Complex Analysis

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Contents

1 Holomorphic functions

We will here review some of the classical results of complex function theory of one variable, namely the following four topics:

- Analytic functions; Cauchy-Riemann equations, harmonic functions.
- Complex integration; Zeroes of analytic functions, winding numbers, Cauchy's formula and theorem, Liouville's theorem, Morera's theorem, open-mapping theorem, Schwarz's lemma.
- Singularities; Classification, Laurent series, Casorati-Weierstrass theorem, residues and applications, meromorphic maps, Rouché's theorem.
- Conformal maps; Möbius transformations, normality and compactness, Riemann mapping theorem.

All this is important as it will build one's intuition of geometry in complex case, which is something we don't learn as early in our studies as, say, real geometry. Of-course this would be of immense use in complex algebraic geometry, some if which we shall cover in later chapters. Moreover, a complex manifold by definition locally looks like $\mathbb{C}^n$, hence it is imperative that we understand the geometry and analysis of complex plane and make it as second nature as the usual geometry over $\mathbb{R}^2$ is to us.

Let $\Omega \subseteq \mathbb{C}$ denote an open subset of the complex plane $\mathbb{C}$ for the rest of this chapter. Consider a function $f : \Omega \to \mathbb{C}$. Motivated by the classical case of real differentiability in one variable, we can define a notion of differentiation for f at $a \in \Omega$.

Definition 1.0.1. (**$\mathbb{C}$-differentiable/holomorphic functions**) A function $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable or holomorphic at $a \in \Omega$ if the following limit exists:

$$\varinjlim_{z \to 0} \frac{f(a+z) - f(a)}{z}$$

in which case it's value is said to be the derivative of f at a and is denoted by $\frac{df}{dz}(a) = f'(a) \in \mathbb{C}$.

Remark 1.0.2. As we shall soon see, this seemingly innocuous definition for some surprising reason gives the following fantastic results:

1. Theorems 1.1.2 and **??** tells us:

$$\{\textit{All } \mathbb{C}\textit{-differentiable maps } f : \Omega \to \mathbb{C}\}$$

$$\cong$$

$$\{\textit{All pairs of differentiable maps } u, v : \Omega \to \mathbb{R}, \textit{ related by CR-equations}\}$$

2. Corollary 1.2.2 and Theorem **??** tells us:

$\mathbb{C}$-differentiable maps are conformal.

3. Theorem **??** tells us:

$\mathbb{C}$-differentiable functions are harmonic.

Moreover, Theorem **??** tells us that if Ω is simply connected, then

$$\begin{array}{c}\{\textit{Harmonic functions } \Omega \subseteq \mathbb{R}^2 \cong \mathbb{C}\} \\ \wr\| \\ \{\mathbb{C}\textit{-differentiable functions on } \Omega \subseteq \mathbb{C}\}\end{array}.$$

4. Theorem **??** tells us:

Contour integral of a $\mathbb{C}$-differentiable map around a loop is 0.

5. Theorem **??** tells us:
A $\mathbb{C}$-differentiable function inside a disc is determined by its values on the disc's boundary.
6. Corollary 2.3.5 tells us:

$$\begin{array}{c}\{\mathbb{C}\textit{-differentiable maps } f : \Omega \to \mathbb{C}\} \\ \wr\| \\ \{\textit{Analytic maps } f : \Omega \to \mathbb{C}\}\end{array}.$$

This shows the sheer importance of the notion of $\mathbb{C}$-differentiability, which we will explore later in a more *local* setting. Our goal in the rest of this chapter is to provide rather quick proofs to these results while portraying the main ideas employed in them.

Let us start by analyzing some elementary properties of holomorphic maps.

1.1 Cauchy-Riemann equations

Let $f : \Omega \to \mathbb{C}$ be a holomorphic map on an open subset $\Omega \subseteq \mathbb{C}$. Now, there is a homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{C}$ given by $(x, y) \mapsto x + iy$. Composing f with this map, we get that f can equivalently be stated as the data of two real valued maps $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x, y) = \Re f(\varphi(x, y))$ and $v(x, y) = \Im f(\varphi(x, y))$.

Like in the case of $\mathbb{R}$-differentiability, in our case we can also define partial differential operators of f w.r.t. x, y and z.

Definition 1.1.1. (**Partial differential operators on** f) Let $f : \Omega \to \mathbb{C}$ be a holomorphic map on an open subset Ω of $\mathbb{C}$. Then, we define the following quantities in an obvious manner:
1. $\frac{\partial f}{\partial x} := \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

2. $\frac{\partial f}{\partial y} := \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$.
3. $\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$.
4. $\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$.

Then the fact that f is holomorphic can be equivalently stated in terms of real differentiability of the maps u and v as the following theorem states.

Theorem 1.1.2. *Suppose $f : \Omega \to \mathbb{C}$ is any $\mathbb{C}$-valued function on an open set Ω of $\mathbb{C}$. Then write $f(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \rightrightarrows \mathbb{R}$.*

1. *$f : \Omega \to \mathbb{C}$ is holomorphic at $z_0 \in \Omega$ if and only if u, v are real differentiable and satisfy any of the following equivalent PDEs at z_0:*
   (a) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.
   (b) $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$.
   (c) $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}$.
   (d) $\frac{\partial f}{\partial \bar{z}} = 0$.
2. *If $u, v : \Omega \to \mathbb{R}$ is a pair of C^1-maps satisfying the CR-equations, then $f := u + iv$ is a holomorphic map.*

Proof. Equivalence of the four PDEs is straightforward. Now let $f : \Omega \to \mathbb{C}$ be a holomorphic map. This means that for any $a \in \Omega$, we have

$$\frac{\partial f}{\partial z}(a) = \varinjlim_{z \to 0} \frac{f(a+z) - f(a)}{z}.$$

The required PDEs for u and v follows by letting z approach 0 first from real axis and then from imaginary axis and deeming them equal.

Next, we may write $R(z) = f(a+z) - f(a) - cz$ for some $c = c_1 + ic_2$ and then $R(z) = R_u(z) + iR_v(z)$ where $R_u(z) = u(a+z) - u(a) - c_1 x + c_2 y$ and $R_v(z) = v(a+z) - v(a) - c_2 x - c_1 y$. Then, f is holomorphic at a with $\frac{df}{dz}(a) = c$ if and only if $\varinjlim_{z \to 0} \frac{R(z)}{z} = 0$. But the latter happens if and only if $\varinjlim_{z \to 0} \frac{R_u(z)}{z} = 0 = \varinjlim_{z \to 0} \frac{R_v(z)}{z}$. Now $\frac{R_u(z)}{z} = 0$ if and only if $c_1 = \frac{\partial u}{\partial x}(a)$ and $c_2 = -\frac{\partial u}{\partial y}(a)$. Similarly, $\varinjlim_{z \to 0} \frac{R_v(z)}{z} = 0$ if and only if $c_2 = \frac{\partial v}{\partial x}(a)$ and $c_1 = \frac{\partial v}{\partial y}(a)$. $\square$

1.2 Conformal maps

We will now show that holomorphic maps "preserves angles". The meaning of angle is not well-defined a-priori on the complex plane, so we will have to develop that first.

A *curve* in $\mathbb{C}$ is a continuous map $\gamma : I \to \mathbb{C}$. It is said to be *differentiable* if $\Re\gamma : I \to \mathbb{R}$ and $\Im\gamma : I \to \mathbb{R}$ are differentiable $\mathbb{R}$-valued functions. It is said to be regular at $t \in I$ if $\gamma'(t) \neq 0 \in \mathbb{C}$. Now, let $\gamma_1, \gamma_2 : I \to \mathbb{C}$ be two curves which intersect at $\gamma_1(t_1) = \gamma_2(t_2)$ for $t_1, t_2 \in I$ such that γ_i is regular at t_i, $i = 1, 2$. Such an intersection is said to be *regular*. Then, the angle of intersection of γ_1 and γ_2 at a regular point is defined to be:

$$\angle\gamma_1(t_1), \gamma_2(t_2) := \arg\gamma_2'(t_2) - \arg\gamma_1'(t_1).$$

A function $f : \Omega \to \mathbb{C}$ is *conformal* at $z_0 \in \Omega$ if f preserves angles of all regular intersections of two curves at z_0.

It is now an easy observation that holomorphic maps will be conformal.

Lemma 1.2.1. *Let $f : \Omega \to \mathbb{C}$ be a holomorphic map on an open set Ω of $\mathbb{C}$. If $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, then for any two curves γ_1, γ_2 such that $\gamma_1(t_1) = z_0 = \gamma_2(t_2)$ and γ_1, γ_2 are regular at t_1, t_2 respectively, then*

$$\angle\gamma_1(t_1), \gamma_2(t_2) = \angle f \circ \gamma_1(t_1), f \circ \gamma_2(t_2).$$

Proof. The result follows from chain rule and the fact that $\arg wz = \arg w + \arg z$. □

A map $f : \Omega \to \mathbb{C}$ is called *conformal* if it preserves angles of all regularly intersecting curves. Thus,

Corollary 1.2.2. *All holomorphic functions are conformal except at those points at which derivative is zero.* □

We will now show that even an arbitrary conformal map $f : \Omega \to \mathbb{C}$ is also holomorphic.

Theorem 1.2.3. *Let $f : \Omega \to \mathbb{C}$ be a conformal map such that $\Re f$ and $\Im f$ are of class C^1. Then,*

1. *f is holomorphic.*
2. *$f'(z) \neq 0$ for all $z \in \Omega$.*

Proof. Simple thus omitted. □

1.3 Harmonic maps

A function $f : \Omega \to \mathbb{C}$ is said to be harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Below are some straightforward implications of Cauchy-Riemann equations.

Lemma 1.3.1. *Let $f = u + iv : \Omega \to \mathbb{C}$ be a function where $u, v : \Omega \rightrightarrows \mathbb{R}$. Then, f is harmonic if and only if u and v are harmonic (in $\mathbb{R}$-sense).*

Lemma 1.3.2. *All holomorphic maps are harmonic.*

Lemma 1.3.3. *All conformal maps are harmonic.*

1.4 Linear fractional transformations

A linear fractional transformation is a map

$$\begin{aligned} \varphi : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{az + b}{cz + d} \end{aligned}$$

where $a, b, c, d \in \mathbb{C}$. These are important as they provide a class of workable examples of rational functions, which are pretty much the bread and butter of algebraic geometry. Moreover, these maps arrange themselves in a group and it then follows that it contains as a subgroup the biholomorphic automorphism group of lots of geometric objects of interest (see Lemmas 1.4.2, 1.4.3). However, these maps makes the most sense on the complex projective line, $\mathbb{C}P^1$, the quotient of $\mathbb{C}^2$ by all lines passing through origin, which is the usual Riemann sphere $\bar{\mathbb{C}}$.

Let us work out this connection in detail. We have the following maps:

$$\alpha : \mathbb{C}^2 \twoheadrightarrow \mathbb{C}P^1 \xrightarrow{\cong} \bar{\mathbb{C}}$$

$$(w,z) \longmapsto [w,z] \longmapsto \tfrac{w}{z}$$

Notice that $L_f(\bar{\mathbb{C}})$, the collection of all linear fractional transforms on $\bar{\mathbb{C}}$ forms a group where the identity is given when $a = 0 = c$. The multiplication of two fractional transforms is again a fractional transform, as can be checked easily. Hence, it follows that $L_f(\bar{\mathbb{C}})$ is a subgroup of all biholomorphic maps of $\bar{\mathbb{C}}$, the $\mathrm{Aut}\left(\bar{C}\right)$. Hence we have a hold on one type of global biholomorphic maps of the Riemann sphere(!)

We then have the following result.

Lemma 1.4.1. *Let $\bar{\mathbb{C}}$ denote the Riemann sphere. Then,*

$$L_f(\bar{\mathbb{C}}) \cong GL_2(\mathbb{C})/\mathbb{C}^\times I_2$$

Proof. There's a natural map

$$\kappa : GL_2(\mathbb{C}) \longrightarrow L_f(\bar{\mathbb{C}})$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{az+b}{cz+d}.$$

This is a group homomorphism, as can be checked easily. The kernel of this homomorphism consists of matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $\frac{az+b}{cz+d} = z$. Unravelling, we get $c = 0 = b$ and $a = d \neq 0$. □

This group is also known by *projective general linear group*, $PGL_2(\mathbb{C}) := L_f(\bar{\mathbb{C}})$. The group $L_f(\bar{\mathbb{C}})$ also has some special subgroups. For example, it consists of all biholomorphic maps of $D^\circ := \{z \in \mathbb{C} \mid |z| < 1\}$.

Lemma 1.4.2. *For the open unit ball D°, we have*

$$\mathrm{Aut}\left(D^\circ\right) \cong \left\{ \frac{t(z-a)}{1-\bar{a}z} \mid |t| = 1 \;\&\; a \in D^\circ \right\}.$$

Similarly, it also contains an isomorphic copy of all biholomorphic maps of upper half plane $\mathbb{H}$.

Lemma 1.4.3. *For the upper half plane $\mathbb{H} \subset \mathbb{C}$, we have*

$$\mathrm{Aut}\left(\mathbb{H}\right) \cong SL_2(\mathbb{R}) \subset GL_2(\mathbb{C}).$$

1.4.1 Properties

Let us now state some basic properties of fractional transforms.

Lemma 1.4.4. *If $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a non-identity fractional transform, then either it has one or two fixed points, but not zero.*

Proof. A non-identity fractional transform $\varphi(z) = \frac{az+b}{cz+d}$ follows that either $b, d \neq 0$ or $a \neq c$. Suppose the former is not the case. Now if $\varphi(z) = z$, then it follows that $cz^2 + (d-a)z - b = 0$ where $b = d = 0$. Thus we obtain $z(cz - a) = 0$, which gives atleast one and atmost two solutions. Similarly, if $a = c$, then $b, d \neq 0$. It then follows that the above quadratic has either one or two solutions. □

Another property of fractional transforms is that they are uniquely determined by how they map on three points.

Lemma 1.4.5. *If z_1, z_2, z_3 and w_1, w_2, w_3 are two pair of distinct points in $\bar{\mathbb{C}}$, then there exists a unique fractional transform $\varphi \in L_f(\bar{\mathbb{C}})$ such that*

$$f(z_i) = w_i \; \forall i = 1, 2, 3.$$

Proof. Uniqueness follows from the fact that if $\varphi, \varpi : \bar{\mathbb{C}} \rightrightarrows \bar{\mathbb{C}}$ are two fractional transforms taking $z_i \mapsto w_i$, then the fractional transform $\varphi \circ \varpi^{-1}$ has 3 fixed points. It follows from Lemma 1.4.4 that $\varphi \circ \varpi^{-1} = \mathrm{id}$.

To show existence, take any arbitrary triple $v_1, v_2, v_3 \in \bar{\mathbb{C}}$. We will show that one can construct a fractional transform depending on v_i mapping as $z_i \mapsto v_i$. Denote then the map φ, $z_i \mapsto v_i$ and ϖ, $w_i \mapsto v_i$. Then $\varpi^{-1} \circ \varphi$ would be the required map. Since v_i can be arbitrary, therefore we choose it as per our convenience. It is perhaps easier to write it for $\infty, 0, 1$. □

One last basic property that may be observed for fractional transforms is that they are conformal.

Lemma 1.4.6. *All fractional transforms $\varphi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ are conformal.*

Proof. Since fractional transforms are holomorphic, therefore by Corollary 1.2.2, we reduce to showing that $\varphi'(z) \neq 0$ for all $z \in \bar{\mathbb{C}}$. Indeed, we have

$$\phi'(z) = \frac{ad - bc}{(cz+d)^2},$$

where since $ad - bc \neq 0$ by definition, therefore $\phi'(z) \neq 0$. □

1.4.2 Example : The Cayley transform

We will discuss here the properties of the following fractional transform, known by Cayley's name:

$$\begin{aligned} \varphi : \bar{\mathbb{C}} &\longrightarrow \bar{\mathbb{C}} \\ z &\longmapsto \frac{z+i}{z-i}. \end{aligned}$$

2 La théorie des cartes holomorphes

The theory of holomorphic maps. We now begin another part of complex function theory which is at the heart of a lot of sources of interest in it. We first consider the line integrals.

2.1 Line integrals

Let $\gamma : [a,b] \to \mathbb{C}$ be a continuous function. Suppose $G \subseteq \mathbb{C}$ is an open subset containing γ and it's interior and let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map $f : G \to \mathbb{C}$. Then, the *line integral* of f along γ is defined as

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

where definite integral of a complex valued function $g : [a,b] \to \mathbb{C}$ where $g = u + iv$ is given simply as the Riemann integral on each of the real and imaginary parts:

$$\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

A continuous map $\gamma : [a,b] \to \mathbb{C}$ is called piecewise C^1 if γ is C^1 at all but finitely many points and where it isn't differentiable, one sided derivative exists.

Few properties of line integrals are in order.

Theorem 2.1.1. *Let $\gamma : [a,b] \to \mathbb{C}$ be a curve in $\mathbb{C}$ and let $G \subseteq \mathbb{C}$ be an open subset containing γ. Let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map over G. Then,*

1. *(FTOC) If γ is piecewise C^1, then*

$$\int_a^b \gamma'(t)dt = \gamma(b) - \gamma(a).$$

2. *If $f \in \mathcal{C}^{\text{hol}}(G)$ where G contains γ, then*

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a)).$$

   So if γ is a closed loop, then integral of f' along it is 0.
3. *If $f \in \mathcal{C}^{\text{hol}}(G)$ and $\tilde{\gamma}$ is a reparametrization of γ, then $\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz$.*
4. *(Estimate) If $f \in \mathcal{C}^{\text{hol}}(G)$ and $M = \sup_{t\in[a,b]} |f(\gamma(t))|$, then*

$$\left|\int_\gamma f(z)dz\right| \leq ML(\gamma)$$

   where $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$ is the arc-length.

Proof. Assuming 1 by FTOC on each piece, all results follows from basic analysis. $\square$

2.2 Cauchy's theorem - I

We will now state the Cauchy's theorems on holomorphic maps and integrals. This will be a special case of the general version, which we shall do later, for we will find almost all of the traditional applications without needing that generality. We will begin with it's infantile version, which is quite simple to state now with line integrals in our pouch.

Theorem 2.2.1. *(Cauchy's theorem) Let $\gamma : [a, b] \to \mathbb{C}$ be a closed piecewise C^1 loop in $\mathbb{C}$ and let $G \subseteq \mathbb{C}$ be a convex open set containing γ and it's interior $\mathrm{Int}(\gamma)$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$, then*

$$\int_\gamma f(z)dz = 0.$$

Then there is the Cauchy integral formula.

Theorem 2.2.2. *(Cauchy's integral formula) Let C be a circle oriented in the counterclockwise manner and let $G \subseteq \mathbb{C}$ be an open set containing C and its interior $\mathrm{Int}(C)$. Then,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

for all $z \in \mathrm{Int}(C)$.

Remark 2.2.3. Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on open $G \subseteq \mathbb{C}$. The integral formula tells us that the value of f at $z \in G$ can be given in terms of line integral of f around a small enough circle C in the CCW orientation centered at z so that $C \subseteq G$. Hence the integral formula tells us that holomorphic maps are pretty much completely determined by taking their line integrals around circles.

We will provide some results which can be derived from them. In particular, using these, we would be able to show that a holomorphic map is analytic (Corollary 2.3.5).

2.2.1 Proof of Cauchy's theorem : Holomorphic maps have primitives

A *primitive* of a holomorphic map f is a holomorphic map g such that $g' = f$. We first state the following theorem without proof using which we will prove the Cauchy's theorem.

Theorem 2.2.4. *(Cauchy's triangle theorem) Let T be a triangle in $\mathbb{C}$ and $G \subseteq \mathbb{C}$ be an open set containing T and $\mathrm{Int}(T)$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$, then*

$$\int_T f(z)dz = 0.$$

Proof. [**??**] [Sarason]. □

Now, we will prove the following lemma using the above triangle theorem.

Lemma 2.2.5. *Let $G \subseteq \mathbb{C}$ be a convex open set and $f \in \mathcal{C}^{\mathrm{hol}}(G)$. Then there exists a map $g \in \mathcal{C}^{\mathrm{hol}}(G)$ such that $g' = f$.*

Proof. For a fixed $z_0 \in G$, define

$$\begin{aligned} g : G &\longrightarrow \mathbb{C} \\ z &\longmapsto \int_{[z_0,z]} f(z)dz \end{aligned}$$

where $[z_0, z]$ denotes the path formed by line joining z_0 and z in G. We claim that for all $z \in G$, $g'(z) = f(z)$. Indeed, pick any $z_1 \in G$ to form triangle $T = (z_0, z_1, z)$ inside G (G is convex). Then, by Theorem 2.2.4, we get the following

$$\begin{aligned} 0 &= \int_T f(w)dw \\ &= \int_{[z_0,z_1]} f(w)dw + \int_{[z_1,z]} f(w)dw + \int_{[z,z_0]} f(w)dw \\ g(z) - g(z_1) &= \int_{[z_1,z]} f(w)dw. \end{aligned}$$

We wish to estimate

$$\begin{aligned} \left| \frac{g(z) - g(z_1)}{z - z_1} - f(z_1) \right| &= \left| \frac{1}{z - z_1} \int_{[z_1,z]} f(w)dw - f(z_1) \right| \\ &= \left| \frac{1}{z - z_1} \int_{[z_1,z]} (f(w) - f(z_1))dw \right| \\ &\leq \frac{1}{|z - z_1|} \int_{[z_1,z]} |f(w) - f(z_1)|\, dw. \end{aligned}$$

Since f is continuous, therefore for any $\epsilon > 0$, there is a $\delta > 0$ such that $|w - z_1| < \delta$ implies $|f(w) - f(z_1)| < \epsilon$. Hence, for $|w - z_1| < \delta$, we get

$$\begin{aligned} &\leq \frac{1}{|z - z_1|} \int_{[z_1,z]} \epsilon dw \\ &= \epsilon. \end{aligned}$$

Hence as $z \to z_1$, the above difference $\to 0$. □

Proof of Theorem 2.2.1. Since $f \in \mathcal{C}^{\mathrm{hol}}(G)$, therefore by Lemma 2.2.5, there exists $g \in \mathcal{C}^{\mathrm{hol}}(G)$ such that $g' = f$. Hence the result follows by Theorem 2.1.1, 2. □

2.2.2 Proof of Cauchy's integral formula : Cauchy integrals

We would like to present the proof of Cauchy integral formula as it portrays how to use the fact that integral of holomorphic maps around closed loops are zero (Theorem 2.2.1).

Proof of Theorem 2.2.2. Pick any $z_0 \in \mathrm{Int}(C)$. We shall show the result for this chosen z_0. We shall use the Cauchy's theorem 2.2.1 in an essential manner. Indeed, consider the following figure on the complex plane inside G: Integrating the holomorphic map $\frac{f(w)}{w - z_0}$ over the each of the four regions will give zero by Theorem 2.2.1. However, summing them up, one can see that we get the difference $\int_C \frac{f(w)}{w-z_0}dw - \int_{C_\epsilon} \frac{f(w)}{w-z_0}dw$, which should thus be zero, yielding us $\int_C \frac{f(w)}{w-z_0}dw = \int_{C_\epsilon} \frac{f(w)}{w-z_0}dw$. Note

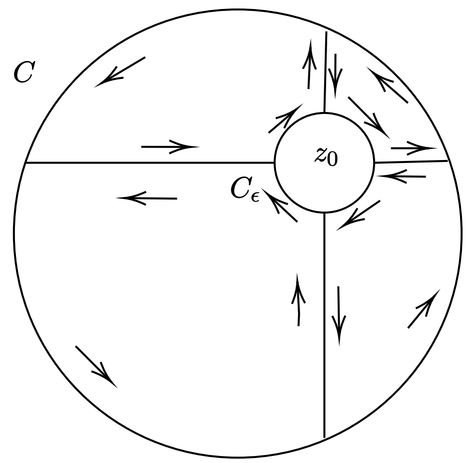

Figure 1: Contour over which to integrate $\frac{f(w)}{w-z_0}$.

this is true for all $\epsilon < d(z_0, C)$.

Now recall that $\int_C \frac{1}{z} dz = 2\pi i$. Hence, we get the following estimate for any chosen $\epsilon < d(z_0, C)$

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}dw - f(z_0)\right| &= \left|\frac{1}{2\pi i}\int_{C_\epsilon} \frac{f(w)}{w-z_0}dw - f(z_0)\right| \\ &= \left|\frac{1}{2\pi i}\int_{C_\epsilon} \frac{f(w)-f(z_0)}{w-z_0}dw\right|\end{aligned}$$

Now, by Theorem 2.1.1, 4, let $M_\epsilon = \sup_{w\in C_\epsilon}\left|\frac{f(w)-f(z_0)}{w-z_0}\right|$ to obtain the following inequality

$$\begin{aligned}&\leq \frac{M_\epsilon}{2\pi} L(C_\epsilon) \\ &= \frac{M_\epsilon}{2\pi} 2\pi\epsilon \\ &= \epsilon M_\epsilon.\end{aligned}$$

Since f is holomorphic, therefore $\varinjlim_{\epsilon\to 0} M_\epsilon = |f'(z_0)|$. Hence, $\varinjlim_{\epsilon\to 0} \epsilon M_\epsilon = 0$, which gives the desired result. □

2.3 Theory of holomorphic maps

We now present applications of the two highly useful results of Cauchy (Theorems 2.2.1, 2.2.2). The results covered here are as follows:

- Mean value property.
- Power series representation of Cauchy integrals.
- Morera's theorem.
- Derivatives.
- Liouville's theorem.

- Identity theorem.
- Maximum modulus theorem.
- Schwarz's lemma.
- Classification of bijective holomorphic maps of open unit ball.
- Open mapping theorem.
- Fundamental theorem of algebra.
- Inverse function theorem.
- Local m^{th} power property.
- Harmonic conjugates.

These results lie at the heart of complex analysis.

Let us begin by understanding the behavior of a holomorphic map around a circle centered at a point.

2.3.1 Mean value property of holomorphic maps

Proposition 2.3.1. *Let $G \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{C}^{\text{hol}}(G)$. Then, for all $z_0 \in G$ and C_r a circle of radius r centered at z_0 contained inside G together with its interior* $\text{Int}(C)$*, we have*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

Proof. Using the integral formula (Theorem 2.2.2) and using $\gamma(t) = z_0 + re^{it}$ as a parameterization of C_r, the result follows. $\square$

2.3.2 Power series representation of Cauchy integrals

We will in this section show that functions defined by Cauchy integrals are analytic. Since holomorphic maps are given by Cauchy integrals, thus we would be able to show that holomorphic maps are analytic.

Definition 2.3.2. (**Maps given by Cauchy integral**) Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise C^1 curve in $\mathbb{C}$ and $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on an open subset $G \subseteq \mathbb{C}$ where G contains $\text{Im}\,(\gamma)$. Define the following map

$$\begin{aligned} \tilde{f} : \mathbb{C} \setminus \text{Im}\,(\gamma) &\longrightarrow \mathbb{C} \\ z &\longmapsto \int_\gamma \frac{f(w)}{w - z} dw. \end{aligned}$$

Then $\tilde{f}$ is called the Cauchy integral associated to $f \in \mathcal{C}^{\text{hol}}(G)$ and $\gamma : [a, b] \to G$.

We first show that holomorphic maps are given by Cauchy integrals.

Lemma 2.3.3. *Let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on an open set $G \subseteq \mathbb{C}$. Then f is locally given by a Cauchy integral.*

Proof. Indeed, by Theorem 2.2.2, we see that for all $z \in G$, choosing a small circle C_z around z and such that C_z and $\mathrm{Int}(C_z)$ are inside G, we can write

$$f(z) = \frac{1}{2\pi i} \int_{C_z} \frac{f(w)}{w - z} dw.$$

Hence locally f looks like a Cauchy integral. □

We now show that Cauchy integrals are analytic, making holomorphic maps analytic by above lemma.

Proposition 2.3.4. *Maps defined by Cauchy integrals are analytic.*

Proof. Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ where G is open and let $\gamma : [a, b] \to G$ be a piecewise C^1 curve. We wish to show that $\tilde{f}$ defined on $\mathbb{C} \setminus \mathrm{Im}\,(\gamma)$ is given locally by power series. Indeed, pick any $z \in \mathbb{C} \setminus \mathrm{Im}\,(\gamma)$. Since $\mathrm{Im}\,(\gamma)$ is closed, therefore there exists a ball of radius r, B_r, such that $B_r \subseteq \mathbb{C} \setminus \mathrm{Im}\,(\gamma)$. In order to expand $\tilde{f}(z) = \int_\gamma \frac{f(w)}{w-z} dw$ as a power series, we first focus on $\frac{1}{w-z}$, where $w \in \mathrm{Im}\,(\gamma)$ and z is as above. Indeed, for any $z_0 \in B_r$, we have $|z - z_0| < r$ and $|w - z_0| > r$, thus yielding that $\left|\frac{z-z_0}{w-z_0}\right| < 1$ and hence we can write

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{(w - z_0)(1 - \frac{z-z_0}{w-z_0})} \\ &= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}. \end{aligned}$$

Moreover the convergence is uniform as we are within the radius of convergence. Now, $f(w) \leq M$ for all $w \in \mathrm{Im}\,(\gamma)$ as $\mathrm{Im}\,(\gamma)$ is compact and f is continuous over it. Hence we get that that following holds for all $w \in \mathrm{Im}\,(\gamma)$

$$\frac{f(w)}{w - z} = \sum_{n=0}^{\infty} \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}}.$$

Taking integral both sides, it thus follows from uniform convergence of above series that

$$\begin{aligned} \tilde{f}(z) = \int_\gamma \frac{f(w)}{w - z} dw &= \int_\gamma \sum_{n=0}^{\infty} \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}} dw \\ &= \sum_{n=0}^{\infty} \int_\gamma \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}} dw \\ &= \sum_{n=0}^{\infty} \left(\int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw \right) (z - z_0)^n. \end{aligned}$$

Hence locally $\tilde{f}$ looks like a power series, i.e. it is analytic. □

Corollary 2.3.5. *Holomorphic maps are analytic.*

Proof. By Lemma 2.3.3, holomorphic maps are given by Cauchy integrals. By Proposition 2.3.4, maps given by Cauchy integrals are analytic. □

2.3.3 Morera's theorem : Converse of Cauchy's triangle theorem

Proposition 2.3.6. *If $f : G \to \mathbb{C}$ is a continuous map on an open set $G \subseteq \mathbb{C}$ such that for all triangles $T \subseteq G$ where $\mathrm{Int}(T) \subseteq G$ as well we get*

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

Proof. □

2.3.4 Derivatives of a holomorphic map

Proposition 2.3.7. *Let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a holomorphic map on an open set $G \subseteq \mathbb{C}$. Then, f is differentiable to all orders and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(w)}{(w-z)^{n+1}} dw$$

where C_r is a circle in CCW orientation of radius r such that $C_r \subseteq G$ and $\mathrm{Int}(C_r) \subseteq G$. Moreover, for all $z \in \mathrm{Int}(C_r)$ with z_0 as center, we have that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \left(\int_{C_r} \frac{f(w)}{(w-z)^{n+1}} dw \right) (z-z_0)^n \end{aligned}$$

2.3.5 Liouville's theorem

A holomorphic map f on the entire complex plane, that is $f \in \mathcal{C}^{\mathrm{hol}}(\mathbb{C})$, is said to be *entire.*

Proposition 2.3.8. *Any entire bounded function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

2.3.6 Zeroes of holomorphic maps

Proposition 2.3.9. *Let $G \subseteq \mathbb{C}$ be an open connected subset of $\mathbb{C}$. If $f \in \mathcal{C}^{\mathrm{hol}}(G)$ is a holomorphic map on G, then the zero set $V(f) = \{z \in G \mid f(z) = 0\}$ has no limit point in G i.e. either $V(f) = G$ or $V(f)$ is discrete.*

2.3.7 Identity theorem

Proposition 2.3.10. *Let $f, g \in \mathcal{C}^{\mathrm{hol}}(G)$ be two holomorphic maps defined on an open connected set $G \subseteq \mathbb{C}$. Then $f = g$ on G if and only if there exists a set $A \subseteq G$ which has a limit point in G such that $f|_A = g|_A$.*

Corollary 2.3.11. *Let f, g be two holomorphic maps on open connected subset $G \subseteq \mathbb{C}$ such that there exists an open set $U \subsetneq G$ contained inside of G such that $\partial U \neq \emptyset$ and $\bar{U} \subseteq G$ and $f|_U = g|_U$. Then $f = g$ on G.*

Proof. Indeed, since any element in ∂U is a limit point of U in G and $f|_U = g|_U$, then the result follows by Proposition 2.3.10. $\square$

Corollary 2.3.12. *Let f, g be two holomorphic maps on open connected subset $G \subseteq \mathbb{C}$ such that there exists a closed ball $B \subset G$ on which $f|_B = g|_B$, then $f = g$ on G.*

Proof. A closed ball has non-empty interior. The result follows by Corollary 2.3.11. $\square$

2.3.8 Maximum modulus principle

Proposition 2.3.13. *Let $G \subseteq \mathbb{C}$ be an open connected set and $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on G. Then $|f|$ doesn't achieves local maxima in G.*

2.3.9 Schwarz's lemma

Lemma 2.3.14. *Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc. If $f \in \mathcal{C}^{\text{hol}}(D)$ is a holomorphic map $f : D \to D$ such that $f(0) = 0$, then*

1. *$|f(z)| \leq |z|$ for all $z \in D$.*
2. *$|f'(0)| \leq 1$.*
3. *If f is not of the form λz for $\lambda \in S^1$, then the inequality in 1. & 2. is strict at all points $z \in D \setminus \{0\}$. In particular, if there exists $z_0 \in D \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, then $f(z) = \lambda z$ for $|\lambda| = 1$ and $\lambda = f'(0)$.*

Proof. Consider the map defined by

$$\begin{aligned} g : D &\longrightarrow \mathbb{C} \\ z &\longmapsto \begin{cases} \frac{f(z)}{z} & \text{if } z \in D \setminus \{0\} \\ f'(0) & \text{if } z = 0. \end{cases} \end{aligned}$$

Clearly g is holomorphic. Now, for any $r \in (0, 1)$, for $C_r \subset D$, by maximum modulus, Proposition 2.3.13, we have

$$|g(z)| < \frac{1}{r}$$

for all $z \in \text{Int}(C_r)$. Taking limit as $r \to 1$, we obtain $|g(z)| \leq 1$ for all $z \in D$. Now, if $\exists w \in D$ such that $|f(w)| = |w|$, then $|g(w)| = 1$. Since $|g(z)| < 1$ for all $z \in D$ as shown above, therefore by another use of maximum modulus, Proposition 2.3.13, it follows that $g(z) = \lambda$ is a constant where $|\lambda| = 1$. Thus $f(z) = \lambda z$. $\square$

Corollary 2.3.15. *(Pick's lemma) Let $f : D \to D$ be a holomorphic map where $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Then, for any two points $z, w \in D$*

$$\left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \leq \left| \frac{z - w}{1 - z\bar{w}} \right|$$

except if f is a linear fractional transform mapping disc onto itself.

Proof. Define for each $w \in D$ the following fractional transform

$$\begin{aligned} g_w : D &\longrightarrow D \\ z &\longmapsto \frac{z-w}{1-z\bar{w}}. \end{aligned}$$

Then apply Schwarz's lemma (Lemma 2.3.14) on $g_{f(w)} \circ f \circ g_w^{-1} : D \to D$ as fractional transforms are biholomorphic. □

2.3.10 Classification of bijective holomorphic maps of open unit ball

We shall classify all bijective holomorphic maps $f : D \to D$ for $D := \{z \in \mathbb{C} \mid |z| < 1\}$ and see that in the process that they are biholomorphic as well. For this, we first define the following important map which we encountered in Pick's lemma (Corollary 2.3.15). Define the following map for each $\alpha \in D$:

$$\begin{aligned} \varphi_\alpha : \bar{D} &\longrightarrow \bar{D} \\ z &\longmapsto \frac{z-\alpha}{1-\bar{\alpha}z}. \end{aligned}$$

This is indeed a holomorphic map over $\bar{D}$. We now see that this is biholomorphic.

Theorem 2.3.16. *For any $\alpha \in D$, the map $\varphi_\alpha : \bar{D} \to \bar{D}$ is such that*

1. *φ_α takes D to D,*
2. *φ_α takes ∂D to ∂D,*
3. *φ_α is injective,*
4. *φ_α is surjective,*
5. *φ_α has a holomorphic inverse given by $\varphi_{-\alpha}$.*

Proof. Fix an $\alpha \in D$. We first show 2. For any $z \in \partial D$, we can write $z = e^{it}$ for $t \in \mathbb{R}$. Thus we have

$$\begin{aligned} \left|\varphi_\alpha(e^{it})\right| &= \left|\frac{e^{it}-\alpha}{1-\bar{\alpha}e^{it}}\right| \\ &= \left|\frac{e^{it}-\alpha}{1-\bar{\bar{\alpha}}\overline{e^{it}}}\right| \\ &= \left|\frac{e^{it}-\alpha}{1-\alpha e^{-it}}\right| \\ &= \left|\frac{e^{it}-\alpha}{e^{it}-\alpha}\right| \\ &= 1. \end{aligned}$$

Thus, $\varphi_\alpha(e^{it}) \in \partial D$. This shows 2. Now we show 1. For this, by maximum modulus (Proposition 2.3.13), we have that $|\varphi_\alpha|$ achieves maxima on ∂D, and by 1., that maxima is 1, hence at every point of ∂D does $|\varphi_\alpha|$ achieves maxima. Hence $\varphi_\alpha(D) \subseteq D$. This shows 1. Next, it is a matter of simple calculation to see that $\varphi_\alpha \circ \varphi_{-\alpha} = \mathrm{id}_{\bar{D}}$ and thus by symmetry $\mathrm{id}_{\bar{D}} = \varphi_{-\alpha} \circ \varphi_\alpha$. Hence, φ_α is a biholomorphic map taking D onto D and ∂D onto ∂D. □

We would now like to see that all biholomorphic maps of open unit ball are given by some unit modulus scalar multiples of φ_α. However, we need an idea to do so, which is provided by the following result.

Proposition 2.3.17. *(Extremality) For fixed $\alpha, \beta \in D$, denote $\mathcal{C}_{\alpha,\beta}$ to be the class of holomorphic maps into the unit disc $f : D \to D$ such that $f(\alpha) = \beta$. Then,*

1. *we have*

$$\sup_{f \in \mathcal{C}_{\alpha,\beta}} |f'(\alpha)| = \frac{1 - |\beta|^2}{1 - |\alpha|^2}.$$

2. *The map $f \in \mathcal{C}_{\alpha,\beta}$ achieving the suprema is given by the following rational map*

$$f = \varphi_{-\beta} \circ \lambda \circ \varphi_\alpha$$

where $\lambda \in \partial D$ is a scalar.

Proof. 1. We need only show that for each $f \in \mathcal{C}_{\alpha,\beta}$, we get

$$|f'(\alpha)| \leq \frac{1 - |\beta|^2}{1 - |\alpha|^2}.$$

Indeed, this simply follows from a similar idea as used in the proof Pick's lemma (Corollary 2.3.15) above; consider the map $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$ and use Schwarz's lemma (Lemma 2.3.14) on it to get the bound $|g'(0)| \leq 1$. Now use chain rule while keeping in mind that $\varphi'(0) = 1 - |\alpha|^2$ and $\varphi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$.

2. From proof of 1, it follows that the equality is achieved if and only if $|g'(0)| = 1$. By Schwarz's lemma (Lemma 2.3.14) this happens only if $g(z) = \lambda z$ for $\lambda \in \partial D$. Rest follows by composing with inverses of φ_β and $\varphi_{-\alpha}$ which we know from Theorem 2.3.16, 5. □

We now come to the real deal. The following shows that all bijective holomorphic maps $D \to D$ are biholomorphic and are given by unit modulus scalar multiples of φ_α for some $\alpha \in D$. However we shall need a topic which we will cover in the next few sections, namely the inverse function theorem for one complex variable (see Section **??**, Theorem **??**). Moreover we shall also need another result which we do only in a further section called Rouché's theorem (Section **??**, Theorem **??**).

Theorem 2.3.18. *(Bijective holomorphic maps $D \to D$) Let $f : D \to D$ be a bijective holomorphic map. Denote $\alpha \in D$ to be the unique element such that $f(\alpha) = 0$. Then, there exists $\lambda \in \partial D$ such that*

$$f = \lambda \varphi_\alpha.$$

Proof. Consider the set-theoretic inverse of f, denoted $g : D \to D$. By Rouché's theorem (Theorem **??**) and by inverse function theorem (Theorem **??**), we obtain that $g \in \mathcal{C}^{\text{hol}}(D)$. Now by chain rule we obtain $g'(f(\alpha))f'(\alpha) = 1$, that is, $g'(0) = 1/f'(\alpha)$. Now by Proposition 2.3.17, we obtain the following inequality for f and g where $f(\alpha) = 0$ and $g(0) = \alpha$:

$$|f(z)| \leq \frac{1}{1-|\alpha|^2}$$
$$|g(z)| \leq 1-|\alpha|^2 .$$

In particular, we obtain that $1-|\alpha|^2 \geq g'(0) = 1/f'(\alpha) \geq 1-|\alpha|^2$, thus $g'(0) = 1-|\alpha|^2$. Similarly, $|f'(\alpha)| = \frac{1}{1-|\alpha|^2}$. Hence f achieves the suprema of Proposition 2.3.17, 1. By Proposition 2.3.17, the result follows. □

Corollary 2.3.19. *There is a bijection*

$$\left\{\textit{Bijective holomorphic maps } f : D \to D\right\}$$
$$\cong$$
$$\left\{\textit{Rational functions of the form } \lambda\tfrac{z-\alpha}{1-\bar{\alpha}z},\ \alpha \in D,\ \lambda \in \partial D\right\}.$$

□

Using this and Schwarz's lemma, we can show that a holomorphic map $f : D \to D$ can have atmost one fixed point.

Corollary 2.3.20. *Let $f : D \to D$ be a holomorphic map. Then f has atmost one fixed point.*

Proof. The idea is quite simple and we have used it already in the proof of Pick's lemma (Corollary 2.3.15). Indeed, we will construct $\varphi_\alpha : D \to D$ in such a manner that Schwarz's lemma can be applied to $\varphi \circ f \circ \varphi^{-1}$ and will use the results about the function φ_α (Theorem 2.3.16).

Suppose $z_1 \neq z_2 \in D$ are two fixed points of f. Consider the map $\varphi_{z_1}(z) := \frac{z-z_1}{1-\bar{z}_1 z}$. This is a biholomorphic mapping $\varphi_{-z_1} : D \to D$. Consider

$$h = \varphi_{z_1} \circ f \circ \varphi_{-z_1}.$$

Then $h : D \to D$ is a holomorphic map and $h(0) = 0$. Applying Schwarz's lemma (Lemma 2.3.14), we obtain that $|h(z)| \leq |z|$. But notice that $h(\varphi_{z_1}(z_2)) = \varphi_{z_1}(z_2)$. Thus $\varphi_{z_1}(z_2)$ is a fixed point of h. Moreover, $\varphi_{z_1}(z_2) \neq 0$ as other wise $z_2 = z_1$, a contradiction. Thus, there exists $w \in D$ such that $|h(w)| = |w|$ (in particular, for $w = \varphi_{z_1}(z_2)$). Hence by contrapositive of Lemma 2.3.14, 3, we obtain that $h(z) = \lambda z$. Since $h(w) = w = \lambda w$, we obtain that $\lambda = 1$. Hence $h = \mathrm{id}$, thus $f = \mathrm{id}$. □

2.3.11 Open mapping theorem

This theorem is quite an important result in the theory of holomorphic maps. It says a very simple thing, *all holomorphic maps on open connected sets are open maps*(!)

Theorem 2.3.21. *Let $G \subseteq \mathbb{C}$ be an open connected subset and let $f \in \mathcal{C}^{\mathrm{hol}}(G)$ be a non-constant holomorphic map $f : G \to \mathbb{C}$. Then f is an open map.*

2.3.12 Fundamental theorem of algebra

Proposition 2.3.22. *Every non-constant polynomial $f(x) \in \mathbb{C}[x]$ can be factored into linear factors.*

Proof. Suppose $f(x) \in \mathbb{C}[x]$ is a non-constant polynomial given by

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Suppose to the contrary that f has no zeros in $\mathbb{C}$. Then $g(x) = \frac{1}{f(x)} : \mathbb{C} \to \mathbb{C}$ is an entire map. We wish to use Liouville's theorem (Proposition 2.3.8) on $g(x)$ in order to obtain a contradiction. Indeed, to get an upper bound for $|g(x)|$, we need a lower bound for $|f(x)|$. To this end we have

$$\begin{aligned}
|f(x)| &\geq |a_n x^n + \cdots + a_1 x + a_0| \\
&\geq |a_n x^n| \left| \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right) \right| \\
&\geq |a_n x^n| \left(1 - \left|\frac{a_{n-1}}{a_n x}\right| - \cdots - \left|\frac{a_1}{a_n x^{n-1}}\right| - \left|\frac{a_0}{a_n x^n}\right|\right)
\end{aligned}$$

where the last inequality comes from triangle inequality. Now write $h(x) = 1 - \left|\frac{a_{n-1}}{a_n x}\right| - \cdots - \left|\frac{a_1}{a_n x^{n-1}}\right| - \left|\frac{a_0}{a_n x^n}\right|$. In order to get a further lower bound for $|f(x)|$, we need to get an upper bound for $h(x)$. Since $h(x) \to 0$ as $x \to \infty$, therefore for some $R > 0$, we shall have $h(x) \leq \frac{1}{3}$ for $|x| > R$. Thus, we get

$$|f(x)| \geq |a_n R^n| \frac{2}{3}$$

for $|x| > R$. Now on $|x| \leq R$, by continuity of $|f|$ on a compact domain, we get that it achieves a minima, and hence $|f|$ is a lower bounded map and hence $g(x)$ is an upper bounded map. $\square$

2.3.13 Inverse function theorem

Remember that for a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$, if $x_0 \in \mathbb{R}^n$ is a point such that Df_{x_0} is invertible, the inverse function theorem tells us that f is a diffeomorphism in some neighborhood around x_0. A similar statement is true for holomorphic maps $f : G \subseteq \mathbb{C} \to \mathbb{C}$.

Theorem 2.3.23. *(Inverse function theorem) Let $G \subseteq \mathbb{C}$ be an open connected set and $\varphi \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on G. If for $z_0 \in G$ we have that $f'(z_0) \neq 0$, then there exists a neighborhood $z_0 \in V \subseteq G$ such that*

1. *$\varphi|_V : V \to \varphi(V)$ is bijective,*
2. *$\varphi(V) \subseteq G$ is open,*
3. *the map $\psi : \varphi(V) \to V$ given by $\varphi(z) \mapsto z$ is in $\mathcal{C}^{\text{hol}}(\varphi(V))$,*
4. *the map $\varphi|_V : V \to \varphi(V)$ is biholomorphic.*

We will now prove it. Let us begin with the following simple lemma.

Lemma 2.3.24. *Let $G \subseteq \mathbb{C}$ be an open connected set. If $f : G \to \mathbb{C}$ is a holomorphic map, then the map defined by*

$$g : G \times G \longrightarrow \mathbb{C}$$
$$(z, w) \longmapsto \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w, \\ f'(z) & \text{if } z = w \end{cases}$$

is continuous.

Proof. Clearly g is continuous for all (z, w) with $z \neq w$. Pick any $a \in G$. We will show that g is continuous at (a, a). For that, we wish to estimate $|g(z, w) - g(a, a)|$. For this, note that we can write $g(z, w)$ as follows where γ is the straight path $\gamma(t) = (1 - t)z + tw$:

$$\begin{aligned} g(z, w) &= \frac{f(z) - f(w)}{z - w} \\ &= \frac{f(\gamma(0)) - f(\gamma(1))}{\gamma(0) - \gamma(1)} \\ &= \frac{1}{w - z} \int_\gamma f'(z) dz \\ &= \int_0^1 f'(\gamma(t)) dt \end{aligned}$$

where the third equality follows from Theorem 2.1.1, 2. Thus we can write

$$\begin{aligned} |g(z, w) - g(a, a)| &= \left| \int_0^1 f'(\gamma(t)) dt - f'(a) \right| \\ &= \left| \int_0^1 f'(\gamma(t)) - f'(a) dt \right| \\ &\leq \int_0^1 \left| f'(\gamma(t)) - f'(a) \right| dt. \end{aligned}$$

Now by continuity of f', the estimate follows. □

We can now prove the inverse function theorem.

Proof of Theorem 2.3.23. 1. The surjectivity is clear. For injectivity, we will show that for two $z_1 \neq z_2 \in V$, $|\varphi(z_1) - \varphi(z_2)| \geq M$ for some $M > 0$ using the lemma just proved. Indeed, using Lemma 2.3.24 and triangle inequality, we obtain for $\epsilon = \frac{1}{2} |\varphi'(z_0)|$ an open set $\tilde{V}$ containing (z_0, z_0) such that for all $(z_1, z_2) \in \tilde{V}$ with $z_1 \neq z_2$ we get the following

$$\left| \left| \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} \right| - |\varphi'(z_0)| \right| \leq \left| \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} - \varphi'(z_0) \right| < \epsilon = \frac{1}{2} |\varphi'(z_0)| .$$

Using this, we obtain that

$$|\varphi(z_1) - \varphi(z_2)| \geq \frac{1}{2} |\varphi'(z_0)| |z_1 - z_2| .$$

Thus for $z_1, z_2 \in V \subseteq G$ where V is obtained by projecting a small open set inside $\tilde{V}$ back to G, we see that on that V φ is injective.

2. This is just open mapping theorem, Theorem 2.3.21.

3. Shrink V enough so that $\varphi'(z) \neq 0$ for all $z \in V$. Then everything is straightforward using

$$|\varphi(z_1) - \varphi(z_2)| \geq \frac{1}{2} \left|\varphi'(z_0)\right| |z_1 - z_2|$$

which we obtained in 1. □

2.3.14 Local m^{th} power property

Any holomorphic map around a point can be represented by the m^{th} power of some other special holomorphic map. Indeed, this is what the following theorem tells us.

Theorem 2.3.25. *Let $G \subseteq \mathbb{C}$ be an open-connected subset of $\mathbb{C}$ and let $f \in \mathcal{C}^{\text{hol}}(G)$ be a holomorphic map on G. Let $z_0 \in G$ and denote $w_0 = f(z_0)$. Let m be the order of zero that $f - w_0$ has at z_0. Then, there exists an open set $z_0 \in V \subseteq G$ and a holomorphic map*

$$\varphi : V \to \mathbb{C}$$

in $\mathcal{C}^{\text{hol}}(G)$ such that

1. *$f(z) = w_0 + (\varphi(z))^m$ for all $z \in V$,*
2. *φ' is nowhere vanishing in V, i.e. has no zero in V,*
3. *there exists $r > 0$ such that φ is biholomorphic onto $D_r(0)$, the open disc of radius r around 0. Thus, $\varphi : V \to D_r(0)$ is bijective.*

Proof. The main point of the proof is to try to represent the desired φ as $\exp \frac{??}{m}$. We just need to fill ?? correctly. Since $f - w_0$ has zero of order m at z_0, therefore there exists $g \in \mathcal{C}^{\text{hol}}(G)$ such that

$$f(z) - w_0 = (z - z_0)^m g(z).$$

Now, by appropriately shrinking G away from zeros of g, we may assume $g \neq 0 \forall z \in G \setminus \{z_0\}$[1]. Thus we have that $\frac{g'}{g}$ is holomorphic on G (this is our V). By Lemma 2.2.5, we get $h \in \mathcal{C}^{\text{hol}}(G)$ such that $h' = \frac{g'}{g}$. We now claim that $g = \exp h$. Indeed, it is a simple matter to see that the derivative of $g \exp -h$ is zero. Thus, by using surjectivity of exp, we can absorb the additive constant into h to obtain the above claim. One then sees that

$$\varphi(z) = (z - z_0) \exp \frac{h(z)}{m}$$

does the job for 1. The rest is straightforward. □

[1] We are implicitly using the isolated zeros theorem (Theorem **??**) which we shall do later.

2.3.15 Harmonic conjugates

We will now show that any real valued harmonic map $u : G \subseteq \mathbb{R}^2 \to \mathbb{R}$ defines a unique (upto some constant) holomorphic map $g : G \to \mathbb{C}$ whose real part is u.

Theorem 2.3.26. *Let $G \subseteq \mathbb{C}$ be a convex open connected set. Let $u : G \to \mathbb{R}$ be a harmonic real valued function. Then, there exists holomorphic map $g : G \to \mathbb{C}$ unique upto an additive constant such that*

$$\Re g = u.$$

Proof. The main idea is to construct a holomorphic map f on G via the data of partial derivatives of u, and then use the Lemma 2.2.5, to get a primitive g, which will do the job. Indeed, we can make f via the following observation: u is harmonic real valued function if and only if $\frac{\partial^2}{\partial \bar{z} \partial z} = 0$. Using this, just define $f = u_x - iu_y$ and to show that f is holomorphic, observe that $\frac{\partial f}{\partial \bar{z}} = 0$. □

In combination with Lemma 1.3.2, we get that

Corollary 2.3.27. *Let $G \subseteq \mathbb{C}$ be open connected. Then,*

$$\{g : G \to \mathbb{C} \text{ is holomorphic}\} \cong \{u : G \to \mathbb{R} \text{ is harmonic}\}$$

where we identify functions upto additive constant. □

3 Singularities

Consider the map $f(z) = 1/z$ on $\mathbb{C}^\times$. It is holomorphic. However, at $z = 0$, it is not holomorphic. Such points are called singularities of f, as we shall define more clearly later. Our goal is to study this phenomenon more carefully in this section. For this, we first need to develop a tool for local analysis of such "bad" points (some may also call it "the" points).

3.1 Laurent series

Definition 3.1.1. (**Laurent series**) A Laurent series centered at $z_0 \in \mathbb{C}$, denoted by $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, is a series of functions defined on some annulus $A_{z_0}(R_1, R_2) := \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$ centered at z_0 for $0 \leq R_1 < R_2$ such that the series converges at all points $z \in A_{z_0}(R_1, R_2)$. That is, the sequence of holomorphic maps $\{\sum_{n=-N}^{n=N} a_n(z - z_0)^n\}_N$ on $A_{z_0}(R_1, R_2)$ converges uniformly and absolutely to a holomorphic function $f : A_{z_0}(R_1, R_2) \to \mathbb{C}$ (by Weierstrass theorem).

For a Laurent series, we can find the coefficents in terms of Cauchy integral of the function it represents.

Lemma 3.1.2. *Let $f(z) = \sum_{n=-\infty}^{n=\infty} a_n(z - z_0)^n$ be a Laurent series around $z_0 \in \mathbb{C}$ in an annulus. Then for all $n \in \mathbb{Z}$*

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

where $R_1 < r < R_2$.

Proof. Use the uniform convergence of the Laurent series on $\frac{f(z)}{(z-z_0)^{n+1}}$ (so to limits out of integrals) and the fact that $\int_{C_r}(z-z_0)^n dz = 2\pi i$. ☐

By Cauchy-Hadamard theorem for calculation of radius of convergence we also get the parameters for the maximum annulus on which a Laurent series can exist.

Lemma 3.1.3. *For a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, the smallest value of R_1 and largest value of R_2 such that $f(z)$ converges on $A_{z_0}(R_1, R_2)$ is given by*

1. $R_1 = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}}$
2. $R_2 = \frac{1}{\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}}$

Proof. Straightforward use of Cauchy-Hadamard. ☐

The following is the main theorem here.

Theorem 3.1.4. *Consider any $0 < R_1 < R_2$ and any $z_0 \in \mathbb{C}$. If $f : A_{z_0}(R_1, R_2) \to \mathbb{C}$ is holomorphic, then it is represented by a Laurent series.*

3.2 Isolated singularities : Removable, poles and essential

We now come to the main matter of the present study, the notion of singularities. A holomorphic function $f : G \to \mathbb{C}$ is said to have an *isolated singularity* at $z_0 \notin G$ if there exists a punctured disc $A_{z_0}(0, r) \hookrightarrow G$. Consequently, by Theorem 3.1.4, we obtain a Laurent series expansion of f in $A_{z_0}(0, r)$. Let us denote it by

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n.$$

We can then classify the isolated singularity z_0 into three types:

1. z_0 is a *removable singularity* if $a_n = 0$ for all $n < 0$,
2. z_0 is a *pole of order m* if $\min\{n < 0 \mid a_n \neq 0\} = m$,
3. z_0 is an *essential singularity* if $\min\{n < 0 \mid a_n \neq 0\} = -\infty$ or unbounded.

There are three characterizing theorems of each of the three kinds of singularities.

Theorem 3.2.1. *(Riemann's extension theorem) Let $f : G \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*

1. *The point $z_0 \in \mathbb{C} \setminus G$ is a removable singularity of f.*
2. *There exists a punctured disc $A_{z_0}(0, r) \hookrightarrow G$ such that f is bounded on it.*

Theorem 3.2.2. *(Criterion for a pole) Let $f : G \to \mathbb{C}$ be a holomorphic map. Then the following are equivalent.*

1. *The point $z_0 \in \mathbb{C} \setminus G$ is a pole of f of some order.*
2. *We have*

$$\varinjlim_{z\to z_0} |f(z)| = \infty.$$

Theorem 3.2.3. *(Casorati-Weierstrauss theorem) Let $f : G \to \mathbb{C}$ be a holomorphic map. If the point $z_0 \in \mathbb{C} \setminus G$ is an essential singularity of f, then there exists a punctured disc $A_{z_0}(0, r) \hookrightarrow G$ such that $f(A_{z_0}(0, r))$ is dense in $\mathbb{C}$.*

The last theorem in particular shows the chaotic behaviour of essential singularities.

4 Cauchy's theorem - II

Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}\,(\gamma)$. We define the *index of* γ to be the following map over Ω:

$$\begin{aligned} \mathrm{Ind}_\gamma(z) : \Omega &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}dw. \end{aligned}$$

Lemma 4.0.1. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}\,(\gamma)$. Then $\mathrm{Ind}_\gamma(z)$ is a holomorphic map on Ω.*

Proof. This follows from Proposition 2.3.4, as $\mathrm{Ind}_\gamma(z)$ is the Cauchy integral of the constant function 1. □

The following is the main theorem that we shall use.

Theorem 4.0.2. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1 curve in $\mathbb{C}$ and let $\Omega = \mathbb{C}^\times \setminus \mathrm{Im}\,(\gamma)$. Then,*

1. *$\mathrm{Ind}_\gamma(z)$ is an integer valued map,*
2. *$\mathrm{Ind}_\gamma(z)$ is constant on each connected component of Ω,*
3. *$\mathrm{Ind}_\gamma(z)$ is 0 on unbounded component of Ω.*

We now introduce the main Cauchy's theorem.

4.1 General Cauchy's theorem

To state the Cauchy's theorem in full generality, we first need to build the small language of chains, which is just a slight generalization of curves. Let $\{\gamma_i : I_i \to \mathbb{C}\}_{i=1}^n$ be a finite collection of piecewise C^1 curves over $\mathbb{C}$. A *chain generated by* $\{\gamma_i\}$ is a formal sum of the form

$$\Gamma = \gamma_1 + \cdots + \gamma_n.$$

One can be more precise here by treating Γ as an element of the free abelian group of all singular 1-chains, but we don't need that technology right now. We denote

$$\mathrm{Im}\,(\Gamma) := \bigcup_{i=1}^n \mathrm{Im}\,(\gamma_i).$$

Moreover, for a continuous map $f : \mathrm{Im}\,(\Gamma) \to \mathbb{C}$, we further denote

$$\int_\Gamma f(z)dz := \sum_{i=1}^n \int_{\gamma_i} f(z)dz.$$

We can further define the *index of a chain* Γ as simply the sum of indices of individual curves:

$$\mathrm{Ind}_\Gamma(z) := \sum_{i=1}^n \mathrm{Ind}_{\gamma_i}(z)$$

for all $z \in \Omega$, where $\Omega = \mathbb{C}^{\times} \setminus \mathrm{Im}(\Gamma)$. Note that the set Ω here will have multiple components if each element of the cycle is a distinct loop. Indeed, if $\Gamma = \gamma_1 + \cdots + \gamma_n$ is a cycle where each γ_i is a closed loop, then we call Γ a *cycle*. The general Cauchy's theorem is then a statement about integral over cycles.

Theorem 4.1.1. *(Cauchy's theorem) Let $\Omega \subseteq \mathbb{C}$ be an open set and $\Gamma \hookrightarrow \Omega$ be a cycle such that*

$$\mathrm{Ind}_{\Gamma}(z) = 0 \ \ \forall z \in \mathbb{C}^{\times} \setminus \Omega.$$

Let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then,

1. *(Integral formula)*

$$\mathrm{Ind}_{\Gamma}(z) f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw$$

   for all $z \in \Omega \setminus \mathrm{Im}(\Gamma)$.
2. *(Integral theorem)*

$$\int_{\Gamma} f(z) dz = 0,$$

3. *if $\Gamma_0, \Gamma_1 \hookrightarrow \Omega$ are two cycles such that $\mathrm{Ind}_{\Gamma_0}(z) = \mathrm{Ind}_{\Gamma_1}(z)$ for all $z \notin \Omega$, then*

$$\int_{\Gamma_0} f(z) dz = \int_{\Gamma_1} f(z) dz.$$

The most important of the above triad of conclusions is the first one, which clearly generalizes the known integral formula.

4.2 Homotopy & Cauchy's theorem

Theorem 4.2.1. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set. If $\gamma_0, \gamma_1 \hookrightarrow \Omega$ are two piecewise C^1 closed loops in Ω such that they are homotopic in Ω, then*

$$\mathrm{Ind}_{\gamma_0}(z) = \mathrm{Ind}_{\gamma_1}(z) \ \ \forall z \notin \Omega.$$

This has some major corollaries in combination with Theorem 4.1.1.

Corollary 4.2.2. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. If $\gamma_0, \gamma_1 \hookrightarrow \Omega$ are two piecewise C^1 closed loops in Ω such that they are homotopic in Ω, then*

$$\int_{\gamma_0} f(z) dz = \int_{\gamma_1} f(z) dz.$$

Corollary 4.2.3. *Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. If $\gamma \hookrightarrow \Omega$ is a piecewise C^1 closed loop in Ω and Ω is simply connected, then*

$$\int_{\gamma} f(z) dz = 0.$$

5 Residues and meromorphic maps

Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and $f : \Omega \to \mathbb{C}$ be holomorphic. Let $z_0 \notin \Omega$ be a point of isolated singularity of f. The *residue of f at z_0* is then defined to be the coefficient a_{-1} of the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

of the map f around z_0. We denote residue of f at z_0 by $\text{res}_{z_0}(f) := a_{-1}$. For example, consider the following integral where C_r is a circle of radius r centered at z_0

$$\int_{C_r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Since all terms $a_n(z-z_0)^n$ are $n \neq -1$ contributes zero integral as the positive parts of holomorphic in the interior of the loop and the negative parts are derivatives of constant 1, which is zero, therefore the only non-zero term is contributed by $n = -1$. Consequently, we have

$$\begin{aligned} \int_{C_r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n &= 2\pi i a_{-1} \\ &= 2\pi i \text{res}_{z_0}(f) \end{aligned}$$

where $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

We now define a class of holomorphic maps which one encounters often in complex analysis.

Definition 5.0.1. (**Meromorphic maps**) Let $\Omega \subseteq \mathbb{C}$ be an open-connected set and $f : \Omega \to \mathbb{C}$ be any function. We say that f is meromorphic if

1. there exists a set $A \subset \Omega$ which has no limit points in Ω,
2. $f : \Omega \setminus A \to \mathbb{C}$ is holomorphic,
3. every point of A is a pole of f.

One often calls the set A as the *set of poles of f*.

There are some observations to be made.

Lemma 5.0.2. *Let $f : \Omega \to \mathbb{C}$ be a meromorphic map on an open-connected set Ω. Then, the set of poles of f is atmost countable.*

Proof. Let $A \subset \Omega$ be the set of poles of f. Covering Ω by countably many compact sets $\{K_i\}$, we observe that intersection of each of $K_i \cap A$ has to be atmost finite, otherwise there exists a sequence in $K_i \cap A$, which consequently admits a convergent subsequence, that is, a limit point in Ω. Consequently, A is a countable union of finite sets. $\square$

Remark 5.0.3. For the purposes of residue of f at $a \in A$, one can replace analysis of f with analysis of f by the analysis of $Q = \sum_{n=-m}^{-1} a_n (z - a)^n$, called the principal part of f at a where m is the order of pole of f at $a \in A$. Clearly, $\text{res}_a Q = \text{res}_a f$. Moreover, one sees that

$$\text{res}_a(f)\text{Ind}_\gamma(a) = \frac{1}{2\pi i} \int_\gamma Q(z) dz$$

where γ is a piecewise C^1-loop centered at a, in $\Omega \setminus A$. This is again a consequence of the fact that all terms inside the integral are zero except the one corresponding to a_{-1}. Indeed, this hints at a general phenomenon, which is clarified by the following theorem.

Theorem 5.0.4. *(The residue theorem) Let $\Omega \subseteq \mathbb{C}$ be an open-connected set. If $f : \Omega \to \mathbb{C}$ is a meromorphic map with $A \subseteq \Omega$ its set of poles and Γ a cycle in $\Omega \setminus A$ such that*

$$\mathrm{Ind}_\Gamma(z) = 0 \ \ \forall z \notin \Omega,$$

then

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{a\in A} \mathrm{res}_a(f)\mathrm{Ind}_\Gamma(a).$$

We now an important result, which gives us information of zeroes of holomorphic maps on certain subsets.

Theorem 5.0.5. *Let $\gamma : I \to \mathbb{C}$ be a piecewise closed C^1-loop in an open-connected set $\Omega \subseteq \mathbb{C}$ such that*

1. $\mathrm{Ind}_\gamma(z) = 0$ *for all* $z \notin \Omega$,
2. $\mathrm{Ind}_\gamma(z) = 0$ *or 1 for all* $z \in \Omega \setminus \mathrm{Im}\,(\gamma)$.

Then we have that for any holomorphic maps $f, g : \Omega \to \mathbb{C}$, denoting $\Omega_1 := \{z \in \Omega\setminus\mathrm{Im}\,(\gamma) \mid \mathrm{Ind}_\gamma(z) = 1\}$ and $N_f = \#Z(f) \cap \Omega_1$, we get that

1. *if f has no zeros on $\mathrm{Im}\,(\gamma) \subseteq \Omega$, then*

$$N_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \mathrm{Ind}_{f\circ\gamma}(0).$$

2. *(Rouché's theorem) if*

$$|f(z) - g(z)| < |g(z)| \ \ \forall z \in \mathrm{Im}\,(\gamma),$$

then $N_g = N_f$.

5.1 Riemann mapping theorem

The following is a very strong rigidity result for holomorphic maps.

Theorem 5.1.1. *Let $\Omega \subsetneq \mathbb{C}$ be a proper simply connected domain. Then Ω is biholomorphic to the open unit disc.*

This is a starting point for the uniformization.