}[G]]$. As $\mathrm{Wh}_0(G) := \mathrm{CoKer}\,(\iota_*)$, thus we deduce that $\mathrm{Wh}_0(G)$ is that part of $K_0(\mathbb{Z}[G])$ which stores the information of non-stably free projective f.g. $\mathbb{Z}[G]$-modules (see Corollary 2.1.3). Thus, we deduce that

$$\mathrm{Wh}_0(G) = 0 \implies K_0(\mathbb{Z}[G]) \cong \mathbb{Z}.$$

Moreover, if $[P] = [Q]$ in $\mathrm{Wh}_0(G)$, then $[P] - [Q] \in K_0(\mathbb{Z})$ and thus $[P] - [Q] = [\mathbb{Z}[G]^n]$ in $K_0(\mathbb{Z}[G])$. It follows that $[P] = [Q \oplus \mathbb{Z}[G]^n]$ and hence by Lemma 2.1.2, we deduce that $P \oplus \mathbb{Z}[G]^k \cong Q \oplus \mathbb{Z}[G]^{n+k}$. We thus deduce that

$$[P] = [Q] \text{ in } \mathrm{Wh}_0(G) \iff \exists n, k \in \mathbb{N} \text{ s.t. } P \oplus \mathbb{Z}[G]^k \cong Q \oplus \mathbb{Z}[G]^{n+k}.$$

The information stored in $\mathrm{Wh}_0(G)$ is thus quite non-trivial (K_0-classes of non-stably free f.g. projective $\mathbb{Z}[G]$-modules).

The following theorem explains the interest in $\mathrm{Wh}_0(G)$. Recall that a CW-complex X is *dominated* by a complex K if there is a map $f : K \to X$ which exhibits X as a homotopy retract of K.

Theorem 2.4.3 (Wall's finiteness obstruction)**.** *Let X be a CW-complex which is dominated by a finite CW-complex K. Denote $G = \pi_1(X)$.*

(1) The tuple (X, K, G) determines an element $w(X) \in \mathrm{Wh}_0(G)$ which is independent of dominating complex K.
(2) The following are equivalent:
  (a) X is homotopy equivalent to a finite CW-complex.
  (b) $w(X) = 0$ in $\mathrm{Wh}_0(X)$.

3. K_1

The Grothendieck group $K_0(R)$ is different to all higher K-groups in the sense that it looks at the spread of projective modules while the rest only look at the eventual behaviour as the size of those modules grows.

Hyman Bass's group $K_1(R)$ is the intended "value group for the determinant" of an invertible matrix over R and can be defined as the abelianization of the direct limit of automorphism groups of finitely generated projective modules. The usual process of Gaussian elimination and Dieudonne's determinant map will help us to compute $K_1(R)$ in familiar examples.

3.1. K_1 of a ring & basic properties.

We begin by defining $K_1(R)$.

Definition 3.1.1 ($K_1(R)$)**.** Let R be a ring. We define $K_1(R)$ to be the abelianisation of $\mathrm{GL}(R)$:

$$K_1(R) := \frac{\mathrm{GL}(R)}{[\mathrm{GL}(R), \mathrm{GL}(R)]}.$$

We immediately have a decomposition of $K_1(R)$ as follows.

Construction 3.1.2 (Dieudonné's det for K_1)**.** Let R be a commutative ring. Consider the map

$$\det : \mathrm{GL}_n(R) \to R^\times$$

which is the determinant map. As $\det \circ \iota_n = \det$ by the determinant of block diagonals, therefore we obtain a map

$$\begin{aligned} \det : \mathrm{GL}(R) &\to R^\times \\ [g_n] &\mapsto \det(g_n). \end{aligned}$$

It is easy to see that this is a group homomorphism. Moreover, this map is surjective. Observe that since det of an element in the commutator is 1, thus by universal property of quotients, we get a surjective group homomorphism

$$\begin{aligned} \det : K_1(R) &\longrightarrow R^\times \\ \overline{[g_n]} &\longmapsto \det(g_n) \end{aligned}$$

which is called Dieudonné's determinant[6]. We further denote its kernel as

$$SK_1(R) := \mathrm{Ker}\left(\det : K_1(R) \twoheadrightarrow R^\times\right).$$

As the composite $R^\times \to K_1(R) \to R^\times$ given by $u \mapsto \overline{[u]} \mapsto \det(u) = u$ is identity, it follows that we have a splitting:

$$K_1(R) \cong R^\times \oplus SK_1(R).$$

Note that the kernel of $\det : \mathrm{GL}_n(R) \to R^\times$ is exactly $\mathrm{SL}_n(R)$.

The K_1 of a ring commutes with product.

[6] We sometimes call $\det : \mathrm{GL}(R) \to R^\times$ as Dieudonné's determinant as well.

Lemma 3.1.3. *Let $R = R_1 \times R_2$ be a ring where R_i are rings. Then*

$$K_1(R) \cong K_1(R_1) \times K_1(R_2).$$

Proof. Since $\mathrm{GL}_n(R) \cong \mathrm{GL}_n(R_1) \times \mathrm{GL}_n(R_2)$ via the map $(m^1_{ij}, m^2_{ij}) \mapsto ((m^1_{ij}), (m^2_{ij}))$, therefore this yields an isomorphism $\mathrm{GL}(R) \cong \mathrm{GL}(R_1) \times \mathrm{GL}(R_2)$. Moreover, under the same isomorphism, it can be checked that $[\mathrm{GL}(R) : \mathrm{GL}(R)] \cong [\mathrm{GL}(R_1) : \mathrm{GL}(R_1)] \times [\mathrm{GL}(R_2) : \mathrm{GL}(R_2)]$. This completes the proof. $\square$

We next see that K_1 of a ring and matrix ring are equivalent.

Proposition 3.1.4. *Let R be a ring and $n \in \mathbb{N}$. Then*

$$K_1(R) \cong K_1(M_n(R)).$$

Proof. To this end, it suffices to show that $\mathrm{GL}(R) \cong \mathrm{GL}(M_n(R))$. We do this via basic Morita theory. We know that R and $M_n(R)$ are Morita equivalent, where the Morita functors are

$$\begin{aligned} \mathcal{M}od(R) &\leftrightarrow \mathcal{M}od(M_n(R)) \\ M &\mapsto M \otimes_R R^n \\ N \otimes_{M_n(R)} R^n &\leftarrow\!\shortmid N \end{aligned}$$

where $M \otimes_R R^n \cong M^n$ is an $M_n(R)$-module by matrix multiplication: $(r_{ij}) \cdot (m_1, \ldots, m_n) = \left(\sum_j r_{1j} m_j, \ldots, \sum_j r_{nj} m_j\right)$. Similarly, $N \otimes_{M_n(R)} R^n$ is an R-module. The equivalence thus takes the $M_n(R)$-module to the R-module $M_n(R)^m$ to $M_n(R)^m \otimes_{M_n(R)} R^n \cong R^{nm} \cong R^n \otimes_R R^m$. Consequently, the equivalence of categories gives an isomorphism of the endomorphism groups $M_{mn}(R) \cong \mathrm{End}_R(R^n \otimes_R R^m) \cong \mathrm{End}_{M_n(R)}(M_n(R)^m) = M_m(M_n(R))$. Thus, the group of units are also isomorphic, yielding

$$\mathrm{GL}_{mn}(R) \cong \mathrm{GL}_m(M_n(R)).$$

As the multiples of n are cofinal in $\mathbb{N}$, thus we get an isomorphism in the direct limit

$$\mathrm{GL}(R) \cong \mathrm{GL}(M_n(R)).$$

Thus the commutators are also isomorphic and the above isomorphism thus extends to the isomorphism $K_1(R) \cong K_1(M_n(R))$. $\square$

We will later see that actually K_1 is a Morita invariant.

Remark 3.1.5 (Information in $K_1(R)$)**.** While $K_0(R)$ tells us about the spread of f.g. projective R-modules, $K_1(R)$ tells us about the eventual spread of automorphisms of f.g. projective modules. The latter may not be evident as of now, but the next few results will showcase exactly this, as will be seen by Whitehead's lemma and Bass' theorem.

Our first task is to show that the commutator of $\mathrm{GL}(R)$ is a familiar object.

Definition 3.1.6 ($E(R)$)**.** Consider the subgroup of $\mathrm{GL}_n(R)$ denoted $E_n(R)$ generated by elementary n-matrices of type 1, that is, the invertible matrices $e_{ij}(r) \in \mathrm{GL}_n(R)$ where $i \neq j$ such that their diagonal is all 1, $e_{ij} = r$ at (i,j) entry and 0 in the rest. Thus we have an injection

$$E_n(R) \hookrightarrow \mathrm{GL}_n(R).$$

Taking direct limits, we get an injection

$$E(R) \hookrightarrow \mathrm{GL}(R).$$

A simple exercise shows that $E_n(F) = \mathrm{SL}_n(F)$ for F a field.

Lemma 3.1.7. *Let R be an Euclidean domain. Then*

$$E_n(R) = \mathrm{SL}_n(R).$$

Consequently, $E(R) = \mathrm{SL}(R)$ in $\mathrm{GL}(R)$.

Proof. We proceed by induction on $n \in \mathbb{N}$. Indeed, for $n = 1$, we have $E_1(R) = \mathrm{SL}_1(R)$. For the inductive step, fix $n \in \mathbb{N}$. We wish to show that $\mathrm{SL}_n(R) \subseteq E_n(R)$. Pick $g \in \mathrm{SL}_n(R)$ and write

$$g = \begin{bmatrix} g_{11} & g_{12} & \cdots \\ g_{21} & g_{22} & \cdots \\ \vdots & & \end{bmatrix}.$$

As R is a Euclidean domain, therefore by Euclidean division, we get

$$\begin{aligned} g_{21} &= g_{11}q_1 + r_1 \\ g_{11} &= r_1 q_2 + r_2 \\ r_1 &= r_2 q_3 + r_3 \\ \vdots &= \vdots \\ r_{k-1} &= r_k q_{k+1} \end{aligned}$$

Note all the above operations can be done by type 1 row operations. Consequently, by doing these row and column operations we can reduce g to

$$ege' = \begin{bmatrix} g'_{11} & 0 & \cdots \\ 0 & g'_{22} & \cdots \\ \vdots & & \end{bmatrix} = \begin{bmatrix} g'_{11} & 0 \\ 0 & M_{n-1} \end{bmatrix}$$

Now, by inductive hypothesis, there exists $e_{n-1}, e'_{n-1} \in E_{n-1}(R) \hookrightarrow E_n(R)$ such that $e_{n-1}M_{n-1}e'_{n-1} = 1_{n-1}$. Taking the image of e_{n-1}, e'_{n-1} in $E_n(R)$, we get $e_n, e'_n \in E_n(R)$ such that

$$e_n e g e' e'_n = \begin{bmatrix} g'_{11} & 0 \\ 0 & I_{n-1} \end{bmatrix}.$$

As $\det(e_n e g e' e'_n) = 1$, therefore we deduce that $g'_{11} = 1$, and thus

$$e_n e g e' e'_n = I_n,$$

hence $g = e^{-1}e_n^{-1}e_n'^{-1}e'^{-1} \in E_n(R)$, as required. This completes the proof. □

Theorem 3.1.8 (Whitehead's lemma)**.** *Let R be a ring. Then $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$. Consequently,*

$$K_1(R) = \frac{\mathrm{GL}(R)}{E(R)}.$$

Remark 3.1.9. It follows that $K_1(R)$ consists of all classes of all infinite invertible matrices which are not similar to each other by type 1 elementary matrices. Recall that two matrices $g, h \in M_n(R)$ are similar to each other if there exists e, e' an elementary matrix (not necessarily only of type 1) such that $g = e'he$. Thus, it is in this sense does $K_1(R)$ measures the eventual spread of automorphisms of free modules. In the next section, we will see that more is true.

Remark 3.1.10 ($E(R)$ is perfect in $\mathrm{GL}(R)$)**.** Observe the following three basic relations for elements in $E_n(R)$ for $n \geq 3$:

$$\begin{aligned} &e_{ij}^{(n)}(\lambda) \cdot e_{ij}^{(n)}(\mu) = e_{ij}^{(n)}(\lambda + \mu) \\ &[e_{ij}^{(n)}(\lambda), e_{kl}^{(n)}(\mu)] = 1, \quad i \neq l,\ j \neq k \\ &[e_{ij}^{(n)}(\lambda), e_{jk}^{(n)}(\mu)] = e_{ik}^{(n)}(\lambda\mu), \quad i \neq k. \end{aligned}$$

The last relation combined with Theorem 3.1.8 immediately tells us that $E_n(R)$ is perfect and hence so is $E(R)$.

3.1.1. *Basic computations.* Using Whitehead's lemma and Dieudonné's determinant, we can do some basic computations.

Lemma 3.1.11. *Let R be an Euclidean domain. Then,*

$$K_1(R) \cong R^{\times}.$$

Proof. Using Dieudonné's determinant, we have a surjective map

$$\det : \mathrm{GL}(R) \twoheadrightarrow R^{\times}.$$

As $\mathrm{Ker}\,(\det) = \mathrm{SL}(R)$ and by Lemma 3.1.7, $\mathrm{SL}(R) = E(R)$, thus by Whitehead's lemma (Theorem 3.1.8), we conclude that $K_1(R) \cong \mathrm{GL}(R)/\mathrm{Ker}\,(\det) \cong R^{\times}$. ☐

Corollary 3.1.12. *Let F be a field. Then $K_1(F) = F^{\times}$.* ☐

Proposition 3.1.13. *Let D be a division ring (a non-commutative field). Then,*

$$K_1(D) \cong \frac{D^{\times}}{[D^{\times}, D^{\times}]}.$$

Proof. This follows from Dieudonné's theorem that for a division ring,

$$\frac{\mathrm{GL}_n(D)}{E_n(D)} \cong \frac{D^{\times}}{[D^{\times}, D^{\times}]}$$

for $n \geq 3$. This induces the required isomorphism. ☐

Proposition 3.1.14. *Let R be a semilocal ring. Then $SK_1(R) = 1$ and thus*

$$K_1(R) \cong R^{\times}.$$

Proof. See Lemma 1.4, pp 202 of [Wei13]. ☐

Corollary 3.1.15. *Let R be an Artinian ring. Then, $K_1(R) \cong R^{\times}$.* ☐

3.2. **More properties of K_1.** We study some properties of $K_1(R)$ which depends on R-modules. We begin by relating group homology and $K_1(R)$.

3.2.1. *Characteristic map & Bass' result.* We now see how $K_1(R)$ is actually the eventual spread of automorphisms of f.g. projective R-modules. We first show that every $P \in \mathcal{P}roj(R)$ yields a group homomorphism $\mathrm{Aut}\,(P) \to K_1(R)$.

Construction 3.2.1 ($\chi_P : \mathrm{Aut}\,(P) \to K_1(R)$)**.** Let P be a f.g. projective R-module. We construct a characteristic map

$$\chi_P : \mathrm{Aut}\,(P) \longrightarrow K_1(R).$$

For each $Q \in \mathcal{P}roj(R)$ and $n \in \mathbb{N}$ such that $P \oplus Q \cong R^n$ via a map $\theta : P \oplus Q \to R^n$, we get a map

$$\begin{aligned} \chi_{\theta,Q,n} : \mathrm{Aut}\,(P) &\longrightarrow K_1(R) \\ \varphi &\longmapsto \overline{[\theta \circ (\varphi \oplus \mathrm{id}_Q) \circ \theta^{-1}]} \end{aligned}$$

We first show that $\chi_{\theta,Q,n}$ is independent of θ if n and Q are fixed. Indeed, it is immediate to see that if we take $\theta' : P \oplus Q \to R^n$ a different isomorphism, then $\chi_{\theta,Q,n}$ and $\chi_{\theta',Q,n}$ are conjugates by an $\theta \circ \theta'^{-1} \in \mathrm{GL}_n(R)$. Thus, in $K_1(R)$, $\chi_{\theta,Q,n}$ determines a unique class for each $\varphi \in \mathrm{Aut}\,(P)$, independent of θ. We may thus write

$$\chi_{Q,n} : \mathrm{Aut}\,(P) \longrightarrow K_1(R).$$

Next we show that $\chi_{Q\oplus R^k,n+k} = \chi_{Q,n}$, that is, χ is stable. Indeed, this is immediate as the image of the above maps factor through $\mathrm{GL}(R)$.

Now suppose we have an isomorphism

$$P \oplus Q' \cong R^m$$

where $m > n$. As $P \oplus Q \cong R^n$, so we get $P \oplus (Q \oplus R^m) \cong R^{m+n} \cong P \oplus (Q' \oplus R^n)$. Thus we get

$$Q \oplus R^{m+k} \cong Q' \oplus R^{n+k}.$$

Hence by previous, we have $\chi_{Q,n} = \chi_{Q\oplus R^{m+k},m+n+k} = \chi_{Q'\oplus R^{n+k},m+n+k} = \chi_{Q',m}$, as required.

Definition 3.2.2 (**Characteristic map**)**.** Let R be a ring and $P \in \mathcal{P}roj(R)$. Then the map $\chi_P : \mathrm{Aut}\,(P) \to K_1(R)$ constructed in Construction 3.2.1 will be called the characteristic map of P.

We now state a classical theorem of Bass, which shows the real intent behind defining $K_1(R)$.

Theorem 3.2.3 (Bass)**.** *Let R be a ring and denote by $t\mathcal{P}roj(R)^{\cong}$ to be the filtered category whose objects are isomorphism classes of f.g. projective R-modules and an arrow $P \to P'$ is an isomorphism class of Q such that $P \oplus Q \cong P'$. Then,*

$$K_1(R) \cong \varinjlim_{P \in t\mathcal{P}roj(R)^{\cong}} \frac{\mathrm{Aut}\,(P)}{[\mathrm{Aut}\,(P), \mathrm{Aut}\,(P)]}.$$

Proof. Note that the mapping $P \mapsto \frac{\mathrm{Aut}(P)}{[\mathrm{Aut}(P),\mathrm{Aut}(P)]}$ is a functor $t\mathcal{P}roj(R)^{\cong} \to \mathcal{A}b$ where for $P \to P'$ given by $\theta : P \oplus Q \xrightarrow{\cong} P'$, the functor maps it to the mapping

$$\begin{aligned} \frac{\mathrm{Aut}\,(P)}{[\mathrm{Aut}\,(P), \mathrm{Aut}\,(P)]} &\longrightarrow \frac{\mathrm{Aut}\,(P')}{[\mathrm{Aut}\,(P'), \mathrm{Aut}\,(P')]} \\ \bar{\varphi} &\longmapsto \overline{\theta \circ (\varphi \oplus \mathrm{id}_Q) \circ \theta^{-1}}. \end{aligned}$$

It can be shown that $t\mathcal{P}roj(R)^{\cong}$ is a filtered category[7]. As we have

$$K_1(R) = \frac{\mathrm{GL}(R)}{[\mathrm{GL}(R), \mathrm{GL}(R)]} \cong \varinjlim_n \frac{\mathrm{GL}_n(R)}{[\mathrm{GL}_n(R), \mathrm{GL}_n(R)]}$$

and free modules are cofinal in the filtered category $t\mathcal{P}roj(R)^{\cong}$, thus we get that

$$\varinjlim_n \frac{\mathrm{GL}_n(R)}{[\mathrm{GL}_n(R), \mathrm{GL}_n(R)]} \cong \varinjlim_{P \in t\mathcal{P}roj(R)^{\cong}} \frac{\mathrm{Aut}\,(P)}{[\mathrm{Aut}\,(P), \mathrm{Aut}\,(P)]},$$

as required. □

Remark 3.2.4. The proof of Theorem 3.2.3 shows that for any f.g. projective module P, the characteristic map

$$\chi_P : \mathrm{Aut}\,(P) \to K_1(R)$$

are the maps into the filtered limit $\varinjlim_P \frac{\mathrm{Aut}(P)}{[\mathrm{Aut}(P),\mathrm{Aut}(P)]}$. Indeed, this is evident from the functor $P \mapsto \frac{\mathrm{Aut}(P)}{[\mathrm{Aut}(P),\mathrm{Aut}(P)]}$ and Construction 3.2.1. In particular the following diagram commutes:

$$\begin{tikzcd} K_1(R) & \varinjlim_{Q \in t\mathcal{P}roj(R)^{\cong}} \frac{\mathrm{Aut}(Q)}{[\mathrm{Aut}(Q),\mathrm{Aut}(Q)]} \arrow[l, "\cong"'] \\ \mathrm{Aut}\,(P) \arrow[u, "\chi_P"] \arrow[r, two heads, "\pi"'] & \frac{\mathrm{Aut}(P)}{[\mathrm{Aut}(P),\mathrm{Aut}(P)]} \arrow[u, "\iota_P"'] \end{tikzcd}.$$

3.2.2. *Generators for $SK_1(R)$.* Recall by Construction 3.1.2 that we have a decomposition

$$K_1(R) \cong R^{\times} \oplus SK_1(R)$$

where $SK_1(R) = \mathrm{Ker}\,(\det : K_1(R) \to R^{\times})$. Hence, to understand $K_1(R)$, it is sufficient to understand the subgroup $SK_1(R)$. Indeed, in certain cases on R, there is a class of elements of $SK_1(R)$ which is known to be its generating set.

[7]For two objects P, P', we have $P \oplus Q \cong R^n$ and $P' \oplus Q' \cong R^m$. Consequently, we have $P \oplus Q \oplus R^m \cong R^{n+m}$ and $P' \oplus Q' \oplus R^n \cong R^{n+m}$. For existence of coequalizers, observe that for $P \oplus Q_1 \cong P' \cong P \oplus Q_2$, we get $[Q_1] = [Q_2]$ in $K_0(R)$. By Lemma 2.1.2, we deduce that $Q_1 \oplus R^n \cong Q_2 \oplus R^n$. Thus, the map

$$P' \xrightarrow{R^n} P' \oplus R^n$$

coequalizes both Q_1 and Q_2, as required.

Definition 3.2.5 (Mennicke symbols). Let R be a commutative ring. A Mennicke symbol is an element of $SK_1(R)$ which is determined by following. Consider $a, b \in R$ such that $aR + bR = R$. Thus there exists $c, d \in R$ such that $ad - bc = 1$. The matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(R)$$

is such that its class in $K_1(R)$ is in $\mathrm{Ker}\,(\det : K_1(R) \to R^\times) = SK_1(R)$. Thus we denote the class of the above matrix in $SK_1(R)$ as $[a, b]$, which we call the Mennicke symbol correspoding to a, b.

The obvious observation here is that Mennicke symbol $[a, b]$ doesn't depend on c, d.

Lemma 3.2.6. *Let R be a commutative ring and let $aR + bR = R$ such that $ad - bc = 1$. Then the Mennicke symbol $[a, b]$ is independent of $c, d \in R$.*

Proof. Indeed, we need only observe that if $ad - bc = 1 = ad' - bc'$, then we have

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c' & d' \end{bmatrix}^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} d' & -b \\ -c' & a \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ cd' - dc' & 1 \end{bmatrix} \in E_2(R) \end{aligned}$$

and thus the class of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c' & d' \end{bmatrix}^{-1}$$

in $K_1(R)$ is contained in $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ (Theorem 3.1.8). Hence in $K_1(R)$ both the classes are same and is in $SK_1(R)$. $\square$

The main point of Mennicke symbols is the following result which clarifies the dimension 1 case.

Theorem 3.2.7. *Let R be a commutative noetherian ring of dimension 1.*

(1) Mennicke symbols generates $SK_1(R)$.

(2) If $\kappa(\mathfrak{m})$ is a finite field for all $\mathfrak{m} \in \mathrm{mSpec}\,(R)$, then every element of $SK_1(R)$ has torsion.

A famous result calculates the $SK_1(\mathcal{O}_F)$ for a number field $F/\mathbb{Q}$.

Theorem 3.2.8 (Serre, Milnor, Bass)**.** *Let $K/\mathbb{Q}$ be a number field. Then,*

$$SK_1(\mathcal{O}_F) = 0.$$

4. K_2

We discuss some basic results about K_2, relegating the more homological discussions about it to higher K-theory.

Definition 4.0.1 (Steinberg group & K_2). Recall that the n^{th}-Steinberg group $\mathrm{St}_n(R)$ is the quotient of the free group generated by the symbols $x_{ij}^{(n)}(\lambda)$, $\lambda \in R$ and $1 \le i \ne j \le n$ by the subgroup generated by the known relations which elementary $n \times n$-matrices of type 1 satisfies:

$$\begin{aligned} &x_{ij}^{(n)}(\lambda) \cdot x_{ij}^{(n)}(\mu) \cdot x_{ij}^{(n)}(\lambda + \mu)^{-1} \\ &[x_{ij}^{(n)}(\lambda), x_{kl}^{(n)}(\mu)], \quad i \ne l,\ j \ne k \\ &[x_{ij}^{(n)}(\lambda), x_{jk}^{(n)}(\mu)] \cdot x_{ik}^{(n)}(\lambda\mu)^{-1}, \quad i \ne k. \end{aligned}$$

We call these the Steinberg relations. Consider the group homomorphism for each $n \in \mathbb{N}$

$$\begin{aligned} \mathrm{St}_n(R) &\longrightarrow \mathrm{St}_{n+1}(R) \\ \overline{x_{ij}^{(n)}(\lambda)} &\longmapsto \overline{x_{ij}^{(n+1)}(\lambda)}. \end{aligned}$$

The Steinberg group is defined to be the direct limit

$$\mathrm{St}(R) = \varinjlim_n \mathrm{St}_n(R)$$

where we denote the class of $x_{ij}^{(n)}(\lambda)$ as $x_{ij}(\lambda)$. As for each $n \in \mathbb{N}$, we have a surjective group homomorphism

$$\begin{aligned} \phi_n : \mathrm{St}_n(R) &\longrightarrow E_n(R) \\ \overline{x_{ij}^{(n)}(\lambda)} &\longmapsto e_{ij}^{(n)}(\lambda), \end{aligned}$$

thus we get a unique surjective map

$$\phi : \mathrm{St}(R) \longrightarrow E(R).$$

We thus define

$$K_2(R) := \mathrm{Ker}\,(\phi : \mathrm{St}(R) \to E(R)).$$

Thus, we have

$$E(R) \cong \frac{\mathrm{St}(R)}{K_2(R)}.$$

4.1. **Central extensions & $K_2(R)$.** Our goal is to show the following theorem, which, amongst other things, says that $K_2(R)$ is an abelian group.

Theorem 4.1.1. *Let R be a ring. Then the extension*

$$1 \to K_2(R) \to \mathrm{St}(R) \xrightarrow{\phi} E(R) \to 1$$

exhibits $\mathrm{St}(R)$ *as a universal central extension of $E(R)$. Moreover, $K_2(R)$ is the center of* $\mathrm{St}(R)$.

Proof. In order to show that this extension is central, we first need to show that $K_2(R)$ is in the center of $\mathrm{St}(R)$. In-fact we see that $K_2(R)$ is the center of $\mathrm{St}(R)$. Indeed, as the center of $E(R)$ is 1 and the center of $\mathrm{St}(R)$ is contained in the inverse image of the center of $E(R)$, thus center of $\mathrm{St}(R) \subseteq K_2(R)$. Conversely, we need to show that $K_2(R)$ is in center of $\mathrm{St}(R)$.

Pick an element $\alpha \in K_2(R)$. We wish to show that every element of $\mathrm{St}(R)$ commutes with α. As every element of $\mathrm{St}(R)$ is a word in elements $x_{ij}(\lambda)$, it suffices to show that $[\alpha, x_{ij}(\lambda)] = 1$. Write $x_{ij}(\lambda)$ as the class of some element $x_{ij}^{(n)}(\lambda) \in \mathrm{St}_n(R)$. By the third relation of $\mathrm{St}_n(R)$, we can write

$$x_{ij}^{(n)}(\lambda) = [x_{in}^{(n)}(\lambda), x_{nj}^{(n)}(1)]$$

in $\mathrm{St}_n(R)$. A little algebra makes it clear that it is sufficient to show that $[\alpha, x_{in}(\lambda)] = 1 = [\alpha, x_{nj}(\lambda)]$. We hence reduce to showing that

$$[\alpha, x_{in}(\lambda)] = 1$$

for any $1 \leq i \leq n-1$, $n \in \mathbb{N}$ and $\lambda \in R$.

We now exploit the fact that $\alpha \in K_2(R) = \mathrm{Ker}\,(\phi)$ by using the map $\phi : \mathrm{St}(R) \to E(R)$. Consider the subgroup

$$G_n = \langle x_{in}(\lambda) \mid 1 \leq i \leq n-1,\ \lambda \in R \rangle \leq \mathrm{St}(R).$$

We wish to understand the structure of G_n. Pick two elements $x_{in}(\lambda), x_{jn}(\mu) \in G_n$. Observe that as $i \neq n$ and $n \neq j$, we deduce by second relation that

$$x_{in}(\lambda) \cdot x_{jn}(\mu) = x_{jn}(\mu) \cdot x_{in}(\lambda),$$

that is, G_n is commutative. Hence, any arbitrary element of G_n is of the form $\beta = x_{1n}(\lambda_1) \cdot \cdots \cdot x_{n-1n}(\lambda_{n-1})$.

We thus see that $\phi|_{G_n} : G_n \to E(R)$ is injective as if $\beta \in G_n$ as above goes to 1 in $E(R)$, then

$$e_{1n}(\lambda_1) \cdot \cdots \cdot e_{n-1n}(\lambda_{n-1}) = \begin{bmatrix} 1 & 0 & \cdots & \lambda_1 \\ 0 & 1 & \cdots & \lambda_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = 1.$$

It follows that $\lambda_i = 0$ and thus $x_{in}(\lambda_i) = 1$, which further implies that $\beta = 1$, as required. Now as shown above, the group G_n is commutative and $\phi|_{G_n}$ is injective. We first claim that α is a normalizer of G_n, that is $\alpha G_n = G_n \alpha$. Indeed, α can be expressed as a product $x_{ij}(\mu)$. Using the Steinberg relations, we first easily see that (recall that $x_{ij}(\mu)^{-1} = x_{ij}(-\mu)$)

$$x_{ij}(\mu) \cdot x_{kn}(\lambda) = \begin{cases} x_{kn}(\lambda) x_{ij}(\mu) & \text{if } j \neq k \\ x_{in}(\lambda\mu) x_{kn}(\lambda) x_{ij}(\mu) & \text{if } j = k. \end{cases}$$

Then, for any $\beta \in G_n$, we have $\alpha \cdot \beta = \beta' \cdot \alpha$ for some $\beta' \in G_n$. Applying the injective map $\varphi|_{G_n}$, we see that $1 \cdot \varphi(\beta) = \varphi(\beta') \cdot 1$, thus $\beta = \beta'$. This shows that α is actually a centralizer of G_n, thus, every element of G_n commutes with α, as required. This shows that the extension is central and $K_2(R)$ is the center of $\mathrm{St}(R)$.

We now show that the extension is moreover universal central. By Theorem 6.9.7 of [Wei94] (Recognition principle), it suffices to prove that $\mathrm{St}(R)$ is perfect and every central extension of $\mathrm{St}(R)$ splits. The fact that $\mathrm{St}(R)$ is perfect follows immediately from the third Steinberg relation. Thus we reduce to showing that every central extension of $\mathrm{St}(R)$ splits. This is the major part of the proof of Theorem 5.10 of [Mil71]. □

Corollary 4.1.2. *Let R be a ring. Then*

$$K_2(R) \cong H_2(E(R); \mathbb{Z}).$$

Proof. Follows from Theorem 4.1.1 and Theorem 6.9.5 of [Wei94]. □

4.2. **G-representations of $K_2(F)$.** Observe that $K_2(R)$ is an abelian group since by Theorem 4.1.1 it is the center of $\mathrm{St}(R)$. Hence, one may try to find a presentation of $K_2(R)$ for any ring R. Matsumoto gives us one such presentation for K_2 of a field. This has intricate connections to algebra.

Theorem 4.2.1 (Matsumoto)**.** *Let F be a field. There is a presentation of $K_2(F)$ (multiplicatively written) as in*

$$1 \to \mathrm{Mats}(F) \to \mathbb{Z}^{\oplus F^\times \times F^\times} \to K_2(F) \to 1$$

where $\mathrm{Mats}(F)$ is the subgroup of $\mathbb{Z}^{\oplus F^\times \times F^\times}$ generated by the following relations for $a_i, b_i, a, b \in F^\times$

$$(a_1a_2, b) = (a_1, b) \cdot (a_1, b)$$
$$(a, b) = (b, a)^{-1}$$
$$(a, 1-a) = 1,\ a \neq 1.$$

We denote the class of (a, b) in $K_2(F)$ by $\{a, b\}$. We call the above relations Matsumoto's relations. Thus $\{a, b\}$ satisfies Matsumoto's relations in $K_2(F)$.

The important technique here is that Matsumoto's theorem allows us to give an equivalence between G-representations of $K_2(F)$ and functions $F^\times \times F^\times \to G$ of certain type, for any abelian group G.

Definition 4.2.2 (**Symbols on a field**)**.** Let F be a field and G be an abelian group. A G-valued symbol over F is a G-valued function $F^\times \times F^\times$, denoted

$$(,) : F^\times \times F^\times \to G$$

which satisfies the Matsumoto relations verbatim as stated in Theorem 4.2.1. We denote by $\mathrm{Symb}_G(F)$ the set of all symbols of F over G.

Here's the equivalence.

Theorem 4.2.3. *Let F be a field and G be an abelian group. Then there is a bijection*

$$\mathrm{Hom}_{\mathcal{A}b}\left(K_2(F), G\right) \cong \mathrm{Symb}_G(F).$$

Proof. Define the bijection as

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}b}\left(K_2(F), G\right) &\longleftrightarrow \mathrm{Symb}_G(F) \\ \varphi &\longmapsto (a, b) := \varphi(\{a, b\}) \\ \tilde{\varphi} &\longleftarrow (,)_s \end{aligned}$$

where $\tilde{\varphi} : K_2(F) \to G$ is obtained by extending the group homomorphism $\mathbb{Z}^{\oplus F^\times \times F^\times} \to G$ given by $(a, b) \mapsto (a, b)_s$ to the unique map $\tilde{\varphi} : K_2(F) \to G$ via universal property of cokernels ($(,)_s$ satsifies Matsumoto's relations) in the presentation of Theorem 4.2.1. ∎

Using this we can immediately find K_2 of a finite field.

Proposition 4.2.4. *Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$. Then*

$$K_2(\mathbb{F}_q) = 1.$$

Proof. By Theorem 4.2.3 and Yoneda's lemma, it suffices to show that for any abelian group G, $\mathrm{Symb}_G(\mathbb{F}_q) = 1$. Let $(,)_s \in \mathrm{Symb}_G(\mathbb{F}_q)$. As $\mathbb{F}_q^\times$ is a finite cyclic group, therefore let $g \in \mathbb{F}_q^\times$ be its generator. Then, any element in $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ is of the form (g^n, g^m). By Matsumoto's relations we deduce that $(g^n, g^m)_s = (g, g^m)_s^n = (g^m, g)_s^{-n} = (g, g)_s^{-nm}$. Moreover, as $(g, g) = (g, g)^{-1}$, therefore (g, g) is an element of order atmost 2 in G. Let us assume that order of (g, g) is precisely 2.

Note that $(-, g) : \mathbb{F}_q^\times \to G$ is a group homomorphism by first Matsumoto's relation. Thus $1 = (1, g) = (g^{|g|}, g) = (g, g)^{|g|}$, so we conclude that $2 \mid |g| = q - 1$. If $p = 2$, then $q - 1$ is odd and thus we have a contradiction. Thus, we may assume $p > 2$.

We wish to show that $(g, g) = 1$. To this end, we use the third Matsumoto's relation. Note that it suffices to show that for some $k \in \mathbb{N}$ odd, we have $(g, g)^k = 1$. Writing $k = nm$, we see that we need $(g^n, g^m) = 1$. Thus we reduce to showing that there exists $h \in \mathbb{F}_q^\times - 1$ such that $h = g^n$, $1 - h = g^m$ and n, m are odd. To this end, observe that

$$\mathbb{F}_q^\times - 1 = \{g, g^2, g^3, \dots, g^{q-2}\}.$$

Thus there are $(q - 3)/2$ squares of g and $(q - 1)/2$ non-squares of g in $\mathbb{F}_q^\times - 1$. Next, observe that the map

$$\begin{aligned} \mathbb{F}_q^\times - 1 &\longrightarrow \mathbb{F}_q^\times - 1 \\ h &\longmapsto 1 - h \end{aligned}$$

is an injective map and since $\mathbb{F}_q^\times$ is finite, thus is a bijection. Thus, if for all $h \in \mathbb{F}_q^\times - 1$ which is non-square, $1 - h$ is a square, then by the above bijection, there are atleast $(q - 1)/2$ squares, which is more than $(q - 3)/2$, a contradiction. Hence there is h such that it is a non-square and $1 - h$ is also a non-square, as required. This completes the proof. $\square$

4.3. **p-divisibility of $K_2(F)$.** Recall that a multiplicatively written abelian group G is a *p-divisible* group if for all $g \in G$, there is $h \in G$ such that $h^p = g$. It is *uniquely p-divisible* if G moreover has no p-torsion element, i.e. no non-trivial element of G has order p. It is *divisible* if G is p-divisible for each prime p. Finally G is *uniquely divisible* if it is uniquely p-divisible for each prime p.

Observe that a uniquely p-divisible group G has a unique p^{th}-root for every element in G. Thus, a uniquely divisible group G has unique p^{th}-root for every element and for every prime p. We would like to show the following result.

Theorem 4.3.1 (Bass-Tate)**.** *Let F be a field and p be a prime such that every polynomial $x^p - a$, $a \in F$ splits in $F[x]$ into linear factors. Then $K_2(F)$ is uniquely p-divisible.*

Proof. We first have to show that $K_2(F)$ is a p-divisible group. As the image of a p-divisible group is p-divisible, it suffices to find a p-divisible group which surjects on $K_2(F)$. Observe that $F^\times$ is a p-divisible group by the given hypothesis. It can be checked easily that tensor product of two uniquely p-divisible groups is unqiuely p-divisible[8]. By Matsumoto's theorem (Theorem 4.2.1), we

[8] Let G, H be two uniquely p-divisible groups. Consider the isomorphism $m_p : G \to G$. Tensoring $0 \to 0 \to G \xrightarrow{m_p} G \to 0$ by H, we get $0 \otimes H \to G \otimes H \xrightarrow{m_p \otimes \mathrm{id}} G \otimes H \to 0$ is exact. As $0 \otimes H = 0$, thus $m_p \otimes \mathrm{id}$ is an isomorphism.

have a surjective homomorphism

$$\varphi : F^\times \otimes_{\mathbb{Z}} F^\times \to K_2(F).$$

This shows that $K_2(F)$ is p-divisible. We need only show that $K_2(F)$ has no p-torsion. To this end, we first relate the unique p-divisibility of $F^\times \otimes_{\mathbb{Z}} F^\times$ to $K_2(F)$. Note that we have the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & F^\times \otimes_{\mathbb{Z}} F^\times \arrow[r, "m_p"] \arrow[d, "\varphi"] & F^\times \otimes_{\mathbb{Z}} F^\times \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & T_p(K_2(F)) \arrow[r] & K_2(F) \arrow[r, "m_p"] & K_2(F) &
\end{tikzcd}.$$

By Snake lemma, we deduce the following exact sequence

$$0 \to \mathrm{Ker}\,(\varphi) \overset{m_p}{\to} \mathrm{Ker}\,(\varphi) \to T_p(K_2(F)) \to 0 \to 0.$$

Thus, we have

$$T_p(K_2(F)) \cong \mathrm{Ker}\,(\varphi)/\mathrm{Im}\,(m_p).$$

We wish to show that $T_p(K_2(F)) = 1$. To this end, it suffices to thus show that $\mathrm{Ker}\,(\varphi)$ is p-divisible.

Note that $\mathrm{Ker}\,(\varphi)$ is generated by elements of form $a \otimes 1 - a$ for $a \in F^\times$ by Matsumoto's theorem (the other relations are trivially satisfied by $\mathrm{Ker}\,(\varphi)$ as it is a subgroup of $F^\times \otimes_{\mathbb{Z}} F^\times$). Thus, it suffices to show that for any $a \in F^\times$, the element $a \otimes (1-a)$ is a multiple of p of some other element in $\mathrm{Ker}\,(\varphi)$. By hypothesis, we have

$$x^p - a = \prod_i (x - b_i)$$

for $b_i \in F^\times$. It follows that $1 - a = \prod_i (1 - b_i)$ and since the notation for tensor product is additive, so we get

$$a \otimes (1-a) = \sum_i a \otimes (1 - b_i) = \sum_i b_j^p \otimes (1 - b_i) = p \sum_i b_j \otimes (1 - b_i).$$

As $\sum_i b_j \otimes (1 - b_i) \in \mathrm{Ker}\,(\varphi)$ as it is a sum of generators of it, hence this completes the proof. □

We have two immediate corollaries.

Corollary 4.3.2. *Let F be a field.*

(1) If F is algebraically closed, then $K_2(F)$ is uniquely divisible.

(2) If F is perfect of characteristic $p > 0$, then $K_2(F)$ is uniquely p-divisible.

Proof. Both follows immediately from Theorem 4.3.1. For the latter, observe that by perfection, for each $a \in F^\times$, there is a $b \in F^\times$ such that $b^p = a$ and thus $x^p - a = (x - b)^p$ by algebra in characteristic $p > 0$. □

4.4. **Brauer group & Galois symbol.** Brauer group is another subtle invariant of a field. Its main uses are in a) algebraic geometry, where it classifies certain type of projective varieties over a field, and in b) algebraic number theory where it is used to construct the Hasse invariant.

We will here construct a representation of $K_2(F)$ for certain fields in the Brauer group of F, denoted $\mathrm{Br}(F)$. For that, we first study a generalization of matrix algebras, *central simple algebras* (CSA) over F. Recall A is a CSA over F if A is a finite dimensional associative unital algebra over F which is simple and whose center is exactly F. The following basic observations is all we require of them.

Proposition 4.4.1. *Let F be a field.*
(1) $M_n(F)$ is a central simple algebra over F.
(2) If A, B are two central simple algebras over F, then so is $A \otimes_F B$.

Proof. 1. The center of $M_n(F)$ is the diagonal matrices $\mathrm{diag}(\lambda, \lambda, \dots, \lambda)$ for $\lambda \in F^\times$. This is immediate by considering elementary matrices. The fact that $M_n(F)$ is simple can be seen by Smith normal form and performing elementary row operations together amongst others to complete the diagonal with non-zero entries.

2. Observe that the map $K \to A \otimes_K B$ mapping $k \mapsto k \otimes 1$ is injective. We first show centrality of $A \otimes_F B$. Let Z be the center of $A \otimes_F B$ and $x = \sum_i a_i \otimes b_i \in Z$. We wish to show that $x \in K$. Indeed, consider $a \in A$. We have $(a \otimes 1) \cdot x = x \cdot (a \otimes 1)$. This yields

$$\sum_i (a_i a - a a_i) \otimes b_i = 0.$$

Write $a_i a - a a_i = \sum_k c_k m_k$ where $\{m_k\}$ forms a basis of A. We may assume that $\{b_i\}$ is a basis of B. It follows that $\sum_{i,k} c_k (m_k \otimes b_i) = 0$. As $\{m_k \otimes b_i\}_{k,i}$ forms a K-basis of $A \otimes_K B$, therefore $c_k = 0$ for all k. It follows at once that $a_i a = a a_i$. Hence $a_i \in K$ for all i. Similarly, $b_i \in K$ for all i. This shows that $x \in K$.

Next we show simplicity of $A \otimes_F B$. Suppose $0 \neq I \leq A \otimes_F B$ is a two-sided non-zero ideal. Let $x = \sum_i a_i \otimes b_i \in I$. As the two-sided ideal generated by b_1 is B by simplicity of B, therefore we have $b', b'' \in B$ such that $b' b_1 b'' = 1$. Similarly, for $a_2 \in A$ we have $a' a_2 a'' = 1$. We thus have that $x \cdot (1 \otimes b) - (1 \otimes b) \cdot x$ is an element in I which has length strictly smaller than n. We may thus put minimality hypothesis on n and hence deduce that $x \cdot (1 \otimes b) = (1 \otimes b) \cdot x$ for each $b \in B$. Similarly, we may get that $x \cdot (a \otimes 1) - (a \otimes 1) \cdot x$ for each $a \in A$. Now, any element of $A \otimes_F B$ is of form $\sum_i (a_i \otimes 1) \cdot (1 \otimes b_i)$. It follows that x is in the center of $A \otimes_F B$ which by above is F. This shows that $x \in I$ is a unit of $A \otimes_F B$, hence I is trivial. ∎

Definition 4.4.2 (**Brauer group of a field**)**.** Let F be a field. The Brauer group of F is defined to be quotient of the free abelian group generated by isomorphism classes of central simple algebras by the subgroup generated by the relations

$$[A \otimes_F B] = [A] \cdot [B]$$
$$[M_n(F)] = 1.$$

We denote the abelian group by $\mathrm{Br}(F)$. We denote the subgroup of n-torsion elements of $\mathrm{Br}(F)$ by ${}_n\mathrm{Br}(F) = \{[A] \in \mathrm{Br}(F) \mid [A]^n = 1\}$.

We will construct a group homomorphism $K_2(F) \to \mathrm{Br}(F)$. The map will take a Matsumoto symbol $\{a, b\}$ to a *cyclic algebra*, which will be a central simple algebra over F.

Definition 4.4.3 (Cyclic algebra). Let F be a field containing a primitive n^{th}-root of unity ζ. Fix two $a, b \in F^\times$. The cyclic algebra generated by a, b is the F-algebra $A_\zeta(a, b)$ generated by two elements x, y subject to the following relations:

$$\begin{aligned} x^n &= a \cdot 1 \\ y^n &= b \cdot 1 \\ yx &= \zeta xy. \end{aligned}$$

The following are few important properties of cyclic algebras.

Proposition 4.4.4. *Let F be a field containing a primitive n^{th}-root of unity ζ and $a, b \in F^\times$. Denote $A = A_\zeta(a, b)$ to be the cyclic algebra generated by a and b.*

(1) A is an n^2-dimensional F-algebra.
(2) A is a central F-algebra.
(3) A is a simple F-algebra.

Proof. 1. Take any element $z \in A$. Then z can be written as a sum of monomials $x^i y^j$ where $1 \leq i, j \leq n - 1$. As each $x^i y^j$ is independent, therefore A has dimension n^2 over F, as required.

2. Let Z be the center of A. It contains F. Let $z \in Z$. We wish to show that $z \in F$. Indeed, we may write

$$z = \sum_{0 \leq i,j \leq n-1} c_{ij} x^i y^j$$

where note that the terms $x^i y^j$ are independent. As z is in the center, we must have $z \cdot x = x \cdot z$ in particular. Expanding this, one yields,

$$\sum_{1 \leq i,j} c_{ij}(\zeta^j - 1) x^{i+1} y^j = 0.$$

We hence deduce that $c_{ij}(\zeta^j - 1) = 0$ for all $1 \leq i, j \leq n - 1$. As ζ is primitive and $j \leq n - 1$, it follows at once that $\zeta^j - 1 \neq 0$ and thus $c_{ij} = 0$ for all $1 \leq i, j \leq n - 1$. We hence deduce that $z = c_{00} \in F$, as required.

3. Pick any non-zero ideal $I \leq A$ and let $z \in I$ be of form $z = \sum_{k=0}^m c_k x^{i_k} y^{j_k}$ of shortest length. We will show that $c_k = 0$ for all k possibly except if $i_k, j_k = 0$. Indeed, we may multiply z by x^{n-i_0} on left and y^{n-j_0} on right to get the first term of $x^{n-i_0} z y^{n-j_0}$ as $c_0 ab$. We may multiply by $a^{-1} b^{-1}$ to further get the first term to be c_0. Now, the difference $\tilde{z} = a^{-1} b^{-1} x^{n-i_0} z y^{n-j_0} - z$ is zero as it is in I and has length strictly smaller than m. By repeating the same on $\tilde{z}$, one can show coefficients of all terms of $\tilde{z}$ are 0, from which we yield that $c_k = 0$ for all k. Since in $\tilde{z}$, the term of z corresponding to $i_k, j_k = 0$ is absent, hence $c_k = 0$ for all k possibly except if $i_k, j_k = 0$, as required. $\square$

An important property of cyclic algebra is that they are *Brauer-torsion*.

Proposition 4.4.5. *Let F be a field containing a primitive n^{th}-root of unity ζ and $a, b \in F^\times$. Denote $A = A_\zeta(a,b)$ to be the cyclic algebra generated by a and b. Then, $A^{\otimes n}$ is isomorphic to a matrix algebra over F.*

We now show the existence of a symbol for fields with enough roots of unity.

Theorem 4.4.6 (Galois symbol-1)**.** *Let F be a field with a primitive n^{th}-root of unity ζ. Then the following map*

$$\begin{aligned} \varphi : K_2(F) &\longrightarrow \mathrm{Br}(F) \\ \{a,b\} &\longmapsto A_\zeta(a,b) \end{aligned}$$

is a homomorphism whose image is in ${}_n\mathrm{Br}(F)$. We call this map the n^{th} power norm residue symbol for F.

We will later do Merkurjev-Suslin theorem, which will tell us that the above map $\tilde{\varphi}$ is an isomorphism. Thus, all n-torsion elements of $\mathrm{Br}(F)$ are precisely the classes of cyclic algebras. We also construct Galois symbols for more general fields in §6.1.

Proof. Consider the function $F^\times \times F^\times \to \mathrm{Br}(F)$ given by $(a,b) \mapsto [A_\zeta(a,b)]$. To get a map from $K_2(F)$, by Matsumoto's theorem (Theorem 4.2.1), it suffices to show that the above map vanishes for Matsumoto's relations. To see that $[A_\zeta(\alpha,\beta)] \cdot [A_\zeta(\alpha,\gamma)] = [A_\zeta(\alpha,\beta\gamma)]$, we first observe the isomorphism $A_\zeta(\alpha,\beta) \otimes A_\zeta(\alpha,\gamma) \cong M_n(A_\zeta(\alpha,\beta\gamma))$ (Ex. 6.12, pp 266 of [Wei94]) and observe that since $M_n(A) \cong A \otimes M_n(F)$, thus, $[M_n(A)] = [A]$ in $\mathrm{Br}(F)$. This shows the first relation. We next wish to show that $A_\zeta(a, 1-a) \cong M_n(F)$ for all $a \neq 1$ in $F^\times$. By the Lemma 4.4.7 below, we reduce to showing that $A_\zeta(a, 1-a)$ contains an n-torsion element. Indeed, as $x^n = a$, $y^n = 1-a$ and since $(x+y)^n = x^n + y^n = a + (1-a) = 1$[9], we thus we win. ∎

Lemma 4.4.7. *Let F be a field containing a primitive n^{th}-root of unity ζ. If A is a center simple algebra over F of dimension n^2 such that there exists $z \in A$ not in the center for which $z^n = 1$, then $A \cong M_n(F)$.*

Proof. Consider the subalgebra of A generated by z, denoted $F[z]$. As $z^n = 1$, therefore we have

$$F[z] \cong \frac{F[x]}{\langle x^n - 1 \rangle}.$$

As $x^n - 1 \prod_{i=0}^{n-1}(x - \zeta^i)$ in $F[x]$ as F contains a primitive n^{th}-root of unity ζ, hence it follows that the quotient

$$F[z] \cong \frac{F[x]}{\langle x^n - 1 \rangle} \cong \prod_{i=0}^{n-1} \frac{F[x]}{\langle x - \zeta^i \rangle}$$

[9]for any cyclic algebra $A_\zeta(a,b)$, the equation $(x+y)^n = a+b$ holds. This can be checked by a calculation involving binomal theorem and then showing that the inner term sums to 0 by multiplying it on left by y and on right by x and observing the resultant pattern.

by Chinese remainder theorem as the ideals $\langle x - \zeta^i \rangle$ are comaximal for $i \neq j$. Consequently, we have a splitting of the algebra $F[z]$ as:

$$F[z] \cong \underbrace{F \times \cdots \times F}_{n-\text{times}}$$

As $A \supseteq F[z]$, thus A has n-idempotents, which further induces a splitting of A into n-subalgebra (e_i are the idempotents of A):

$$A = e_1 A \times \cdots \times e_n A.$$

By Wedderburn-Artin theorem, every finite dimensional simple algebra B is isomorphic to $M_k(S)$ for some division ring S and B can atmost split into k-many right ideals. Consequently, $A \cong M_k(S)$ for some division algebra S over F and $n \leq k$. As $\dim_F A = n^2 = (\dim_F S) \cdot (\dim_S M_k(S)) = k^2 \cdot \dim_F S$, therefore we deduce that $k = n$ and $\dim_F S = 1$, yielding $A \cong M_n(F)$, as needed. $\square$

5. Higher K-theory of rings-I

We now wish to define higher K-groups for an associative ring R with 1. To this end, we will construct a space whose lower homotopy groups will agree with the K_1 and K_2 as defined above, and will then define higher K-groups to be the higher homotopy groups of this space. The K_0 case will need some special attention. We will require a lot of topological information to thoroughly discuss the definitions and their motivations, for which we may often refer to Foundational Homotopy Theory, Chapter 10 of [FoG].

We now make a series of observations to motivate the definition of what we need.

- Let R be a ring. The classifying space $\mathrm{BGL}(R)$ of $\mathrm{GL}(R)$ is such that

  $$\pi_1(\mathrm{BGL}(R)) \cong \mathrm{GL}(R)$$

  and thus it has a perfect subgroup $E(R)$ (Remark 3.1.10).
- Moreover, the homology of $\mathrm{BGL}(R)$ satisfies the following for any $\mathrm{GL}(R)$-module M:

  $$H_\bullet(\mathrm{BGL}(R); M) \cong H_\bullet(\mathrm{GL}(R); M)$$

  where on the right we have group homology of $\mathrm{GL}(R)$ with coefficients in M. We also know that $K_1(R) \cong H_1(\mathrm{GL}(R); \mathbb{Z})$ and $K_2(R) \cong H_2(E(R); \mathbb{Z})$ (Corollary 4.1.2). In particular, we have

  $$K_1(R) \cong H_1(\mathrm{BGL}(R); \mathbb{Z}).$$

- As we have a natural quotient map

  $$\pi : \mathrm{GL}(R) \to \frac{\mathrm{GL}(R)}{E(R)} = K_1(R),$$

  hence, we may ask whether there is a space X and a map

  $$i : \mathrm{BGL}(R) \to X$$

  such that $\pi_1(X) \cong K_1(R)$ and the map

  $$i_* : \pi_1(\mathrm{BGL}(R)) \to \pi_1(X)$$

  is exactly the quotient map π?
  In such a scenario, $\pi_1(X)$ is abelian and hence by Hurewicz, we must have

  $$K_1(R) \cong \pi_1(X) \cong H_1(X; \mathbb{Z}).$$

  Since we also have a map

  $$i_* : H_1(\mathrm{BGL}(R); \mathbb{Z}) \to H_1(X; \mathbb{Z})$$

  and by the previous observation $H_1(\mathrm{BGL}(R); \mathbb{Z}) \cong K_1(R)$, hence we have a map

  $$i_* : K_1(R) \to K_1(R).$$

  We would naturally like this to be an isomorphism. Hence we may wonder whether X and $i : \mathrm{BGL}(R) \to X$ can also be made such that

  $$i_* : H_\bullet(\mathrm{BGL}(R); M) \to H_\bullet(X; M)$$

  for any $\mathrm{GL}(R)$-module M is an isomorphism?

In conclusion, we want the following:

(Q) Construct a space X and map $i : \mathrm{BGL}(R) \to X$ such that
1. $\pi_1(X) \cong K_1(R)$,
2. $i_* : \pi_1(\mathrm{BGL}(R)) \to \pi_1(X)$ is the map $\pi : \mathrm{GL}(R) \twoheadrightarrow K_1(R)$,
3. For all $\mathrm{GL}(R)$-modules M, $i_* : H_\bullet(\mathrm{BGL}(R); M) \to H_\bullet(X; M)$ is an isomorphism.

We call the above three requirements to be the (Q)-criterion. What Quillen found that the space X can be constructed, but it will be well defined only upto homotopy equivalence (as the conditions we want is only about the homotopy groups of X). Consequently, there are many ways to construct X from $\mathrm{BGL}(R)$, all yielding homotopy equivalent spaces. We first discuss the definition of $\mathrm{BGL}(R)^+$.

5.1. **The homotopy type** $\mathrm{BGL}(R)^+$**.** The definition of $\mathrm{BGL}(R)^+$ is not enlightening, in-fact the criterion (Q) is what we will call a $\mathrm{BGL}(R)^+$.

Definition 5.1.1 ($\mathrm{BGL}(R)^+$)**.** Let R be an associative ring with 1. A pair (X, i) of a CW-complex X and map $i : \mathrm{BGL}(R) \to X$ satisfying the (Q)-criterion above is called a model of $\mathrm{BGL}(R)^+$. For a model X of $\mathrm{BGL}(R)^+$, we define the homotopy type of X to be the $\mathrm{BGL}(R)^+$. We will abuse the notation and sometimes write $\mathrm{BGL}(R)^+$ as a model of $\mathrm{BGL}(R)^+$ as well!

Remark 5.1.2 ($\mathrm{BGL}(R)^+$ is defined upto homotopy)**.** As the definition shows, the space $\mathrm{BGL}(R)^+$ is a homotopy type, not really a space. Of-course, we need to show that any two spaces X and Y satisfying (Q)-criterion are homotopy equivalent. This will require some work.

From now on, as was the case earlier, we will assume all the rings are associative with 1.

Theorem 5.1.3 (Quillen)**.** *Let R be a ring. If X and Y are two models for $\mathrm{BGL}(R)^+$, then $X \simeq Y$, that is, they are homotopy equivalent.*

This is proved later in Corollary 5.1.17. Before proving this, we first need to establish that our definition of homotopy type $\mathrm{BGL}(R)^+$ actually does serve the purpose. That is, we wish to show that $\pi_2(\mathrm{BGL}(R)^+) \cong K_2(R)$.

Theorem 5.1.4. *Let R be a ring. Then*

$$\pi_2(\mathrm{BGL}(R)^+) \cong K_2(R).$$

This is proved in Theorem 5.1.8. We now embark on a small homotopical study of $\mathrm{BGL}(R)^+$, to extract information about $\mathrm{BGL}(R)^+$ which will yield the results stated above.

We fix a model of $\mathrm{BGL}(R)^+$ and also call it $\mathrm{BGL}(R)^+$ in the following. Of-course, we have not proved existence of a model yet, but we will do so soon, after realizing that indeed the homotopy type $\mathrm{BGL}(R)^+$ works for our need.

5.1.1. *Acyclic fiber, acyclic maps.* As the map $i : \mathrm{BGL}(R) \to \mathrm{BGL}(R)^+$ is a quasi-isomorphism, thus it becomes a special map by the following important theorem.

Theorem 5.1.5 (Acyclic fiber theorem)**.** *Let $f : X \to Y$ be a based map of connected CW-complexes. Then the following are equivalent:*

(1) For all $k \geq 0$, we have

$$f_* : H_k(X; M) \xrightarrow{\cong} H_k(Y; M)$$

for every $\pi_1(Y)$-module M[10]*.*

(2) The homotopy fiber Ff of f is acyclic[11]*.*

Proof. (1. $\Rightarrow$ 2.) By replacing X by the fibration replacement of f (see Construction 10.2.1.11 of [FoG]), we may assume that we have a fibration $Ff \xrightarrow{i} X \xrightarrow{f} Y$. Assume that $\pi_1(Y) = 0$, so that we have a Serre spectral sequence $E^2_{pq} = H_p(Y; H_q(Ff)) \Rightarrow H_{p+q}(X)$ and for the trivial fibration pt. $\to Y \xrightarrow{\mathrm{id}} Y$ which gives another Serre spectral sequence ${}'E^2_{pq} = H_p(Y; H_q(\mathrm{pt.})) \Rightarrow H_{p+q}(Y)$. We have a commutative diagram:

$$\begin{tikzcd} Ff \arrow[r, "i"] \arrow[d] & X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\mathrm{id}"'] \\ \mathrm{pt.} \arrow[r] & Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}.$$

By comparison theorem (Proposition 5.13 of [Hat04]), we deduce that Ff is acyclic. lt follows that if Y is simply connected and f induces isomorphism on integral homology, then homotopy fiber of f is acyclic.

Now suppose $\pi_1(Y) \neq 0$. The main idea is to reduce to the simply connected case by going to universal cover of Y. Indeed, if $\tilde{Y}$ is the universal cover of Y, then we have the following pullback diagrams (by Lemma 10.2.1.2 of [FoG], we have that $\tilde{f}$ is a fibration):

$$\begin{tikzcd} F\tilde{f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_Y \tilde{Y} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{Y} \arrow[d, "p"] \\ Ff \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd}.$$

Denote $\tilde{X} = X \times_Y \tilde{Y}$. It then follows by maps constructed by unique path lifting that $F\tilde{f} \cong Ff$. It thus suffices to show that $F\tilde{f}$ is acyclic. To this end, by above, we reduce to showing that we have an isomorphism $\tilde{f}_* : H_k(\tilde{X}; \mathbb{Z}) \to H_k(\tilde{Y}; \mathbb{Z})$ for all $k \geq 0$. This follows from the following comutative square with vertical maps being isomorphisms:

$$\begin{tikzcd} H_k(\tilde{X}; \mathbb{Z}) \arrow[r, "\tilde{f}_*"] \arrow[d, "\cong"'] & H_k(\tilde{Y}; \mathbb{Z}) \arrow[d, "\cong"] \\ H_k(X; \mathbb{Z}[\pi_1(Y)]) \arrow[r, "f_*"'] & H_k(X; \mathbb{Z}[\pi_1(Y)]) \end{tikzcd}.$$

As f_* is an isomorphism by hypothesis, we win.

(2. $\Rightarrow$ 1.) As before, we may assume that $Ff \xrightarrow{i} X \xrightarrow{f} Y$ is a fibration. Fix a $\pi_1(Y)$-module

[10]That is, M is a left $\mathbb{Z}[\pi_1(Y)]$-module.

[11]that is, Ff has homology of a point.

M. Observe that the E^2-page of Serre spectral sequence $E^2_{pq} = H_p(Y; H_q(Ff; M)) \Rightarrow H_{p+q}(X; M)$ is all 0 except possibly the bottom row (which consists of $H_q(Y; M)$) since $H_q(Ff; M) = 0$ forall $q \geq 1$ and $H_0(Ff; M) = M$ by a simple use of universal coefficients theorem. It follows that E collapses on the E^2-page, so that $H_n(X; M) \cong H_n(Y; M)$. In particular, this isomprhism comes from f_* as the above isomorphim is by the edge homomorphism which we know in Serre spectral sequence is via the map $f : X \to Y$ (see Addendum 2, Theorem 5.3.2 of [Wei94]). ☐

Corollary 5.1.6. *Let R be a ring. Then the homotopy fiber of $i : \mathrm{BGL}(R) \to \mathrm{BGL}(R)^+$ is acyclic.*

Proof. Follows from definition and Theorem 5.1.5. ☐

Acyclicity is both homological and cohomological.

Lemma 5.1.7. *Let $f : X \to Y$ be a map of connected spaces and π be an abelian group. If $f_* : H_q(X; \pi) \to H_q(Y; \pi)$ is an isomorphism for all $q \geq 0$, then $f^* : H^q(Y; \pi) \to H^q(X; \pi)$ is an isomorphism for all $q \geq 0$.*

Proof. By universal coefficient theorem for cohomology, we have the following commutative diagram where rows are exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), \pi) \arrow[r] & H^n(X; \pi) \arrow[r] & \mathrm{Hom}\,(H_n(X), \pi) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(Y), \pi) \arrow[r] \arrow[u] & H^n(Y; \pi) \arrow[r] \arrow[u, "f^*"'] & \mathrm{Hom}\,(H_n(Y), \pi) \arrow[r] \arrow[u] & 0 \end{tikzcd}.$$

As the vertical arrows on left and right are induced by $f_* : H_q(X) \to H_q(Y)$ which is an isomorphism, therefore they are also isomorphisms. By 5-lemma, we conclude that f^* is an isomorphism. ☐

5.1.2. $K_2(R)$ *&* $\pi_2(\mathrm{BGL}(R)^+)$. We may now see that indeed $K_2(R) \cong \pi_2(\mathrm{BGL}(R)^+)$!

Theorem 5.1.8. *Let R be a ring. Then*

$$\pi_2(\mathrm{BGL}(R)^+) \cong H_2(E(R); \mathbb{Z}).$$

Proof. Main idea is to exhibit $\pi_2(\mathrm{BGL}(R)^+)$ in a universal central extension of $E(R)$ as follows:

$$0 \to \pi_2(\mathrm{BGL}(R)^+) \to ?? \to E(R) \to 1,$$

so that uniqueness of universal central extension would yield the proof together with Theorem 6.9.5 of [Wei94].

We may employ the long exact sequence of homotopy groups associated to a map (see Corollary 10.2.3.7 of [FoG]). Indeed, we have the following part of a l.e.s:

$$\pi_2(\mathrm{BGL}(R)) \to \pi_2(\mathrm{BGL}(R)^+) \to \pi_1(Fi) \to \pi_1(\mathrm{BGL}(R)) \xrightarrow{i_*} \pi_1(\mathrm{BGL}(R)^+) \to \pi_0(Fi).$$

As Fi is acyclic, therefore $\pi_0(Fi) = 0$. As $\mathrm{BGL}(R)$ is $K(\mathrm{GL}(R), 1)$, therefore $\pi_2(\mathrm{BGL}(R)) = 0$. Moreover, $\pi_1(\mathrm{BGL}(R)) = \mathrm{GL}(R)$, $\pi_1(\mathrm{BGL}(R)^+) = K_1(R)$ and $i_* = \pi$ where $\pi : \mathrm{GL}(R) \to \frac{\mathrm{GL}(R)}{E(R)}$, by definition.

It follows that we have the following s.e.s.:

$$0 \to \pi_2(\mathrm{BGL}(R)^+) \xrightarrow{\rho} \pi_1(Fi) \to \mathrm{GL}(R) \xrightarrow{\pi} K_1(R) \to 0.$$

We then further deduce the following s.e.s.:

$$0 \to \pi_2(\mathrm{BGL}(R)^+) \xrightarrow{\rho} \pi_1(Fi) \to E(R) \to 1.$$

To complete the proof, we need only show that the above is a universal central extension of $E(R)$. To this end, we first have to show that image of ρ is in the center of $\pi_1(Fi)$. This follows from Corollary IV.3.5 of [Whi78]. Finally we wish to see that the above is universal central. To this end, by Recognition Criterion 6.9.7 of [Wei94], it suffices to show that $\pi_1(Fi)$ is perfect and every central extension of $\pi_1(Fi)$ splits. The former is true as Fi is acyclic, so $H_1(Fi) = 0$ and thus $\pi_1(Fi)$ is perfect. For the latter, by Corollary 6.9.9 of [Wei94], it suffices to show that $H_2(\pi_1(Fi); \mathbb{Z}) = 0$. This follows from the Propsotioin 5.1.9 mentioned below. □

Proposition 5.1.9. *Let E be an acyclic space with fundamental group G. Then $H_2(G; \mathbb{Z}) = 0$.*

Proof. Note that BG is a space with fundamental group G as well and moreover $H_2(BG; \mathbb{Z}) \cong H_2(G; \mathbb{Z})$. It hence suffices to show that $H_2(BG; \mathbb{Z}) = 0$. As BG is obtained by attaching cells to X, therefore we have a natural map $f : E \to BG$ which induces isomorphism on π_1. By considering the fibration replacement, we may assume f is a fibration and

$$Ff \to E \xrightarrow{f} BG$$

a fibration sequence. Considering the Serre spectral sequence of this fibration, we obtain $E^2_{pq} = H_p(BG; H_q(Ff)) \Rightarrow H_{p+q}(E) = 0$. Consequently, we have $E^\infty_{pq} = 0$ for all p, q except $p, q = 0$. An immediate observation of relevant differentials yield that $E^\infty_{00} = E^2_{00}$, from which it follows that $E^2_{00} = H_0(BG; H_0(Ff)) \cong H_0(E) = \mathbb{Z}$. As BG is path-connected, therefore $H_0(Ff) \cong \mathbb{Z}$, showing that Ff is path-connected. Another simple analysis of differentials at E^2_{20} yields that if $E^2_{01} = 0$, then $E^\infty_{20} = E^2_{20} = H_2(BG; H_0(Ff)) = H_2(BG; \mathbb{Z})$, so that it will follow that $H_2(BG; \mathbb{Z}) = 0$, as required. We thus reduce to showing that $E^2_{01} = H_0(BG; H_1(Ff)) \cong H_1(Ff)$ is 0.

From the fibration long exact sequence obtained from f and the fact that $\pi_k(BG) = 0$ for all $k \neq 1$, and $\pi_1(BG) = G$, we deduce that that map $i : Ff \to E$ induces an isomorphism on π_k for all $k \geq 2$ and moreover $\pi_0(Ff) = \pi_1(Ff) = 0$ as $f_* : \pi_1(E) \to G$ is an isomorphism. Thus, Ff and $\tilde{E}$, the universal cover of E has same homotopy groups via the map $Ff \to E$. One can then show by unique lifting criterion that there is a map $\tilde{i} : Ff \to \tilde{E}$ which is a weak equivalence. By Whitehead's theorem (Ff is a CW-complex as well), it follows that $\tilde{i}$ is a homotopy equivalence. Hence $H_1(Ff) \cong H_1(\tilde{E})$ and since $\pi_1(\tilde{E}) = 0$, by Hurewicz, we have $H_1(\tilde{E}) = 0$, as required. This completes the proof. □

5.1.3. *The +-construction & uniqueness.* We next wish to prove Theorem 5.1.3. That is, any two models of $\mathrm{BGL}(R)^+$ are homotopy equivalent. To this end, we first abstract out the necessary conditions from the definition of $\mathrm{BGL}(R)^+$.

Definition 5.1.10 (**+-construction**). Let X be a based connected CW-complex and G be a perfect normal subgroup of $\pi_1(X)$. Then a map of CW-complexes $f : X \to Y$ is called a +-construction on X w.r.t. G if f is acyclic and $\mathrm{Ker}\,(f_* : \pi_1(X) \to \pi_1(Y)) = G$.

Remark 5.1.11. Let $f : X \to Y$ be a +-construction w.r.t. $P \leq \pi_1(X)$ perfect normal subgroup. By homotopy long exact sequence corresponding to map $Ff \to X \xrightarrow{f} Y$, we can immediately get following exact sequence:

$$\pi_1(Ff) \to \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \to \pi_0(Ff).$$

By Theorem 5.1.5, Ff is acyclic and thus $\pi_0(Ff) = 0$. Thus we have the exact sequence:

$$0 \to G \to \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \to 0.$$

Construction 5.1.12 (The construction of X^+)**.** Let X be a based connected CW-complex and $G \leq \pi_1(X)$ a perfect normal subgroup. We construct an inclusion $i : X \to X^+$ which is a +-construction of X w.r.t. G. To this end, the main strategy is as follows:

(1) First attach 2-cells to X to kill G in $\pi_1(X)$.
(2) Then attach 3-cells to remove the extra homology classes added by step 1.

Let us denote G in generators as follows:

$$G = \langle g_\alpha \mid \alpha \in I \rangle.$$

As $g_\alpha \in \pi_1(X)$, therefore we may interpret them as loops

$$g_\alpha : S^1 \to X.$$

Now attach 2-cells to X along each of the g_α:

(A1)
$$\begin{tikzcd} X' \arrow[dr, phantom, "\ulcorner", very near start] & \amalg_\alpha D^2 \arrow[l] \\ X \arrow[u, hook, "j_0"] & \amalg_\alpha S^1 \arrow[l, "\amalg_\alpha g_\alpha"] \arrow[u, hook, "i_0"'] \end{tikzcd} .$$

We first claim that $\pi_1(X')$ is $\pi_1(X)/G$ via j_0. Indeed, the map

$$j_{0*} : \pi_1(X) \longrightarrow \pi_1(X')$$

is surjective since any element $h : S^1 \to X'$ in $\pi_1(X')$ by cellular approximation theorem factors through the inclusion j_0. In particular, the 1-skeleton of X' is same as that of X. Consequently to prove our claim, we need only show that $\operatorname{Ker}(j_{0*}) = G$. Clearly, $\operatorname{Ker}(j_{0*}) \supseteq G$ by construction. Furthermore, if $k : S^1 \to X$ is null-homotopic in X', then k extends to $k' : D^2 \to X'$. By cellular approximation, we may assume that k' is a cellular map, so that k' is mapping in the 2-skeleton of X'. It follows at once that if k is not in G, then k (which we assume, by cellular approximation, that it is in 1-skeleton of X) on composition with j_0 gives a non-contractible loop as X' only trivializes all loops in G, a contradiction.

This shows that

$$\pi_1(X') = \pi_1(X)/G.$$

To complete the proof, we have to now kill all "new" homology classes of X' with an arbitrary choice of coefficient system $\mathcal{L}$ whose groups are isomorphic to L. To this end, we will attach 3-cells to X' to obtain the space X^+.

To illustrate the idea, suppose we have constructed X^+ by attaching 3-cells to X'. Our goal is then to show that $H_k(X^+;\mathcal{L}) \cong H_k(X;\mathcal{L})$. We thus have a triplet (X^+, X', X). By homology l.e.s. for the pair (X^+, X) , it suffices to show that

$$H_k(X^+, X; \mathcal{L}) = 0$$

for all $k \geq 0$. Recall that the homology of pair (X^+, X') with coefficient $\mathcal{L}$ is given by the homology of complex $L \otimes_{\mathbb{Z}[\pi_1(X)/G]} C_\bullet(\widetilde{X^+}, \hat{X})$ where $\hat{X}$ is the pullback of $\widetilde{X^+}$ along $X \to X^+$. It is thus sufficient to show that $C_\bullet(\widetilde{X^+}, \hat{X})$ is an acyclic complex (whose homology in every degree is 0). As $\widetilde{X^+}/\hat{X}$ will be a 3-dimensional CW-complex with no 1-cells, it is thus sufficient to show that the differential

$$d : C_3(\widetilde{X^+}, \hat{X}) \to C_2(\widetilde{X^+}, \hat{X})$$

is an isomorphism.

Now since we have isomorphisms $C_3(\widetilde{X^+}, \hat{X}) \cong C_3(\widetilde{X^+}, \widetilde{X'}) \cong H_3(\widetilde{X^+}, \widetilde{X'})$ and $C_2(\widetilde{X^+}, \hat{X}) \cong C_2(\widetilde{X'}, \hat{X}) \cong H_2(\widetilde{X'}, \hat{X})$ by the fact that cells of universal cover are obtained by lifting, therefore we have to show that the boundary map obtained by the triplet l.e.s. for $(\widetilde{X^+}, \widetilde{X'}, \hat{X})$ is an isomorphism. This is how we construct X^+ and then show that for this construction the above actually holds.

In order to construct X^+, we need maps $S^2 \to X'$ through which we can attach 3-cells. In particular, these are elements of $\pi_2(X')$. Consider the following pullback square

$$\begin{tikzcd} \hat{X} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X}' \arrow[d, "\pi"] \\ X \arrow[r, hook, "j_0"'] & X' \end{tikzcd}$$

where $\tilde{X}' \to X'$ is the universal cover. As pullback of covering is a covering, thus the map $\hat{X} \to X$ is a covering. Now, it is clear that $\hat{X} = \pi^{-1}(X)$, thus the inclusion $\hat{X} \hookrightarrow \tilde{X}'$ is also induced by attaching 2-cells to $\hat{X}$. It follows that $\pi_1(\hat{X}) \cong G$.

Next, observe that in the homology l.e.s. of $(\tilde{X}', \hat{X})$, we get the following isomorphism by Hurewicz (as $\tilde{X}'$ is 1-connected)

$$\pi_2(\tilde{X}') \xrightarrow{\cong} H_2(\tilde{X}') \xrightarrow{j_*} H_2(\tilde{X}', \hat{X}) \longrightarrow H_1(\hat{X}).$$

Again, by Hurewicz, we have

$$H_1(\hat{X}) \cong \pi_1(\hat{X})^{ab} = G^{ab} = 0$$

as G is perfect. Hence the above sequence becomes

$$\pi_2(\tilde{X}') \xrightarrow{\cong} H_2(\tilde{X}') \overset{j_*}{\twoheadrightarrow} H_2(\tilde{X}', \hat{X}).$$

Using the above, we have a surjection $\pi_2(\tilde{X}') \twoheadrightarrow H_2(\tilde{X}', \hat{X})$. For each homology class $[c_\beta] \in H_2(\tilde{X}', \hat{X})$ in a fixed generating set, choose one and only element in the fiber $[\tilde{h}_\beta] \in \pi_2(\tilde{X}')$. We thus have a collection of maps $\{\tilde{h}_\beta : S^2 \to \tilde{X}'\}_\beta$. Composing them with $\pi : \tilde{X}' \to X'$ yields maps $\{h_\beta : S^2 \to X'\}_\beta$. We use these maps to attach 3-cells to X'. Indeed, consider the pushout space:

(A2)
$$\begin{tikzcd} X^+ \arrow[dr, phantom, "\ulcorner", very near start] & \amalg_\beta D^3 \arrow[l, "\amalg_\beta d_\beta"'] \\ X' \arrow[u, hook, "k_0"] & \amalg_\beta S^2 \arrow[l, "\amalg_\beta h_\beta"] \arrow[u, hook] \end{tikzcd}.$$

We thus have the following inclusions of subcomplexes of X^+:

$$X \overset{j_0}{\hookrightarrow} X' \overset{k_0}{\hookrightarrow} X^+.$$

We again pass to universal cover of X^+ in order and take pullback along $X' \hookrightarrow X^+$ to have better algebraic control via Hurewicz:

$$\begin{tikzcd} \hat{X}' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X^+} \arrow[d, "\pi"] \\ X' \arrow[r, hook, "k_0"'] & X^+ \end{tikzcd}.$$

But $\pi_1(\hat{X}') = 0$ since k_{0*} is an isomorphism on π_1 and $\pi_1(\widetilde{X^+}) = 0$. Hence, we deduce that

$$\hat{X}' \cong \tilde{X}',$$

that is, $\hat{X}'$ is the universal cover of X'.

By naturality of Hurewicz, we have a map between the long exact sequences of homotopy groups induced by the map $\hat{X}' \hookrightarrow \widetilde{X^+}$ to that of homology groups

$$\begin{tikzcd} \cdots & \pi_{n+1}(\widetilde{X^+}, \hat{X}') \arrow[r] \arrow[d] & \pi_n(\hat{X}') \arrow[r] \arrow[d] & \pi_n(\widetilde{X^+}) \arrow[r] \arrow[d] & \pi_n(\widetilde{X^+}, \hat{X}') \arrow[d] & \cdots \\ \cdots & H_{n+1}(\widetilde{X^+}, \hat{X}') \arrow[r] & H_n(\hat{X}') \arrow[r] & H_n(\widetilde{X^+}) \arrow[r] & H_n(\widetilde{X^+}, \hat{X}') & \cdots \end{tikzcd}.$$

For $n = 3$, we get the following sequence from the above

$$\begin{tikzcd} \pi_3(\widetilde{X^+}, \widetilde{X'}) \arrow[r] \arrow[d] & \pi_2(\widetilde{X'}) \arrow[d, "\cong"] & \\ H_3(\widetilde{X^+}, \widetilde{X'}) \arrow[r, "\tilde{\partial}"'] & H_2(\widetilde{X'}) \arrow[r, two heads, "j_*"'] & H_2(\widetilde{X'}, \hat{X}) \end{tikzcd}.$$

We claim that $j_* \circ \tilde{\partial}$ is an isomorphism. Note that this is the boundary map of cellular complex. Indeed, observe that $H_3(\widetilde{X^+}, \widetilde{X'})$ is a free abelian group generated by the lift of 3-cells attached by $\tilde{h}_\beta$. We thus need only show that $j_* \circ \tilde{\partial}$ maps this bijectively onto the generators of $H_2(\widetilde{X'}, \hat{X})$ which we know are $[c_\beta]$. We know that the lifted map $\tilde{h}_\beta : S^2 \to \widetilde{X'}$ determines an element in $\pi_3(\widetilde{X^+}, \widetilde{X'})$ by definition of relative homotopy, whose image in $\pi_2(\widetilde{X'})$ is exactly $[\tilde{h}_\beta]$. Moreover, the class determined by $\tilde{h}_\beta$ in $\pi_3(\widetilde{X^+}, \widetilde{X'})$, under the Hurewicz map, determines a class $[l_\beta] \in H_3(\widetilde{X^+}, \widetilde{X'})$. By commutativity of above, it follows that $j_* \circ \tilde{\partial}$ maps $[l_\beta] \mapsto [c_\beta]$. As for each generator $[c_\beta] \in H_2(\widetilde{X'}, \hat{X})$, the element $[l_\beta]$ is unique by construction, we get that $j_* \circ \tilde{\partial}$ is an isomorphism, as required.

Remark 5.1.13. While it is rarely that we will use the explicit construction above, it is still good to keep in mind the precise way in which we found the 3-cells to attach to X' to get X^+. In particular, the attaching steps (A1) and (A2) are good to keep in mind.

Example 5.1.14 (+-construction of homology spheres)**.** Let X be a based connected CW-complex which is a homology n-sphere for $n > 1$ so that $\pi_1(X)$ is perfect. For $P = \pi_1(X)$, we claim that any +-construction of X w.r.t. P, $f : X \to X^+$, is such that $S^n \simeq X^+$.

Indeed, observe that $\pi_1(X)$ is perfect as X is a homology n-sphere. As f is a +-construction, therefore $\pi_1(X^+)$ is $\pi_1(X)/\pi_1(X) = 0$ by Remark 5.1.11. Moreover, X^+ itself is a homology n-sphere as $f : X \to X^+$ is acyclic. We now find a map $g : S^n \to X$ such that g is a weak equivalence, so that by Whitehead's theorem we will conclude that g is a homotopy equivalence, as required.

Indeed, observe that since X^+ is 1-connected, therefore by Hurewicz's theorem, we have $\pi_2(X^+) \cong H_2(X^+)$. If $n \neq 2$, then $\pi_2(X^+) = 0$ as X^+ is also a homology n-sphere. By induction and using Hurewicz repeatedly, we get that $\pi_k(X^+) = 0$ for all $0 \leq k \leq n-1$, so that X^+ is $n-1$-connected and thus by another application of Hurewicz, we have $\pi_n(X^+) \cong H_n(X^+) = \mathbb{Z}$. We thus have a non-trivial map $g : S^n \to X^+$ whose homology class is the generator. We finally claim that g induces an isomorphism in integral homology, which will complete the proof by Theorem 7.5.9 of [Spa66] (Whitehead's theorem). To this end, as X^+ is a also a homology n-sphere, thus we need only show that $g_* : H_n(S^n) = \mathbb{Z} \to H_n(X^+) = \mathbb{Z}$ takes $[\mathrm{id}] \mapsto [g]$. Indeed, we have $g_*([\mathrm{id}]) = [g \circ \mathrm{id}] = [g] \in H_n(X^+)$, as needed.

Proposition 5.1.15. *Let $i : X \to X^+$ and $j : Y \to Y^+$ be +-constructions w.r.t. perfect normal subgroups $G \leq \pi_1(X)$ and $H \leq \pi_1(Y)$. Then*

$$i \times j : X \times Y \to X^+ \times Y^+$$

is a +-construction of $X \times Y$ w.r.t. the perfect normal subgroup $G \times H \leq \pi_1(X \times Y)$.

Proof. We first show acyclicity of $i \times j$. By unravelling definitions, one reduces to showing that $F(i \times j) \cong F(i) \times F(j)$ is acyclic. To this end, use Künneth formula to deduce that if X, Y are acyclic, then so is $X \times Y$. The fact that kernel of $(i \times j)_*$ is $G \times H$ follows from $(i \times j)_* = i_* \times j_* : \pi_1(X) \times \pi_1(Y) \to \pi_1(X^+) \times \pi_1(Y^+)$, as required. □

The following universal property of Quillen tells us what we need, and then some more.

Theorem 5.1.16 (Quillen)**.** *Let X be a CW-complex and P be a perfect normal subgroup of $\pi_1(X)$. Let $f : X \to Y$ be a +-construction on X w.r.t. P. If $g : X \to Z$ is a map such that*

$$P \subseteq \mathrm{Ker}\,(g_* : \pi_1(X) \to \pi_1(Z)),$$

then there exists a map $h : Y \to Z$ such that the following diagram of spaces commutes

$$\begin{tikzcd} Y \arrow[r, dashed, "h"] & Z \\ X \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

and h is unique upto homotopy.

An immediate corollary is what we seek.

Corollary 5.1.17 (Uniqueness of +-construction)**.** *Let X be a CW-complex and P be a perfect normal subgroup of $\pi_1(X)$. If $f : X \to Y$ and $g : X \to Z$ are two +-constructions, then there is a homotopy equivalence $h : Y \xrightarrow{\simeq} Z$.* □

Another important consequence is that we have maps in +-construction.

Lemma 5.1.18. *Let X, Y be two connected CW-complexes and $i : X \to X^+$ and $j : Y \to Y^+$ be +-constructions w.r.t. perfect normal subgroups $G \leq \pi_1(X)$ and $H \leq \pi_1(Y)$ respectively. If $f : X \to Y$ is a map such that $f_* : \pi_1(X) \to \pi_1(Y)$ maps G into H, then there exists a map $\tilde{f} : X^+ \to Y^+$ unique upto homotopy w.r.t. the commutativity of the following square of spaces:*

$$\begin{tikzcd} X^+ \arrow[r, dashed, "\tilde{f}"] & Y^+ \\ X \arrow[u, "i"] \arrow[r, "f"'] & Y \arrow[u, "j"'] \end{tikzcd} .$$

Proof. The map $j \circ f$ on π_1 takes G to 0, so by Theorem 5.1.16 gives the required map unique upto homotopy. □

We shall prove Theorem 5.1.16 by using obstruction theory as developed in [Whi78], Chapter VI.

Proof of Theorem 5.1.16. Consider the based connected CW-complex X^+ obtained by Construction 5.1.12. Let $g : X \to Z$ be a map such that

$$P \subseteq \operatorname{Ker}(g_* : \pi_1(X) \to \pi_1(Z)).$$

We wish to extend g to $\tilde{g} : X^+ \to Z$. Consider the map $\theta : \pi_1(X)/P \to \pi_1(Z)$ as in the triangle below which exists by hypothesis on g_*:

$$\begin{tikzcd} \pi_1(X)/P \arrow[r, dashed, "\theta"] & \pi_1(Z) \\ \pi_1(X) \arrow[u, "i_*"] \arrow[ur, "g_*"'] & \end{tikzcd} .$$

We wish to show that g extends to $\tilde{g} : X^+ \to Z$ such that $\tilde{g}_* = \theta$. To this end, by obstruction theory, it is sufficient to show that

$$H^q(X^+, X; \mathcal{L}) = 0$$

for all $q \geq 3$ and all local coefficient systems $\mathcal{L}$ on X^+. Fix a local coefficient system $\mathcal{L}$ with group G. Note that we have

$$H^q(X^+, X; \mathcal{L}) \cong H^q\left(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X^+)]}\left(C_\bullet(\widetilde{X^+}, \hat{X}), G\right)\right)$$

where we have the following pullback of the universal cover of X^+:

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{X^+} \arrow[d] \\ X \arrow[r, hook] & X^+ \end{tikzcd}.$$

Now note from the Construction 5.1.12 that

$$C_k(\widetilde{X^+}, \hat{X}) = 0$$

for all $k \neq 2, 3$ and $d : C_3(\widetilde{X^+}, \hat{X}) \to C_2(\widetilde{X^+}, \hat{X})$ is an isomorphism. It follows at once that $H^q(X^+, X; \mathcal{L}) = 0$ for all $q \geq 0$, as required.

For uniqueness upto homotopy, obstruction theory further gives us a sufficient criterion that $H^2(X^+, X; \mathcal{L}) = 0$. Hence we are done. Moreover, by the long exact sequence of pairs for cohomology with local coefficients, we deduce that the map $i : X \hookrightarrow X^+$ induces isomorphism

$$i^* : H^q(X^+; \mathcal{L}) \to H^q(X; i^*\mathcal{L}),$$

that is, $i : X \to X^+$ is cohomologically acyclic as well. This shows the universal property for the explicit construction. We now show that any $+$-construction on X w.r.t. P is homotopy equivalent to the explicit one. This will then complete the proof.

Let $f : X \to Y$ be a $+$-construction w.r.t. P. Then by above there exists a map $\tilde{f} : X^+ \to Y$ as in the following triangle

$$\begin{tikzcd} X^+ \arrow[r, dashed, "\tilde{f}"] & Y \\ X \arrow[u, "i"] \arrow[ur, "f"'] \end{tikzcd}.$$

We claim that the map $\tilde{f}$ is a homotopy equivalence. By Whitehead's theorem, it is sufficient to show that $\tilde{f}$ is a weak-equivalence. Observe that as i and f are homologically acyclic, it follows at once that $\tilde{f}$ is also acyclic. Moreover, $\tilde{f}$ induces isomorphism in fundamental groups. By acyclic fiber theorem (Theorem 5.1.5), it follows that the homotopy fiber $F\tilde{f}$ is acyclic. We further claim that $F\tilde{f}$ is 1-connected. Indeed, from the long exact sequence for homotopy groups for $\tilde{f}$ and that $\tilde{f}_* : \pi_1(X^+) \to \pi_1(Y)$ is an isomorphism, it follows that the map $\pi_1(F\tilde{f}) \to \pi_1(X^+)$ is the zero map. It suffices to show that the transgression $\pi_2(Y) \to \pi_1(F\tilde{f})$, which is surjective by exactness, is the zero map as well. As $F\tilde{f}$ is acyclic, therefore $\pi_1(F\tilde{f})$ is a perfect group. By above, it is also abelian, and thus the zero group, as required.

Hence $F\tilde{f}$ is a 1-connected acyclic space, so that by Hurewicz's theorem, all homotopy groups of $F\tilde{f}$ are 0. By homotopy long exact sequence of $\tilde{f}$, it follows that $\tilde{f}$ is a weak-equivalence, as required. This also proves Corollary 5.1.17. $\square$

5.1.4. *K-theory space.* We now finally define higher K-groups as follows:

Definition 5.1.19 (**K-theory space & K-groups**)**.** Let R be a ring and let $\mathrm{BGL}(R)^+$ be as in Definition 5.1.1. Then for all $n \geq 1$, define

$$K_n(R) = \pi_n(\mathcal{K}(R))$$

where $\mathcal{K}(R) = K_0(R) \times \mathrm{BGL}(R)^+$ and $K_0(R)$ has discrete topology. The space $\mathcal{K}(R)$ is called the K-theory space of R.

We first have to show that $\pi_n(\mathcal{K}(R))$ is really independent of basepoints and that we are really computing homotopy groups of $\mathrm{BGL}(R)^+$ only, not of something else.

Lemma 5.1.20. *Let R be a ring and $\mathcal{K}(R)$ be the K-theory space of R.*
(1) $\pi_0(\mathcal{K}(R)) = K_0(R)$ in the usual sense.
(2) $\pi_n(\mathcal{K}(R)) = \pi_n(\mathrm{BGL}(R)^+)$.

Proof. Picking any base point of $\mathcal{K}(R)$, the path component of that point is homeomorphic to $\mathrm{BGL}(R)^+$, so the homotopy groups for different base points are all isomorphic. □

We immediately have maps in K-theory space.

Proposition 5.1.21 (Maps in K-theory)**.** *Let $f : R \to S$ be a ring homomorphism of commutative rings. Then there is an induced map*

$$\mathcal{K}(f) : \mathcal{K}(R) \longrightarrow \mathcal{K}(S)$$

unique upto homotopy w.r.t. commutativity of the following square:

$$\begin{tikzcd} \mathcal{K}(R) \arrow[r, dashed, "\mathcal{K}(f)"] & \mathcal{K}(S) \\ K_0(R) \times \mathrm{BGL}(R) \arrow[u, "\mathrm{id}\times i"] \arrow[r, "\mathrm{id}\times Bf"'] & K_0(S) \times \mathrm{BGL}(S) \arrow[u, "\mathrm{id}\times j"'] \end{tikzcd} .$$

We denote the maps in K_n-groups by

$$f_* : K_n(R) \to K_n(S).$$

Proof. As f induces a group homomorphism $f : \mathrm{GL}_p(R) \to \mathrm{GL}_p(S)$ by taking a matrix A to $f(A)$ where f is applied on each entry and $f(A)$ is invertible as product of matrices is a polynomial in each entry. Taking direct limits both side yields map

$$f : \mathrm{GL}(R) \to \mathrm{GL}(S).$$

Applying $B(-)$ yields a map

$$Bf : \mathrm{BGL}(R) \to \mathrm{BGL}(S)$$

such that $(Bf)_* = f$. Note that f maps $E(R)$ into $E(S)$. It follows by Lemma 5.1.18 that we have a map

$$\tilde{B}f : \mathrm{BGL}(R)^+ \to \mathrm{BGL}(S)^+$$

unique upto homotopy. As we already have a map $f_* : K_0(R) \to K_0(S)$ by Construction 2.1.4, it follows we have a continuous map

$$\mathcal{K}(f) : \mathcal{K}(R) \longrightarrow \mathcal{K}(S)$$

unique upto homotopy, as required. □

An important property of $\mathrm{BGL}(R)^+$ which will be exploited later is that it has a homotopy associative, unital and commutative operation.

Theorem 5.1.22 (H-spaces & $\mathrm{BGL}(R)^+$)**.** *Let R be a ring. Then $\mathrm{BGL}(R)^+$ is a homotopy commutative H-group.*

5.2. **Cup product in K-theory.**

Construction 5.2.1 (Loday's product)**.** Let R, S be two rings. We wish to define a product ($\otimes$ over $\mathbb{Z}$)

$$K_p(R) \otimes K_q(S) \longrightarrow K_{p+q}(R \otimes S).$$

Indeed, recall that we have a product on homotopy groups induced by smash product

$$\pi_p(X) \otimes \pi_q(Y) \longrightarrow \pi_{p+q}(X \wedge Y)$$

for any spaces X and Y. Hence it is sufficient to define a map

$$\psi : \mathrm{BGL}(R)^+ \wedge \mathrm{BGL}(S)^+ \longrightarrow \mathrm{BGL}(R \otimes S)^+.$$

This is the main content of Loday's construction.

To construct ψ, we first construct a map

$$\tilde{\varphi}_{pq} : \mathrm{BGL}_p(R) \times \mathrm{BGL}_q(S) \longrightarrow \mathrm{BGL}_{pq}(R \otimes S)$$

as follows. Note we have a homomorphism

$$\begin{aligned} \theta_{pq} : \mathrm{GL}_p(R) \times \mathrm{GL}_q(S) &\longrightarrow \mathrm{GL}_{pq}(R \otimes S) \\ (A, B) &\longmapsto A \otimes B \end{aligned}$$

where $A \otimes B$ is the usual tensor product of matrices (A and B represent a class in $\mathrm{GL}(R)$ and $\mathrm{GL}(S)$ respectively). In more details, the map is given by

$$\left(\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \right) \mapsto \begin{bmatrix} a_{11} \otimes b_{11} & a_{11} \otimes b_{12} & \cdots \\ a_{11} \otimes b_{21} & a_{11} \otimes b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Now observe that θ maps $E(R) \times E(S)$ into $E(R \otimes S)$, and thus we get maps φ_{pq} and $\tilde{\varphi}_{pq}$ by Theorem 5.1.16, unique upto homotopy, so that the following square commutes (see Propsition 5.1.15):

$$\begin{tikzcd} \mathrm{BGL}_p(R)^+ \times \mathrm{BGL}_q(S)^+ \arrow[r, dashed, "\tilde{\varphi}_{pq}"] & \mathrm{BGL}_{pq}(R \otimes S)^+ \\ \mathrm{BGL}_p(R) \times \mathrm{BGL}_q(S) \arrow[u, "i \times j"] \arrow[r, "\varphi_{pq}"'] & \mathrm{BGL}_{pq}(R \otimes S) \arrow[u] \end{tikzcd}.$$

To get a map in $\mathrm{BGL}_p(R)^+ \wedge \mathrm{BGL}_q(S)^+ \to \mathrm{BGL}_{pq}(R \otimes S)^+$, we need to show that the subspace $\{e\} \times \mathrm{BGL}_q(S)^+ \cup \mathrm{BGL}_p(R)^+ \times \{e\}$ is mapped to the basepoint of $\mathrm{BGL}_{pq}(R \otimes S)^+$ under the map $\tilde{\varphi}_{pq}$. However, the map $\tilde{\varphi}_{pq}$ may not satisfy this condition. So we exploit the H-space structure of $\mathrm{BGL}(R \otimes S)^+$. First, consider the map $\tilde{\varphi}$ with domain and codomain stabilized, which exists by another application of Theorem 5.1.16:

$$\begin{tikzcd}
\mathrm{BGL}(R)^+ \times \mathrm{BGL}(S)^+ \arrow[r, dashed, "\tilde{\varphi}"] & \mathrm{BGL}(R \otimes S)^+ \\
\mathrm{BGL}_p(R)^+ \times \mathrm{BGL}_q(S)^+ \arrow[u] \arrow[r, "\tilde{\varphi}_{pq}"'] & \mathrm{BGL}_{pq}(R \otimes S)^+ \arrow[u]
\end{tikzcd}.$$

Now construct the following map

$$\begin{aligned}
\psi : \mathrm{BGL}(R)^+ \times \mathrm{BGL}(S)^+ &\longrightarrow \mathrm{BGL}(R \otimes S)^+ \\
(x, y) &\longmapsto \tilde{\varphi}(x, y) - \tilde{\varphi}(x, \mathrm{pt.}) - \tilde{\varphi}(\mathrm{pt.}, y)
\end{aligned}$$

where pt. denotes the relevant basepoint and the addition is the homotopy commutative group operation on $\mathrm{BGL}(R \otimes S)^+$ by Theorem 5.1.22. Note that this map ψ is constant on the subspace $\mathrm{BGL}(R)^+ \wedge \mathrm{BGL}(S)^+$, so that it finally induces a map (which we again call ψ)

$$\psi : \mathrm{BGL}(R)^+ \wedge \mathrm{BGL}(S)^+ \longrightarrow \mathrm{BGL}(R \otimes S)^+.$$

This is the required map ψ. Thus, we get a map

$$\pi_p(\mathrm{BGL}(R)^+) \otimes \pi_q(\mathrm{BGL}(S)^+) \longrightarrow \pi_{p+q}(\mathrm{BGL}(R)^+ \wedge \mathrm{BGL}(S)^+) \xrightarrow{\psi_*} \pi_{p+q}(\mathrm{BGL}(R \otimes S)^+),$$

which in other words is

$$K_p(R) \otimes K_q(S) \longrightarrow K_{p+q}(R \otimes S),$$

as required.

The following is then the main theorem, which is suggestive of the heuristic that Loday's product behaves like cup product in cohomology. That is, the K-theory of schemes will come equipped with a cup product[12]!

Theorem 5.2.2 (Loday's theorem)**.** *Let R, S be rings.*

(1) The Loday's product $\psi : K_p(R) \otimes K_q(S) \to K_{p+q}(R \otimes S)$ is an associative and bilinear map.

(2) If R is commutative, then the following Loday's product:

$$K_p(R) \otimes K_q(R) \xrightarrow{\psi} K_{p+q}(R \otimes_{\mathbb{Z}} R) \longrightarrow K_{p+q}(R)$$

where $R \otimes_{\mathbb{Z}} R \to R$ is the structure map of ring R, makes $K(R) = \bigoplus_{i=0}^{\infty} K_i(R)$ into a graded-commutative ring. That is, if $x \in K_p(R)$ and $y \in K_q(R)$, then

$$x \cdot y = (-1)^{pq} y \cdot x.$$

[12]and hence suggests how to do intersection theory on non-smooth schemes.

5.3. **Relative exact sequence for K_n.** One of the main benefits of defining higher K-groups as homotopy groups of the K-theory space is that we get the familiar relative exact sequence for free.

Definition 5.3.1 (**Relative K_n**)**.** Let R, S be commutative rings and $f : R \to S$ be a ring homomorphism. Then we get a map $\mathcal{K}f : \mathcal{K}(R) \to \mathcal{K}(S)$. Consider the homotopy fiber

$$F(\mathcal{K}f) \longrightarrow \mathcal{K}(R) \xrightarrow{\mathcal{K}f} \mathcal{K}(S).$$

Denote

$$K_i(f) := \pi_i(F(\mathcal{K}f))$$

which we call the relative K-group of map f. Note that then by homotopy l.e.s. associated to a map, we get the following:

$$\begin{tikzcd}
 & & {} \arrow[dll, dash] \\
K_{i-1}(f) \arrow[r] & K_{i-1}(R) \arrow[r, "f_*"] & K_{i-1}(S) \arrow[dll] \\
K_i(f) \arrow[r] & K_i(R) \arrow[r, "f_*"] & K_i(S) \arrow[dll, dash] \\
{} & &
\end{tikzcd}.$$

Lemma 5.3.2. *Let R be a commutative ring, $I \leq R$ be an ideal and $\pi : R \twoheadrightarrow R/I$ be the corresponding quotient map. Then,*

$$K_0(R, I) \cong K_0(\pi)$$

where $K_0(R, I)$ is as defined in Definition 2.3.4.

Proof. (*Sketch*) Note $K_0(\pi) = \pi_0(F\mathcal{K}\pi)$, where we have the following fiber sequence

$$F\mathcal{K}\pi \to K_0(R) \times \mathrm{BGL}(R)^+ \overset{\mathcal{K}\pi}{\to} K_0(R/I) \times \mathrm{BGL}(R/I)^+.$$

By definition, we have

$$F\mathcal{K}\pi = \mathrm{Ker}\,(\pi_* : K_0(R) \to K_0(R/I)) \times F\widetilde{B\pi}$$

where $\widetilde{B\pi} : \mathrm{BGL}(R)^+ \to \mathrm{BGL}(R/I)^+$. Thus, we have

$$K_0(\pi) = \mathrm{Ker}\,(\pi_*) \times \pi_0(F\widetilde{B\pi}).$$

On the other hand, consider $p : R \oplus I \to R$ as in Definition 2.3.4. We then get the fiber sequence

$$F\mathcal{K}p \to K_0(R \oplus I) \times \mathrm{BGL}(R \oplus I)^+ \overset{\mathcal{K}p}{\to} K_0(R) \times \mathrm{BGL}(R)^+.$$

Again, we have

$$\begin{aligned} F\mathcal{K}p &= \operatorname{Ker}(p_* : K_0(R \oplus I) \to K_0(R)) \times F\widetilde{Bp} \\ &= K_0(R, I) \times F\widetilde{Bp} \end{aligned}$$

where $\widetilde{Bp} : \mathrm{BGL}(R \oplus I)^+ \to \mathrm{BGL}(R)^+$. Thus, we have

$$K_0(p) = K_0(R, I) \times \pi_0(F\widetilde{Bp}).$$

We claim that the fiber $F\widetilde{Bp}$ is connected, so that $K_0(p) = K_0(R, I)$. Indeed, observe by homotopy l.e.s. corresponding to $F\mathcal{K}p$ that it suffices to show that $p_* : K_1(R \oplus I) \to K_1(R)$ is a surjection. This follows immediately from functoriality of π_1 and the splitting $R \to R \oplus I \to R$. This shows that

$$K_0(p) = K_0(R, I).$$

One then reduces to showing that $K_0(p) \cong K_0(\pi)$, for which one has to do more work. □

5.4. **Finite coefficients.**

Construction 5.4.1 (Moore spectrum)**.** Let G be an abelian group. We have an associated suspension CW-spectrum called Moore spectrum $P^\infty G$ whose n^{th}-term is $P(G, n)$, the unique homotopy type whose n^{th} reduced cohomology is G and rest are 0. Recall that for any compactly generated based spaces X, Y, the based homotopy classes of maps $[\Sigma X, Y]$ is a group and $[\Sigma^2 X, Y]$ is an abelian group. Since $P(G, n) \simeq \Sigma^{n-1} P(G, 1)$, it follows at once that

$$[P(G, n), X] = [\Sigma^{n-2} P(G, 2), X]$$

is a group for $n = 3$ and an abelian group for $n \geq 4$. For $n = 2$ it may not be a group.

Using Moore spectrum, we define mod l homotopy groups for any integer l as follows.

Definition 5.4.2 ($\pi_n(X; \mathbb{Z}/l)$)**.** Let X be a based CW-complex and $P^\infty \mathbb{Z}/l$ be the Moore spectrum for group $\mathbb{Z}/l$ for some integer l. Then mod l homotopy groups of X are defined as the following homotopy class of maps

$$\pi_n(X; \mathbb{Z}/l) = [P(\mathbb{Z}/l, n), X].$$

As noted in Construction 5.4.1, $\pi_3(X; \mathbb{Z}/l)$ is a group and $\pi_n(X; \mathbb{Z}/l)$ is an abelian group for $n \geq 4$. For a map $f : X \to Y$, we get by composition maps $f_* : \pi_n(X; \mathbb{Z}/l) \to \pi_n(Y; \mathbb{Z}/l)$ which is a group homomorphism for $n \geq 3$. Thus $\pi_n(-; \mathbb{Z}/l)$ is a functor on $\mathcal{C}W_*$.

Remark 5.4.3 (Main facts about mod l homotopy)**.** The following three facts is what we shall use in the discussion later, most of which are quite familiar.

(1) Every fibration sequence $F \to E \to B$ induces a long exact sequence in mod l homotopy groups.
(2) For every complex X, there is a s.e.s.

$$0 \to \pi_n(X) \otimes \mathbb{Z}/l \to \pi_n(X; \mathbb{Z}/l) \to T_l(\pi_{n-1}(X)) \to 0$$

which is split exact if $l \neq 2 \mod 4$, where $T_l(G) = \{g \in G \mid l \cdot g = 0\}$.

(3) There is a natural mod l Hurewicz map

$$h_i : \pi_i(X;\mathbb{Z}/l) \longrightarrow H_i(X;\mathbb{Z}/l)$$

which is an isomorphism for all $1 \leq i \leq n$ if X is an $(n-1)$ connected nilpotent space.

All these can results in more generality can be found in [Nei80].

The following observation tells us why we are looking at mod l-homotopy groups.

Proposition 5.4.4. *Let X be an H-space and consider the map*

$$m_l : X \to X$$

which is multiplication by $l \in \mathbb{Z}$[13]. Denote F to be the homotopy fiber of the map m_l. Then

$$\pi_n(X;\mathbb{Z}/l) \cong \pi_{n-1}(F)\ \forall n \geq 2.$$

Proof. A model of Moore space $P^{n+1}(\mathbb{Z}/l)$ is obtained by gluing an $n+1$-cell to S^n by a degree l-map. Equivalently, this CW-complex is homeomorphic to the homotopy cofiber

$$S^n \xrightarrow{f_l} S^n \longrightarrow Cf_l \cong P^{n+1}(\mathbb{Z}/l)$$

where f_l is a degree l map. Recall that the inclusion $S^n \to Cf_l$ is a cofibration, thus the above is a short cofiber sequence. We also have a short fiber sequence

$$F \longrightarrow X \xrightarrow{m_l} X$$

where $F \to X$ is a fibration.

It is well-known that $\mathrm{Map}_*(-,X) : \mathcal{T}op_*^{cg} \to \mathcal{T}op_*^{cg}$ takes cofibrations to fibrations and $\mathrm{Map}_*(X,-)$ takes fibrations to fibrations. Furthermore, for any based space X, we have the smash-map duality (akin to $\otimes$-hom duality):

$$\pi_k(\mathrm{Map}_*(S^n,X)) = [S^k, \mathrm{Map}_*(S^n,X)] \cong [S^k \wedge S^n, X] \cong [S^{k+n}, X] = \pi_{k+n}(X).$$

Similarly for $P^{n+1}(\mathbb{Z}/l)$. Now with this information, we apply $\mathrm{Map}_*(-,X)$ onto the first cofiber sequence and $\mathrm{Map}_*(S^n,-)$ onto the second fiber sequence to get two fiber sequences for homotopic maps c(S^n is an H-cogroup and X is an H-group):

$$\mathrm{Map}_*(P^{n+1}(\mathbb{Z}/l),X) \to \mathrm{Map}_*(S^n,X) \xrightarrow{-\circ f_l} \mathrm{Map}_*(S^n,X)$$
$$\mathrm{Map}_*(S^n,F) \to \mathrm{Map}_*(S^n,X) \xrightarrow{m_l\circ -} \mathrm{Map}_*(S^n,X).$$

Hence $\mathrm{Map}_*(P^{n+1}(\mathbb{Z}/l),X) \simeq \mathrm{Map}_*(S^n,F)$, which yields the proof after taking homotopy groups and using the above mentioned fact. $\square$

Definition 5.4.5 ($K_n(R;\mathbb{Z}/l)$)**.** Let $l \in \mathbb{Z}$ be an integer and R be a ring. Then mod l K-groups of R are defined as

$$K_n(R) := \pi_n(\mathrm{BGL}(R)^+;\mathbb{Z}/l).$$

[13]this makes sense as ΩA has a canonical H-space structure.

Construction 5.4.6 (Bott element in $K_2(R;\mathbb{Z}/l)$)**.** Let R be any ring containing a primitive l^{th}-root of unity ζ. Note that the class of the diagonal matrix consisting of ζ in $\mathrm{GL}(R)/E(R) = K_1(R)$ is l-torsion and is thus in $T_l(K_1(R))$, which we denote by $[\zeta] \in K_1(R)$. Then, by universal coefficients theorem (Remark 5.4.3, item 2), we get an l-torsion element $\beta \in K_2(R;\mathbb{Z}/l)$ which maps to $\zeta \in K_1(R)$. This element β is called a Bott element of $K_2(R;\mathbb{Z}/l)$.

By the calculation for finite fields done by Quillen, one can deduce the following by basic algebra.

Theorem 5.4.7. *Let p be a prime and consider the algebraic closure $\bar{\mathbb{F}}_p$.*

(1) We have

$$K_n(\bar{\mathbb{F}}_p) = \begin{cases} (\mathbb{Q}/\mathbb{Z})[1/p] & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

(2) We have for $l \in \mathbb{Z}$ coprime to p

$$K_n(\bar{\mathbb{F}}_p;\mathbb{Z}/l) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z}/l & \text{if } n \text{ is even.} \end{cases}$$

6. K-theory & étale cohomology

We discuss some fundamental connections between K-theory and algebraic geometry.

6.1. The case of fields : K_2 & Galois cohomology.

In §4.4, we constructed a representation of $K_2(F)$ in the Brauer group $\mathrm{Br}(F)$ for fields which contains a primitive root of unity. Our goal in this section is to construct Galois symbol for fields not necessarily containing primitive roots of unity. We will thus replace the role of ζ by $\mu_m(F)$, the group of m^{th}-roots of unity in F.

The main idea is to replace Brauer group from the codomain of φ of Theorem 4.4.6 to an object which is more cohomological w.r.t. F and thus will have bilinear pairing from $F^\times \otimes F^\times$, which will, hopefully, satisfy Matsumoto relations. Indeed, such an object exists and is the content of Galois cohomology.

Construction 6.1.1 (Galois cohomology)**.** Let F be a field. Denote F_{sep} to be the separable closure of F (in the algebraic closure). Thus, F_{sep}/F is a separable normal extension and hence Galois. Let $G = \mathrm{Gal}\,(F_{\text{sep}}/F)$ be the Galois group. As F_{sep}/F may not be finite, thus G may not be a finite group. Recall the fundamental Galois theorem for finite case

$$\begin{array}{c} \{L \mid F_{\text{sep}}/L/F \text{ is a finite intermediate extension}\} \\ {\scriptstyle F_{\text{sep}}^{(-)}}\Big\uparrow \quad \Big\downarrow{\scriptstyle \mathrm{Gal}(F_{\text{sep}}/-)} \\ \{H \mid H \leq G \text{ is a subgroup}\} \end{array} .$$

Our goal is to study this relationship more "homologically" in the the case of separable closure. In particular, note that G here is acting on field F_{sep} by the obvious action, and the whole classical Galois theory is the study of this action together with its orbits and stabilizers.

Consider the following collection of subgroups of G:

$$G_E = \mathrm{Gal}\,(F_{\text{sep}}/E)\,,\ F_{\text{sep}}/E/F \ \&\ E/F \text{ is finite.}$$

Observe that $\{G_E\}_{F_{\mathrm{sep}}/E/F}$ forms a basis of topology on G, as we have

$$\bigcup_{F_{\mathrm{sep}}/E/F} G_E = G \qquad \text{[as } E = F \text{ is possible]}$$
$$G_E \cap G_{E'} \supseteq G_{E\cdot E'} \qquad \text{[by definition].}$$

Thus G is a topological group. We will now study left modules over the group ring $\mathbb{Z}[G]$, also called G-modules. Note that for any G-module M, we will have a left multiplication map $G \times M \to M$. We will call M *discrete* if this map is continuous, where M has discrete topology.

Now fix a G-module M and as usual in Galois theory, consider the G-invariant subgroup $M^G := \{m \in M \mid g \cdot m = m \ \forall g \in G\}$. Observe that invariant subgroup construction is functorial on the category of discrete G-modules:

$$(-)^G : \mathcal{M}od_c(G) \longrightarrow \mathcal{A}b$$
$$M \longmapsto M^G$$

and for a map of G-modules $f : M \to N$, we get a map $f^G : M^G \to N^G$ as $f(g \cdot m) = g \cdot f(m)$ for all $m \in M$. An easy observation tells us that $(-)^G$ is also left-exact. Observe that the category of discrete G-modules is abelian with enough injectives (Lemma 6.11.10 of [Wei94]). We may thus right derive the functor $(-)^G$, to obtain the *Galois cohomology groups*

$$H^i_{\text{ét}}(F; M) := (R^i(-)^G)(M).$$

Observe that $H^0_{\text{ét}}(F; M) = M^G$ (Lemma 26.2.3.3 of [FoG]).

Remark 6.1.2 (G-modules)**.** Consider the notation of Construction 6.1.1. Here are some elementary examples of discrete G-modules:

(1) $\mathbb{G}_m := F_{\mathrm{sep}}^{\times}$, the abelian group of units is a discrete G-module as $\sigma \in G$ acts on $x \in \mathbb{G}_m$ by $(\sigma, x) \mapsto \sigma(x)$ and since σ is an automorphism, thus every fiber of the above map is open, thus the action being continuous.
(2) μ_n, the subgroup of $\mathbb{G}_m$ of all n^{th}-roots of unity. This is discrete for the same reason as above.
(3) $M \otimes_{\mathbb{Z}} N$, the tensor product of two G-modules. Indeed, we define the action of $\sigma \in G$ on a simple tensor $m \otimes n \in M \otimes_{\mathbb{Z}} N$ by $(\sigma, m \otimes n) \mapsto (\sigma \cdot m) \otimes (\sigma \cdot n)$. This is continuous since the individual actions on M and N are continuous. In particular, $\mu_n^{\otimes 2} = \mu_n \otimes \mu_n$ is a discrete G-module.

The cohomology of G-groups $\mathbb{G}_m, \mu_n$ and $\mu_n^{\otimes 2}$ is very interesting. We begin by seeing this for μ_n.

Remark 6.1.3. Let F be a field, F_{sep} be its separable closure and $G = \mathrm{Gal}\,(F_{\mathrm{sep}}/F)$. Denote $\mu_n \leq \mathbb{G}_m$ to be the group of all n^{th}-roots of unity in F_{sep}. Consider the map $\mathbb{G}_m \to \mathbb{G}_m$ mapping $g \mapsto$

Construction 6.1.4 (Kummer sequence)**.** Let F be a field of characteristic $p > 0$ and let n be coprime to p. Denote F_{sep} to be the separable closure of F and $\mathbb{G}_m = F_{\mathrm{sep}}^{\times}$. We obtain the following short exact sequence:

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \stackrel{g \mapsto g^n}{\longrightarrow} \mathbb{G}_m \longrightarrow 1.$$

Indeed, the only part that needs to be shown is the surjectivity of the above map. To this end, pick any $a \in \mathbb{G}_m$ and consider the polynomial $x^n - a \in F_{\text{sep}}[x]$. As the derivative of $x^n - a$ is nx^{n-1} which is not zero as $p \nmid n$, thus $x^n - a$ is separable. Let $b \in \overline{F}$ be a root of $x^n - a$. By Proposition 23.6.8.7 of [FoG], it follows that $\overline{F}/F_{\text{sep}}$ is purely inseparable. If $b \notin F_{\text{sep}}$, then the minimal polynomial of b in F_{sep} is $m_{b,F_{\text{sep}}}(x) = x^{p^k} - c$ for some $c \in F_{\text{sep}}$ (Theorem 23.6.8.3 of [FoG]). As $m_{b,F_{\text{sep}}}(x) | x^n - a$ and $m_{b,F_{\text{sep}}}$ is not separable (in-fact it has only one root which repeats), therefore $x^n - a$ is not separable as well, a contradiction.

The above short-exact sequence is called the Kummer sequence of F.

As the above short exact sequence is that of discrete G-modules, therefore in right derived functors we get a long exact sequence.

Lemma 6.1.5 (Kummer cohomology sequence)**.** *Let F be a field of characteristic $p > 0$ and let n be coprime to p. Denote F_{sep} to be the separable closure of F, $\mathbb{G}_m = F_{\text{sep}}^\times$ and $G = \operatorname{Gal}(F_{\text{sep}}/F)$. Then the following is a long exact sequence:*

$$\begin{tikzcd}
1 \arrow[r] & \mu_n(F) \arrow[r] & F^\times \arrow[r, "g \mapsto g^n"] & F^\times \arrow[dll] \\
& H^1_{\text{ét}}(F;\mu_n) \arrow[r] & H^1_{\text{ét}}(F;\mathbb{G}_m) \arrow[r] & H^1_{\text{ét}}(F;\mathbb{G}_m) \arrow[dll] \\
& H^2_{\text{ét}}(F;\mu_n) \arrow[r] & H^2_{\text{ét}}(F;\mathbb{G}_m) \arrow[r] & H^2_{\text{ét}}(F;\mathbb{G}_m) \arrow[dll, dash] \\
& {} & &
\end{tikzcd}.$$

Proof. Follows from Theorem 26.2.3.5 of [FoG]. ∎

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