Foundational Geometry

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Contents

1 Locally ringed spaces and manifolds

We will define the notion of a real and complex manifold. Some foundational constructions are made on them. We will take a rather modern viewpoint on the matter. We will make very fluid use of sheaves. Let us begin by the foundational structure in all of geometry, a (locally)ringed space.

Definition 1.0.1. (**Ringed and locally ringed spaces**) A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative R-algebras. The space $(X, \mathcal{O}_X)$ is locally ringed if the stalk $\mathcal{O}_{X,x}$ at each point $x \in X$ is a local ring. The sheaf $\mathcal{O}_X$ is called the structure sheaf of X.

In order to understand the relation between two such spaces, we next have to understand the morphism of (locally)ringed spaces. For a motivation, see Example **??**.

Definition 1.0.2. (**Morphism of ringed and locally ringed spaces**) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and a map of sheaves over X denoted $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed, then for $(f, f^\sharp)$ to be morphism of locally ringed spaces has to satisfy an additional condition that the induced map on stalks is a map of local rings. That is, for each $x \in X$, the induced map on stalks

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is such that $(f_x^\sharp)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$ (see Special Topics, Remark **??**). We call this map the co-morphism at $x \in X$. In particular, this map is given by the unique map obtained by universality of direct limits under question: consider any open $V \ni f(x)$ in Y, we then obtain the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{X,x} & & \mathcal{O}_{Y,f(x)} \arrow[ll, dashed, "=:f_x^\sharp"'] \\
\mathcal{O}_X(f^{-1}(V)) \arrow[u, "\iota_{f^{-1}(V)}"] & & \mathcal{O}_Y(V) \arrow[ll, "f_V^\flat"] \arrow[u, "\iota_V"'] \arrow[llu, "\iota_{f^{-1}(V)} \circ f_V^\flat"]
\end{tikzcd}.$$

In most of our purposes, the map $f^\flat$ will be given on sections by composing with f. In such situations, the map on stalks being local corresponds to the geometric intuition that all non-invertible functions around some open subset of $f(x)$ comes from non-invertible maps around x. This in some sense makes sure that the local data around $f(x)$ is completely available via f.

Definition 1.0.3. (**Composition**) Composition of two maps of locally ringed spaces is defined in the obvious manner. For $X \xrightarrow{g} Y \xrightarrow{f} Z$, we get maps $g^\flat : \mathcal{O}_Y \to g_*\mathcal{O}_X$ and $f^\sharp : f^{-1}\mathcal{O}_Z \to \mathcal{O}_Y$. Then, the map $f \circ g : X \to Z$ is defined on space level by just the composite $f \circ g$ of the continuous maps and on the sheaf level as the corresponding flat and sharp maps of $f \circ g : X \to Z$:

$$h^\flat : \mathcal{O}_Z \longrightarrow (f \circ g)_*\mathcal{O}_X$$
$$h^\sharp : (f \circ g)^{-1}\mathcal{O}_Z \longrightarrow \mathcal{O}_X.$$

In particular, for an open set $U \subseteq Z$, the corresponding map on local sections $h^\flat_U$ is given by the following composite:

$$\begin{tikzcd}
\mathcal{O}_Z(U) \arrow[r, dashed, "h^\flat_U"] \arrow[d, "f^\flat_U"] & (f_* g_* \mathcal{O}_X)(U) \arrow[r, equal] & \mathcal{O}_X(g^{-1}f^{-1}(U)) \\
(f_*\mathcal{O}_Y)(U) \arrow[rr, equal] & & \mathcal{O}_Y(f^{-1}(U)) \arrow[u, "g^\flat_{f^{-1}(U)}"']
\end{tikzcd} .$$

Similarly, the corresponding morphism of stalks given by $h^\sharp_x$ is given by the usual

$$h^\sharp_x : (g^{-1}f^{-1}\mathcal{O}_Z)_x \cong \mathcal{O}_{Z,f(g(x))} \longrightarrow \mathcal{O}_{X,x}$$

which is the composite

$$\begin{tikzcd}
\mathcal{O}_{Z,h(x)} \arrow[d, "f^\sharp_{g(x)}"'] & \mathcal{O}_Z(W) \arrow[l] \arrow[d, "f^\flat_W"] \\
\mathcal{O}_{Y,g(x)} \arrow[d, "g^\sharp_x"'] & \mathcal{O}_Y(f^{-1}(W)) \arrow[l] \arrow[d, "g^\flat_{f^{-1}(W)}"] \\
\mathcal{O}_{X,x} & \mathcal{O}_X(g^{-1}(f^{-1}(W))) \arrow[l]
\end{tikzcd} .$$

Lemma 1.0.4. *Let $h : X \xrightarrow{g} Y \xrightarrow{f} Z$ be a morphism of ringed spaces. Consider the base change functors corresponding to maps g and f:*

$$\begin{aligned} g^{-1} &: \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X) \\ f_* &: \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(Z). \end{aligned}$$

and consider the following composite in $\mathbf{Sh}(Y)$

$$f^{-1}\mathcal{O}_Z \xrightarrow{f^\sharp} \mathcal{O}_Y \xrightarrow{g^\flat} g_*\mathcal{O}_X \ .$$

Then,

1. $g^{-1}(g^\flat \circ f^\sharp) \cong h^\sharp$,
2. $f_*(g^\flat \circ f^\sharp) \cong h^\flat$.

Proof. These are cumbersome but straightforward identities. For example, one has to observe that $f_*(f^\sharp) \cong f^\flat$ and that for an open set $U \subseteq Z$, we have $(f_*(g^\flat))_U = g^\flat_{f^{-1}(U)}$. □

We have a simple lemma for isomorphism of ringed spaces.

Lemma 1.0.5. *Let $f : X \to Y$ be a morphism of ringed spaces. Then, f is an isomorphism if and only if $f : X \to Y$ is a homeomorphism and $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

Proof. (L $\Rightarrow$ R) Use Theorem **??**, 3 and 4.
(R $\Rightarrow$ L) One can explicitly construct a map of sheaves in the other direction in a straightforward manner. □

An open subspace of a ringed space also inherits the structure of a ringed space.

Definition 1.0.6. (**Open subspace and embedding**) Let $(X, \mathcal{O}_X)$ be a (locally) ringed space. An open subspace of $(X, \mathcal{O}_X)$ is an open subset $i : U \hookrightarrow X$ together with the inverse image sheaf $i^{-1}\mathcal{O}_X = \mathcal{O}_{X|U}$[1]. The pair $(U, \mathcal{O}_{X|U})$ is called an open subspace, $(U, \mathcal{O}_{X|U}) \hookrightarrow (X, \mathcal{O}_X)$. A map $(j, j^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is an open embedding if $U := j(Z) \hookrightarrow X$ is open and $(j, j^\sharp) : (Z, \mathcal{O}_Z) \to (U, \mathcal{O}_{X|U})$ is an isomorphism of ringed spaces.

An important concept is of local isomorphism of ringed spaces, which will prove it's worth while defining manifolds.

Definition 1.0.7. (**Local isomorphism**) Let $f : X \to Y$ be a morphism of ringed spaces. One calls f to be a local isomorphism if there exists an open cover $\{U_i\}_{i\in I}$ of X such that $f|_{U_i} : U_i \to Y$ is an open embedding for all $i \in I$.

1.1 Local models and manifolds

Before we proceed further, we have to clearly state some of our local model spaces that we are going to use while defining the manifolds. Therefore the following example of ringed spaces are foundational.

Example 1.1.1. (*Sheaf of C^α-maps*) Let $X \subseteq \mathbb{R}^n$ be an open set and $\alpha \in \mathbb{N}^\infty$. One defines the following presheaf

$$\mathcal{C}^\alpha_{X;\mathbb{R}^m} := \{f : X \to \mathbb{R}^m \mid f \text{ is } C^\alpha\}$$

where the restriction maps are usual functional restrictions. Then, $\mathcal{C}^\alpha_{X;\mathbb{R}^m}$ forms a sheaf, called the sheaf of C^α maps on X. This sheaf has stalks as local rings which can be seen quite easily (set of all functions defined in *some* neighborhood of $x \in X$ has a ring structure with maximal ideal being all those functions taking value 0 at x). Hence, $(X, \mathcal{C}^\alpha_{\mathbb{R}^m})$ is a locally ringed space, where we dropped the subscript X for notational convenience.

Example 1.1.2. (*Sheaf of holomorphic maps*) Let $X \subseteq \mathbb{C}^n$ be an open set. One defines the following presheaf

$$\mathcal{C}^{\text{hol}}_{X;\mathbb{C}^m} := \{f : X \to \mathbb{C}^m \mid f \text{ is holomorphic}\}$$

where the restriction maps are the usual functional restriction. This is easily seen to be a sheaf, called the sheaf of holomorphic functions over X. This endows $(X, \mathcal{C}^{\text{hol}}_{\mathbb{C}^m})$ with the structure of a locally ringed space.

With these two examples, we can come to the notion of real and complex manifolds as follows.

Definition 1.1.3. (**Real and complex manifolds**) Let X be a Hausdorff and second-countable topological space. Then,

[1] It's a trivial matter to observe that inverse image of a sheaf to an open inclusion will be the restriction sheaf (see Lemma **??**).

1. A locally $\mathbb{R}$-ringed space $(X, \mathcal{O}_X)$ is a real C^α-manifold if there exists an open covering $\{U_i\}_{i\in I}$ of X and for each $i \in I$, there exists a positive integer $n_i \in \mathbb{N}$ and an isomorphism of locally $\mathbb{R}$-ringed spaces $\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^\alpha_{\mathbb{R}})$ for some open $Y_i \subseteq \mathbb{R}^{n_i}$. Hence a real C^α-manifold structure on X is the following tuple of data:
$$\left(X, \mathcal{O}_X, \{U_i\}_{i\in I}, \{Y_i \subseteq \mathbb{R}^{n_i}\}_{i\in I}, \{\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^\alpha_{\mathbb{R}})\}_{i\in I}\right)$$
2. A locally $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ is a complex manifold if there exists an open covering $\{U_i\}_{i\in I}$ of X and for each $i \in I$ there exists $n_i \in \mathbb{N}$ and an isomorphism of locally $\mathbb{C}$-ringed spaces $\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^{\mathrm{hol}}_{\mathbb{C}})$ for some open $Y_i \subseteq \mathbb{C}^{n_i}$. Hence a complex manifold structure on X is the following tuple of data:
$$\left(X, \mathcal{O}_X, \{U_i\}_{i\in I}, \{Y_i \subseteq \mathbb{C}^{n_i}\}_{i\in I}, \{\varphi_i : (U_i, \mathcal{O}_{X|U_i}) \xrightarrow{\cong} (Y_i, \mathcal{C}^{\mathrm{hol}}_{\mathbb{C}})\}_{i\in I}\right)$$

In both of these, the isomorphisms $\{\varphi_i\}$ are called *charts* of the manifold and the sheaf $\mathcal{O}_X$ the structure sheaf of the manifold. Also, we can rather consider $\{\varphi_i\}_{i\in I}$ to be open embeddings. A map of manifolds is just defined to be a map of locally ringed spaces. Let $\mathbf{Mfd}^{\mathbb{R}}_\alpha$ and $\mathbf{Mfd}^{\mathbb{C}}$ denote the category of real C^α and complex manifolds respectively. A map of manifolds are just locally ringed maps between them. Isomorphisms in them are called C^α-diffeomorphism and biholomorphic maps respectively.

Let us now dwell into some of the immediate observations and remarks coming out of this definition. Let us first ease some notations. Let $(X, \mathcal{O}_X)$ be a real or complex manifold. The local chart (U_i, φ_i) is usually denoted by (U_i, x) where $x : U_i \to \mathbb{R}^n$ is a local embedding of locally ($\mathbb{R}$ or $\mathbb{C}$)-ringed spaces, where n depends on U_i. We usually suppress all the sheaves and their morphisms unless necessary (we will soon see why that's the case). For a local chart (U_i, x), the n component maps $\pi_j \circ x : U_i \to \mathbb{R}$ are denoted by x^j. Moreover, since $x : U \to x(U)$ is an isomorphism, therefore we denote $x^{-1} : x(U) \to U$ to be its inverse. All this will come in handy when we will start doing geometry over $(X, \mathcal{O}_X)$.

Let $(X, \mathcal{O}_X)$ be a real or complex manifold. We call an open subspace $(U, \mathcal{O}_{X|U}) \hookrightarrow (X, \mathcal{O}_X)$ an open submanifold.

One now sees that any morphism of manifolds as locally ringed spaces is completely determined by what happens at the level of points. In-fact, the sheaf allowed on X is also restricted if its a manifold. This is why we usually completely suppress the map of sheaves from our notation as that will be vacuous as long as we are working with map of manifolds. Let $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ be two manifolds ($\mathbb{R}$ or $\mathbb{C}$, but both of same type). We can define a sheaf $\mathcal{O}_{M;N}$ on M given by following sections: for some open $U \subseteq M$, we have a sheaf
$$\mathcal{O}_{M;N}(U) := \{f : (U, \mathcal{O}_{X|U}) \to (N, \mathcal{O}_N) \mid f \text{ is a map of manifolds}\}.$$
Now we show a foundational result which says that the notion of morphism of locally ringed spaces are nothing new in the classical world of $\mathbb{R}^n$ or $\mathbb{C}^n$. We place high importance on the following result as it becomes our point of departure (and thus a point of motivation) as to why the notion of a morphism of locally ringed spaces is defined as what it is; because it is the right notion of a "geometric map" in more abstract situations.

Theorem 1.1.4. *Let K be either $\mathbb{R}$ or $\mathbb{C}$, $X \subseteq K^n$ and $Y \subseteq K^m$ be two open subsets of the standard spaces. If $f : (X, \mathcal{C}^\alpha_X) \to (Y, \mathcal{C}^\alpha_Y)$ is a map of locally ringed spaces, then*

1. *$f^\flat : \mathcal{C}^\alpha_Y \to f_* \mathcal{C}^\alpha_X$ is given on an open set $V \subseteq Y$ by the standard composition map*
$$\begin{aligned} f^\flat_V : \mathcal{C}^\alpha_Y(V) &\longrightarrow \mathcal{C}^\alpha_X(f^{-1}(V)) \\ V \xrightarrow{t} K &\longmapsto f^{-1}(V) \xrightarrow{f} V \xrightarrow{t} K, \end{aligned}$$
2. *f is a C^α-map.*

Remark 1.1.5. As a slogan, we may remember the above theorem as the following principle:

In $\mathbb{R}^n$ or $\mathbb{C}^n$, locally ringed maps are exactly real C^α or holomorphic maps.

As a consequence of this, whenever we would like to consider C^α maps from, say $\mathbb{R}^n$ to $\mathbb{R}^m$, we might as well ask to produce a map of locally ringed spaces $(\mathbb{R}^n, \mathcal{C}^\alpha_{\mathbb{R}^n})$ to $(\mathbb{R}^m, \mathcal{C}^\alpha_{\mathbb{R}^m})$, which again shows how much geometric information is hidden in the notion of sheaves.

Proof of Theorem 1.1.4. [2] Pick any open $V \subseteq Y$ and any $t \in \mathcal{C}^\alpha_Y(V)$. We wish to show that $f^\flat_V(t) = t \circ f$ as a map $f^{-1}(V) \to K$. Consequently, pick any point $p \in f^{-1}(V)$. We wish to show that $f^\flat_V(t)(p) = t(f(p))$. To this end, we consider the evaluation homomorphism which are available at stalks. Observe that we have the following commutative square of K-algebras:

$$\begin{tikzcd} \mathcal{C}^\alpha_Y(V) \arrow[r, "f^\flat_V"] \arrow[d] & \mathcal{C}^\alpha_X(f^{-1}(V)) \arrow[d] \\ \mathcal{C}^\alpha_{Y,f(p)} \arrow[r, "f^\sharp_p"'] & \mathcal{C}^\alpha_{X,p} \end{tikzcd} .$$

In order to show $f^\flat_V(t)(p) = t(f(p))$, it is sufficient to show that the following triangle commutes:

$$\begin{tikzcd} \mathcal{C}^\alpha_{Y,f(p)} \arrow[rr, "f^\sharp_p"] \arrow[dr, "\mathrm{ev}_{f(p)}"'] & & \mathcal{C}^\alpha_{X,p} \arrow[dl, "\mathrm{ev}_p"] \\ & K & \end{tikzcd} .$$

But this is immediate from the fact that the K-algebra homomorphism $f^\sharp_p$ is a local ring homomorphism and the kernels of the evaluation maps are exactly the corresponding unique maximal ideals, so by quotienting by the maximal ideals, we obtain a K-algebra homomorphism $K \to K$ which necessarily is identity as it is a K-algebra homomorphism. Hence the triangle indeed commutes.

In order to show that the map f is a C^α-map, we need only show that the m projection maps $\pi_i : K^m \to K$ when composed with f yields C^α maps given by $X \to K$, but that is immediate from 1. □

Using the above result, one can show that any manifold essentially has a unique structure sheaf of the form $\mathcal{O}_{X;\mathbb{R}}$ or $\mathcal{O}_{X;\mathbb{C}}$.

[2] First proof in my new creator of meaning!

Proposition 1.1.6. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. If $(X, \mathcal{O}_X)$ is a real or complex manifold, then $\mathcal{O}_X \cong \mathcal{O}_{X;\mathbb{R}}$ or $\mathcal{O}_X \cong \mathcal{O}_{X;\mathbb{C}}$.*

Proof. We wish to show that there is an isomorphism of sheaves $\varphi : \mathcal{O}_{X;\mathbb{R}} \to \mathcal{O}_X$. For an open set $U \subseteq X$, we define φ_U as follows:

$$\begin{aligned}\varphi_U : \mathcal{O}_{X;\mathbb{R}}(U) &\longrightarrow \mathcal{O}_X(U)\\ t : (U, \mathcal{O}_{X|U}) \to (\mathbb{R}, \mathcal{C}^\alpha_{\mathbb{R}}) &\longmapsto t^\flat_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}).\end{aligned}$$

We claim that this map of sheaves is an isomorphism. We need only show that the map on stalks $\varphi_x : \mathcal{O}_{X;\mathbb{R},x} \to \mathcal{O}_{X,x}$ is an isomorphism. So we may assume that X has a global chart $\eta : (X, \mathcal{O}_X) \cong (W, \mathcal{C}^\alpha_{W;\mathbb{R}})$ where $W \subseteq \mathbb{R}^n$ is an open subset. Consequently, we have $\eta^\sharp_x : \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)} \cong \mathcal{O}_{X,x}$. Furthermore, $\mathcal{O}_{X;\mathbb{R},x} \cong \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)}$. Consequently, we wish to show that $\varphi_x : \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)} \to \mathcal{C}^\alpha_{W;\mathbb{R},\eta(x)}$ given by $(W, t : W \to \mathbb{R})_{\eta(x)} \mapsto (W, t^\flat_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}}))_{\eta(x)}$ is an isomorphism. Since by Theorem 1.1.4, 1, the map $t^\flat_{\mathbb{R}}$ is given by precomposition by t, therefore $t^\flat_{\mathbb{R}}(\mathrm{id}_{\mathbb{R}})$ is just t. Consequently, φ_x is identity, which proves the result. $\square$

Remark 1.1.7. By virtue of Proposition 1.1.6, we can assume that any C^α-manifold is a locally ringed space of the form $(X, \mathcal{O}_{X;\mathbb{R}})$ (similarly for $\mathbb{C}$-manifolds).

1.2 Sheaves & atlases

We have defined a manifold to be a space with an open covering by a model locally ringed spaces. There is a traditional definition, whereas, which is used heavily in traditional geometry because we really care about the charts (which is usually not done in algebraic geometry). This elucidates how one has to undertake a different viewpoint of geometry in algebraic geometry.

We wish to show that giving a manifold structure on a second countable Hausdorff space X as defined above is equivalent to giving an atlas in the classical sense. Indeed, for each atlas on X, we first define a sheaf on X.

Definition 1.2.1 (Atlas sheaf). Let X be a second countable Hausdorff space and $\mathcal{A} = (U_i, x_i)_{i \in I}$ be a C^α-atlas on X where $x_i : U_i \to \mathbb{C}^{n_i}$ is an open embedding. Consider the following assignment for each open $V \subseteq X$:

$$\mathcal{O}_{\mathcal{A}}(V) := \{f : V \to K \mid f \circ x_i^{-1} : x_i(U_i \cap V) \to K \text{ is } C^\alpha\text{-map}\}.$$

Then $\mathcal{O}_{\mathcal{A}}$ is a sheaf of $\mathbb{R}$-algebras, called the sheaf of atlas $\mathcal{A}$. Similarly for the holomorphic case.

We first observe that equivalent atlases give same atlas sheaves.

Lemma 1.2.2. *Let X be a second-countable Hausdorff space with $\mathcal{A} = (U_i, x_i)_i$ and $\mathcal{B} = (V_i, y_i)_i$ being two equivalent C^α or holomorphic atlases on X. Then the atlas sheaves $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ are isomorphic.*

Proof. Indeed, for each open $W \subseteq X$, define the map

$$\begin{aligned}\varphi_W : \mathcal{O}_{\mathcal{A}}(W) &\longrightarrow \mathcal{O}_{\mathcal{B}}(W)\\ f : W \to K &\longmapsto f : W \to K.\end{aligned}$$

To show that this is well-defined, we have to show that $f \in \mathcal{O}_{\mathcal{B}}(W)$. Indeed, pick any chart $y_i : V_i \to K$ of $\mathcal{B}$. We wish to show that $f \circ y_i^{-1} : y_i(V_i \cap W) \to K$ is C^α or holomorphic. As either condition is local on domain, so pick any point in $y_i(V_i \cap W)$. Pick a chart $x_i : U_i \to x_i(U_i)$ containing that point. Note that it is sufficent to show $f \circ y_i^{-1} : y_i(V_i \cap U_i \cap W) \to K$ is C^α or holomorphic. Indeed, we can write this as

$$f \circ y_i^{-1} = (f \circ x_i^{-1}) \circ (x_i \circ y_i^{-1}) : y_i(U_i \cap V_i \cap W) \to K.$$

Since $\mathcal{A}$ and $\mathcal{B}$ are equivalent and $f \in \mathcal{O}_{\mathcal{A}}$, it follows repsectively that $(x_i \circ y_i^{-1})$ and $(f \circ x_i^{-1})$ are C^α or holomorphic, as required.

Thus $\varphi : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$ is a sheaf map, which is identity, hence both sheaves are same. □

We next see that a C^α or holomorphic atlas sheaf on a space X gives a C^α or $\mathbb{C}$ manifold structure on X.

Proposition 1.2.3. *Let $(X, \mathcal{O}_{X;\mathbb{C}})$ be a locally ringed space and $Y \subseteq \mathbb{C}^n$ be open. If $\varphi : (X, \mathcal{O}_{X;\mathbb{C}}) \to (Y, \mathcal{C}^{\text{hol}}_{Y;\mathbb{C}})$ is a map of locally ringed spaces, then $\varphi^\flat$ on open $V \subseteq Y$ is given by*

$$\begin{aligned} \varphi^\flat_V : \mathcal{C}^{\text{hol}}_{Y;\mathbb{C}}(V) &\longrightarrow \mathcal{O}_{X;\mathbb{C}}(\varphi^{-1}(V)) \\ t : V \to \mathbb{C} &\longmapsto t \circ \varphi : \varphi^{-1}(V) \to \mathbb{C}. \end{aligned}$$

Moreover, the following are equivalent:
1. *$\varphi : (X, \mathcal{O}_{X;\mathbb{C}}) \to (Y, \mathcal{C}^{\text{hol}}_{Y;\mathbb{C}})$ is an isomorphism of locally ringed spaces.*
2. *$\varphi : X \to Y$ is a homeomorphism such that for any open $U \subseteq X$ and any $f : U \to \mathbb{C}$ in $\mathcal{O}_X(U)$, $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is a holomorphic map.*

The same conclusions hold true for C^α-manifolds as well.

Proof. The proof of the first statement is exactly same as that of Theorem 1.1.4, hence is omitted. We now show the equivalence of items 1 and 2.

(1. ⇒ 2.) This is immediate as the map $\varphi^\flat$ is an isomorphism, so in particular a bijection on sections.

(2. ⇒ 1.) Pick any open $V \subseteq Y$. Then $\varphi^\flat_V$ is injective as φ is an isomorphism. It is also surjective by the given hypothesis and homeomorphism φ. This shows that $\varphi^\flat$ is an isomorphism. □

Theorem 1.2.4. *Let X be a second-countable Hausdorff space and $(X, \mathcal{O}_X)$ be a locally ringed space. Then the following are equivalent.*
1. *$(X, \mathcal{O}_X)$ is a C^α/complex manifold.*
2. *$\mathcal{O}_X$ is a C^α/complex atlas sheaf.*

To avoid repetitions, we will do the complex case only, as there is no change in the proof for the real case.

Proof. (1. ⇒ 2.) By Proposition 1.1.6, we may assume that $\mathcal{O}_X$ is just $\mathcal{O}_{X;\mathbb{C}}$, the sheaf of locally ringed maps from X to $\mathbb{C}$. We have an open cover $\{U_i\}_{i \in I}$ of X and isomorphisms of locally ringed spaces $\varphi_i : (U_i, \mathcal{O}_{U_i;\mathbb{C}}) \to (Y_i, \mathcal{C}^{\text{hol}}_{\mathbb{C}})$. This makes (U_i, φ_i) into an usual atlas as follows. For any i, j such that $U_i \cap U_j \neq \emptyset$, we obtain that the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

This is holomorphic since $\varphi_j : U_i \cap U_j \to \mathbb{C}$ is a map of locally ringed spaces in $\mathcal{O}_{X;\mathbb{C}}(U_i \cap U_j)$. Now, $\varphi_i : (U_i, \mathcal{O}_{U_i;\mathbb{C}}) \to (Y_i, \mathcal{C}^{\text{hol}}_{Y_i;\mathbb{C}})$ is an isomorphism, therefore by Proposition 1.2.3, it follows that $\varphi_j \circ \varphi_i^{-1}$ is a holomorphic map, as required.

We claim that this makes $\mathcal{O}_X$ into an atlas sheaf. Indeed, observe that $f \in \mathcal{O}_X(V)$ is a locally ringed map $f : (V, \mathcal{O}_{V;\mathbb{C}}) \to (Y, \mathcal{C}^{\text{hol}}_{\mathbb{C}})$. We claim that the data of f is equivalent to saying that $f \circ \varphi_i^{-1} : \varphi_i(V \cap U_i) \to \mathbb{C}$ is holomorphic. Indeed, this is the content of Proposition 1.2.3.

(2. $\Rightarrow$ 1.) Let $\mathcal{A} = (U_i, \varphi_i)$ be a complex atlas where $\varphi_i : U_i \to Y_i$ for open $Y_i \subseteq \mathbb{C}^{n_i}$ is a homeomorphism with holomorphic transitions. We need only show the item 2 of Proposition 1.2.3 for φ_i as then it would follow that $\varphi_i : (U_i, \mathcal{O}_{U_i;\mathbb{C}}) \to (Y_i, \mathcal{C}^{\text{hol}}_{Y_i;\mathbb{C}})$ is an isomorphism of locally ringed spaces, completing the proof. Indeed, pick any open $U \subseteq X$ and any $f : U \to \mathbb{C}$ in $\mathcal{O}_X(U)$. As $\mathcal{O}_X$ is the atlas sheaf of $\mathcal{A}$, therefore for φ_i in particular, we have that $f \circ \varphi_i^{-1} : \varphi_i(U \cap U_i) \to \mathbb{C}$ is a holomorphic map, as required. This completes the proof. $\square$

2 Global algebra

Let $(X, \mathcal{O}_X)$ be a locally ringed space. We will discuss here the operations on and properties of $\mathbf{Mod}(\mathcal{O}_X)$, the category of $\mathcal{O}_X$-modules [3]. An $\mathcal{O}_X$-module is a sheaf $\mathcal{M}$ on X such that $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module and the restriction maps of $\mathcal{M}$ are given as module homomorphism w.r.t the corresponding restriction map of $\mathcal{O}_X$ (more precisely below). There are several important constructions and properties that one can make with these. In-fact, just like one understands a ring R by understanding R-modules, one can understand $\mathcal{O}_X$ by understanding $\mathcal{O}_X$-modules. The similarity runs deeper as we can also define in certain cases the very same constructions we do in module, but in the case of $\mathcal{O}_X$-modules, and these constructions and operations becomes indispensable in doing geometry over locally ringed spaces of special kind, like schemes. A lot of such phenomenon is merely due to the fact that $\mathbf{Mod}(\mathcal{O}_X)$ is an abelian category. In-fact, notice that for each singleton space $X = \{\text{pt.}\}$, a ring R can be seen as the structure sheaf $\mathcal{O}_X$ over X and any R-module as a $\mathcal{O}_X$-module. Hence one may also think of the concept of $\mathcal{O}_X$-modules as the global version of classical commutative algebra.

Needless to say, this is an indispensable section for the purposes of geometry in general.

Let us first observe that over any topological space X, the product of two sheaves $\mathcal{F}, \mathcal{G}$ over X defined by $(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$ is indeed a sheaf with restriction maps as products of the restrictions. This allows us to define $\mathcal{O}_X$-modules very naturally.

For the rest of this section, we fix a ringed space $(X, \mathcal{O}_X)$.

Definition 2.0.1. ($\mathcal{O}_X$**-modules**) An abelian sheaf $\mathcal{F}$ over X is an $\mathcal{O}_X$-module if there is a map of sheaves

$$\begin{aligned} \mathcal{O}_X \times \mathcal{F} &\longrightarrow \mathcal{F} \\ (c, s) &\longmapsto cs \end{aligned}$$

[3] we will give some general constructions for arbitrary sheaves over a topological case at times, before specializing to $\mathcal{O}_X$-module case.

where $c \in \mathcal{O}_X(U), s \in \mathcal{F}(U)$ for all open $U \subseteq X$ which endows $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module structure.

An $\mathcal{O}_X$-linear map of $\mathcal{O}_X$-modules is defined as a sheaf map $\varphi : \mathcal{F} \to \mathcal{G}$ between $\mathcal{O}_X$-modules such that for each open $U \subseteq X$, the map $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-linear map and that the restrictions preserves the respective module structures.

The above definition, when unravelled, yields that the scalar multiplication of each $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ commutes with restrictions; for $c \in \mathcal{O}_X(U)$, $s \in \mathcal{F}(U)$ and an open subset $V \subseteq U$, we have $(c \cdot s)|_V = c|_V \cdot s|_V$.

Remark 2.0.2. For an $\mathcal{O}_X$-module $\mathcal{F}$ we have the following easy observations:

1. $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module for all $x \in X$. Indeed, this follows from the universal property of direct limits and the fact that direct limits commutes with product; we have the following diagram

$$\begin{tikzcd}
\mathcal{O}_{X,x} \times \mathcal{F}_x \arrow[rrdd, dashed, "\tilde{m}"] & & \\
\varinjlim_{x \in V} \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[u, "\cong"] & & \\
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[u] \arrow[r, "m_U"] & \mathcal{F}(U) \arrow[r] & \mathcal{F}_x
\end{tikzcd} .$$

   Explicitly, the $\mathcal{O}_{X,x}$-module structure on $\mathcal{F}_x$ is given by

$$\begin{aligned} \mathcal{O}_{X,x} \times \mathcal{F}_x &\longrightarrow \mathcal{F}_x \\ ((U,c)_x, (U,s)_x) &\longmapsto (U, c \cdot s)_x \end{aligned}$$

   where we may assume c and s are defined on same open neighborhood of x by appropriately restricting.
2. For a homomorphism $f : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules, we get a $\mathcal{O}_{X,x}$-module homomorphism $f_x : \mathcal{F}_x \to \mathcal{G}_x$ mapping as $(U,s)_x \mapsto (U, f_U(s))_x$ for each $x \in X$,
3. Let X be locally ringed space. Then, $\mathcal{F}_x/\mathfrak{m}_{X,x}\mathcal{F}_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a $\kappa(x)$-vector space. This is called the *fiber of module $\mathcal{F}$ over x*, denoted by $\mathcal{F}(x)$. Recall this is how the fiber of a module over a prime ideal of the ring is defined.

We first give few basic constructions, which is useful to keep in mind.

Definition 2.0.3. (**Support of a sheaf**) Let X be a topological space and $\mathcal{F}$ be an abelian sheaf over X. Let $U \subseteq X$ be an open set. For $s \in \mathcal{F}(U)$, we define the support of s as the subset

$$\mathrm{Supp}\,(s) := \{x \in U \mid (U,s)_x \neq 0 \text{ in } \mathcal{F}_x\}.$$

We further define the support of the sheaf as

$$\mathrm{Supp}\,(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Support of a section is always a closed subset, but the support of a sheaf may not be closed.

Lemma 2.0.4. [4] *Let X be a space and $\mathcal{F}$ be a sheaf over X with $s \in \mathcal{F}(U)$ for an open set $U \subseteq X$. Then* $\mathrm{Supp}\,(s) \subseteq U$ *is a closed subset of U.*

[4] Exercise II.1.14 of Hartshorne.

Proof. Take any point $y \in U \setminus \operatorname{Supp}(s)$. We will find an open set $W \subseteq U \setminus \operatorname{Supp}(s)$ with $W \ni y$. Indeed, as $(U, s)_y = 0$, therefore we get a $W \subseteq U$ with $s|_W = 0$. For any $z \in W$, one further checks that $(U, s)_z = (W, s|_W)_z = 0$. Thus, $z \notin \operatorname{Supp}(s)$ and consequently, $W \subseteq U \setminus \operatorname{Supp}(s)$. □

Do skyscraper and subsheaf with support (Exercises 1.17 and 1.20 in Hartshorne.)

2.1 Global algebra : The algebra of $\mathcal{O}_X$-modules

In our quest to do geometry over schemes, we will make heavy use of the algebra of sheaves, especially that of exact sequences, so we give a lot of constructions that we may have to make out in the wild. We will make heavy use of sheafification (Theorem **??**) in the sequel. An important question that arises is whether sheafification of an algebraic construction over collection of $\mathcal{O}_X$-modules actually is again an $\mathcal{O}_X$-module or not? The answer is yes, as can be easily checked by explicitly looking at sections of sheafification directly (see Remark **??** to observe that its not difficult, anyways we will show the explicit checks consistently).

Caution 2.1.1. The following pages might seem to be filled with *unnecessary details* about checking whether a given construction on $\mathcal{O}_X$-modules results in an $\mathcal{O}_X$-module or not. While for some this might be unnecessary, but working this out in experience has been satisfying and tends to give a deeper understanding of the various module structures that gets associated with an $\mathcal{O}_X$-module $\mathcal{F}$ and how they interrelate. Indeed, we will see that with more elaborate constructions, we get more and more module structures to handle with. Thus it is necessary to work some details out of this. At any rate, we will be using notions presented in the sequel quite frequently in algebraic geometry and in particular while doing cohomology (Cěch cohomology in particular!) so we need a good knowledge of the $\mathcal{O}_X$-modules and their internal technicalities.

Remark 2.1.2. Since there are a lot of constructions in the sequel, so to have a sense of mental clarity, let us list them here:

- Submodules and ideals of $\mathcal{O}_X$.✓
- Quotient of modules.✓
- Image and kernel modules.✓
- Exact sequences of modules.✓
- The $\Gamma(\mathcal{O}_X, X)$-module $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.✓
- $\mathcal{H}om_{\mathcal{O}_X}$ module.✓
- Direct sum of modules.✓
- Direct product of modules.✓
- Tensor product of modules.✓
- Free, locally free & finite locally free $\mathcal{O}_X$-modules.✓
- Invertible modules and the Picard group.✓
- Direct and inverse image modules.✓
- Sums & intersections of submodules.
- Modules generated by sections.
- Inverse limit.
- Direct limit.
- Tensor, symmetric & exterior algebras.
- $\mathcal{E}xt$ module.
- $\mathcal{T}or$ module.

Remark 2.1.3. Let $\mathbf{V}$ be the category of abelian groups and X be a locally ringed space. Consider a functor $F : \mathbf{V} \times \cdots \times \mathbf{V} \to \mathbf{V}$. Given abelian sheaves $\mathcal{F}_1, \ldots, \mathcal{F}_k$ over X, we obtain a sheaf $\mathcal{F}_F := F(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ by the following procedure: first define the presheaf $\mathcal{F}_F^-$ on X given by $U \mapsto F(\mathcal{F}_1(U), \ldots, \mathcal{F}_k(U))$, then define the sheaf $\mathcal{F}_F = (\mathcal{F}_F^-)^{++}$ to be the sheafification of $\mathcal{F}_F^-$. We will follow this general strategy in all the constructions in the following.

2.1.1 Submodules and ideals of $\mathcal{O}_X$

Definition 2.1.4. (**Submodules and ideals**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *submodule* of $\mathcal{F}$ is an $\mathcal{O}_X$-module which is a subsheaf $\mathcal{G} \subseteq \mathcal{F}$ such that for all open $U \subseteq X$, the inclusion

$$\mathcal{G}(U) \hookrightarrow \mathcal{F}(U)$$

is an $\mathcal{O}_X(U)$-module homomorphism. Since $\mathcal{O}_X$ is an $\mathcal{O}_X$-module, thus, to be in line with usual terminology, we define submodules of $\mathcal{O}_X$ as *ideals* of $\mathcal{O}_X$.

Remark 2.1.5. Note that for any $\mathcal{O}_X$ submodule $\mathcal{G} \subseteq \mathcal{F}$, we get a submodule $\mathcal{G}_x \subseteq \mathcal{F}_x$ of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.

2.1.2 Quotient of modules

Definition 2.1.6. (**Quotient modules**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and $\mathcal{G}$ be a submodule of $\mathcal{F}$. The *quotient module* is the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$, denoted by $\mathcal{F}/\mathcal{G}$ (see Definition **??**). Indeed, $\mathcal{F}/\mathcal{G}$ is an $\mathcal{O}_X$-module by the following lemma.

Lemma 2.1.7. *$\mathcal{F}/\mathcal{G}$ is an $\mathcal{O}_X$-module.*

Proof. We will use the definition of sheafification as given in Remark **??**. For each open set $U \subseteq X$, consider the following map:

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times (\mathcal{F}/\mathcal{G})(U) &\longrightarrow (\mathcal{F}/\mathcal{G})(U) \\ (c, s) &\longmapsto \eta_U(c, s) : U \to \amalg_{x \in U} \mathcal{F}_x/\mathcal{G}_x \end{aligned}$$

where $\eta_U(c, s)(x) := c_x \cdot s(x)$ where $c_x \in \mathcal{O}_{X,x}$ and $s(x) \in \mathcal{F}_x/\mathcal{G}_x$ and the multiplication $c_x \cdot s(x)$ is coming from the $\mathcal{O}_{X,x}$-module structure that $\mathcal{F}_x/\mathcal{G}_x$ has. We now need to show following two statements:

1. $\eta_U(c, s)$ is indeed in $(\mathcal{F}/\mathcal{G})(U)$,
2. $\eta : \mathcal{O}_X \times \mathcal{F}/\mathcal{G} \to \mathcal{F}/\mathcal{G}$ is a sheaf map.

For statement 1, we need to show that for each $x \in U$, there exists an open set $x \in V \subseteq U$ and there exists $r \in \mathcal{F}(U)/\mathcal{G}(U)$ such that for all $y \in V$ we have the equality $c_y \cdot s(y) = r_y$ in $\mathcal{F}_y/\mathcal{G}_y$. Indeed, this can easily be seen via the fact that $s \in (\mathcal{F}/\mathcal{G})(U)$. Statement 2 is immediate after drawing the relevant square whose commutativity is under investigation. □

Remark 2.1.8. Note further that we get a natural map

$$\mathcal{F} \to \mathcal{F}/\mathcal{G}$$

which factors through the inclusion of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$ into the sheaf $\mathcal{F}/\mathcal{G}$.

2.1.3 Image and kernel modules

Definition 2.1.9. (**Image and kernel modules**) Let $f : \mathcal{F} \to \mathcal{G}$ be a $\mathcal{O}_X$-module homomorphism. We then get the image sheaf $\mathrm{Im}\,(f)$ and the kernel sheaf $\mathrm{Ker}\,(f)$ by Definition **??**. Indeed, both of these are $\mathcal{O}_X$-modules as the following lemma shows.

Lemma 2.1.10. $\mathrm{Im}\,(f)$ *and* $\mathrm{Ker}\,(f)$ *are* $\mathcal{O}_X$*-modules.*

Proof. $\mathrm{Ker}\,(f)$ is straightforward. For $\mathrm{Im}\,(f)$, we first observe that if we denote $\mathrm{Im}\,(f) = (\mathrm{im}\,(f))^{++}$, then $(\mathrm{im}\,(f))_x = f_x(\mathcal{F}_x)$. We thus define the $\mathcal{O}_X$-module structure on $\mathrm{Im}\,(f)$ as follows:

$$\begin{aligned}\eta_U : \mathcal{O}_X(U) \times \mathrm{Im}\,(f)(U) &\longrightarrow \mathrm{Im}\,(f)(U)\\ (c, s : U \to \amalg_{x\in U} f_x(\mathcal{F}_x)) &\longmapsto \eta_U(c,s)\end{aligned}$$

where $\eta_U(c,s)(x) = c_x \cdot s(x)$ where $s(x) \in f_x(\mathcal{F}_x) \subseteq \mathcal{G}_x$. One checks like for quotient modules that this defines an $\mathcal{O}_X$-module structure on $\mathrm{Im}\,(f)$. Further, it is clear that $\mathrm{Im}\,(f) \subseteq \mathcal{G}$. □

Corollary 2.1.11. *For a* $\mathcal{O}_X$*-module homomorphism* $f : \mathcal{F} \to \mathcal{G}$*, we have* $\mathrm{Ker}\,(f) \leq \mathcal{F}$ *and* $\mathrm{Im}\,(f) \leq \mathcal{G}$ *are submodules.*

Proof. Use Remark **??** to get this immediately. □

We have a "first isomorphism theorem" for modules then.

Lemma 2.1.12. *For a map* $f : \mathcal{F} \to \mathcal{G}$ *of* $\mathcal{O}_X$*-modules, we obtain an isomorphism*

$$\mathcal{F}/\mathrm{Ker}\,(f) \cong \mathrm{Im}\,(f).$$

Proof. For each $x \in X$ let $\varphi_x : \mathcal{F}_x/\ker f_x \xrightarrow{\cong} \mathrm{im}\,(f_x)$. Then we define the following for any $U \subseteq X$ open

$$\begin{aligned}(\mathcal{F}/\mathrm{Ker}\,(f))(U) &\longrightarrow \mathrm{Im}\,(f)(U)\\ s : U \to \amalg_{x\in U}\mathcal{F}_x/\ker f_x &\mapsto \varphi \circ s\end{aligned}$$

where $(\varphi \circ s)(x) = \varphi_x(s(x))$. This is immediately an isomorphism by going to stalks (Theorem **??**, 3). □

2.1.4 Exact sequences of modules

Definition 2.1.13. (**Exact sequences**) A sequence of $\mathcal{O}_X$-modules

$$\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$$

is said to be *exact* if $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$.

Remark 2.1.14. By Lemma **??**, $\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$ is exact if and only if $\mathrm{Ker}\,(g_x) = \mathrm{Im}\,(f_x)$ at all points $x \in X$.

2.1.5 The $\Gamma(\mathcal{O}_X, X)$-module $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$

We now consider the set of all $\mathcal{O}_X$-module homomorphisms $f : \mathcal{F} \to \mathcal{G}$ and observe very easily that it has a $\Gamma(\mathcal{O}_X, X)$-module structure. This generalizes the fact that under point-wise addition and scalar multiplication, the set $\mathrm{Hom}_R(M, N)$ for two R-modules M, N is again an R-module.

Definition 2.1.15. ($\Gamma(\mathcal{O}_X, X)$**-module** $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then the collection of all $\mathcal{O}_X$-module homomorphisms $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a $\Gamma(X, \mathcal{O}_X)$-module. Indeed, for two $f, g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $c \in \Gamma(\mathcal{O}_X, X)$, we define $f + g : \mathcal{F} \to \mathcal{G}$ by $s \mapsto f(s) + g(s)$ and we define $c \cdot f : \mathcal{F} \to \mathcal{G}$ by $s \mapsto \rho_{X,U}(s) \cdot f(s)$ for any open set $U \subseteq X$ and $s \in \mathcal{F}(U)$.

We will now globalize the construction of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to obtain an $\mathcal{O}_X$-module out of it.

2.1.6 $\mathcal{H}om_{\mathcal{O}_X}$ module

Definition 2.1.16. (**Hom module** $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then the following presheaf

$$U \mapsto \mathcal{H}om_{\mathcal{O}_{X|U}}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$$

with restriction given by restriction of sheaf maps, is an $\mathcal{O}_X$-module denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, as the following lemma shows.

Lemma 2.1.17. *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an $\mathcal{O}_X$-module*

Proof. The fact that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf can be seen immediately. The $\mathcal{O}_X$-module structure is defined as follows: pick any open $U \subseteq X$

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U}) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U}) \\ (c, f) &\longmapsto cf \end{aligned}$$

where $cf : \mathcal{F}_{|U} \to \mathcal{G}_{|U}$ is given on an open set $V \subseteq U$ by

$$\begin{aligned} (cf)_V : \mathcal{F}(V) &\longmapsto \mathcal{G}(V) \\ s &\longmapsto \rho_{U,V}(c) \cdot f_V(s). \end{aligned}$$

One easily check that η is a well-defined natural map of sheaves, thus making $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ into an $\mathcal{O}_X$-module. $\square$

Remark 2.1.18. It is in general NOT true that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$.

We now define the dual of a module in the obvious manner.

Definition 2.1.19. (**Dual module**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The dual of $\mathcal{F}$ is defined to be the module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. We denote the dual by $\mathcal{F}^\vee$.

There are some isomorphisms regarding $\mathcal{H}om$ that is akin to their usual algebraic counterparts. We outline them in the following lemma.

Lemma 2.1.20. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,*

1. $\mathcal{H}om(\mathcal{O}_X^n, \mathcal{F}) \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n$,
2. $\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$.

Proof. In both cases we construct a map and its inverses and it is straightforward to see that they are well-defined, natural and indeed inverses of each other.
1. Consider the map

$$\mathcal{H}om(\mathcal{O}_X^n, \mathcal{F}) \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n$$

which on an open set $U \subseteq X$ maps as

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}^n, \mathcal{F}_{|U}\right) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right)^n \\ f : \mathcal{O}_{X|U}^n \to \mathcal{F}_{|U} &\longmapsto (f_i)_{i=1,\dots,n} \end{aligned}$$

where for $V \subseteq U$, we have that $f_{i,V} : \mathcal{O}_X(V) \to \mathcal{F}(V)$ maps as $s \mapsto s \cdot f_V(e_i) = f_V(s \cdot e_i)$ where e_i is i^{th} standard vector in $\mathcal{O}_X(V)^n$. Conversely, define the map

$$\mathcal{H}om(\mathcal{O}_X, \mathcal{F})^n \longrightarrow \mathcal{H}om(\mathcal{O}_X^n, \mathcal{F})$$

which on $U \subseteq X$ open maps as

$$\left(g_i : \mathcal{O}_{X|U} \to \mathcal{F}_{|U}\right)_{i=1,\dots,n} \longmapsto g : \mathcal{O}_{X|U}^n \to \mathcal{F}_{|U}$$

where on $V \subseteq U$ open, we define $g_V : \mathcal{O}_X(V)^n \to \mathcal{F}(V)$ as $(s_1, \dots, s_n) \mapsto \sum_{i=1}^n g_{i,V}(s_i) = \sum_{i=1}^n s_i \cdot g_{i,V}(e_i)$.
2. Define the map

$$\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \longrightarrow \mathcal{F}$$

on open $U \subseteq X$ by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right) &\longrightarrow \mathcal{F}(U) \\ f : \mathcal{O}_{X|U} \to \mathcal{F}_{|U} &\longmapsto f_U(1). \end{aligned}$$

Define the inverse

$$\mathcal{F} \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{F})$$

on an open set $U \subseteq X$ by

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right) \\ s &\longmapsto f : \mathcal{O}_{X|U} \to \mathcal{F}_{|U} \end{aligned}$$

where for an open set $V \subseteq U$, we define $f_V(t) = f_V(t \cdot 1) := t \cdot s$. □

The following the usual adjunction from algebra.

Proposition 2.1.21 ($\otimes$-hom adjunction)**.** *For any $\mathcal{O}_X$-modules $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, we have*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})\right).$$

Proof. Let $R = \Gamma(\mathcal{O}_X, X)$. We construct an R-linear map $\varphi : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$ as follows: for any $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G})$, define $\varphi(f) : \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})$ by the following on open $U \subseteq X$:

$$\varphi(f) : s \mapsto \varphi(f)(s) : t \mapsto f(s \otimes t).$$

This is R-linear by construction. To show its a bijection, we construct an inverse as follows: for any $f : \mathcal{F} \to \mathcal{H}om(\mathcal{E}, \mathcal{G})$, define $\theta(f) : \mathcal{E} \otimes \mathcal{F} \to \mathcal{G}$ as the unique map corresponding to the map on presheaves:

$$\begin{aligned}(\mathcal{E} \otimes \mathcal{F})^- &\longrightarrow \mathcal{G}\\ s \otimes t &\mapsto f(t)(s).\end{aligned}$$

It is easy to see that this is an inverse of φ, as required. □

2.1.7 Direct sum of modules

Definition 2.1.22. (**Direct sum of modules**) Let $\{\mathcal{F}_i\}_{i \in I}$ be a family of $\mathcal{O}_X$-modules. The direct sum of $\mathcal{F}_i$ is the sheafification of the presheaf

$$U \mapsto \bigoplus_{i \in I} \mathcal{F}_i(U)$$

whose restriction is the direct sum of the corresponding restrictions. We denote this sheaf by $\bigoplus_{i \in I} \mathcal{F}_i$ and it is an $\mathcal{O}_X$-module by the following lemma. If for all $i \in I$, we have $\mathcal{F}_i = \mathcal{F}$, then we write

$$\bigoplus_{i \in I} \mathcal{F} = \mathcal{F}^{\oplus I} = \mathcal{F}^{(I)}$$

as usually is done in algebra.

Lemma 2.1.23. *$\bigoplus_{i \in I} \mathcal{F}_i$ is an $\mathcal{O}_X$-module and $\left(\bigoplus_{i \in I} \mathcal{F}_i\right)_x \cong \bigoplus_{i \in I} \mathcal{F}_{i,x}$ for all $x \in X$.*

Proof. Since stalks functor is left adjoint (to skyscraper, we didn't covered this but this is a basic known fact), therefore it preserves all colimits and thus $\left(\bigoplus_{i \in I} \mathcal{F}_i\right)_x \cong \bigoplus_{i \in I} \mathcal{F}_{i,x}$. Now, the $\mathcal{O}_X$-module structure over $\bigoplus_{i \in I} \mathcal{F}_i$ is obtained as follows: pick any $U \subseteq X$ open and consider the map

$$\begin{aligned}\eta_U : \mathcal{O}_X(U) \times \left(\bigoplus_{i \in I} \mathcal{F}_i\right)(U) &\longrightarrow \left(\bigoplus_{i \in I} \mathcal{F}_i\right)(U)\\ (c, s : U \to \amalg_{x \in U} \oplus_{i \in I} \mathcal{F}_{i,x}) &\longmapsto cs\end{aligned}$$

where $cs(x) = c_x \cdot s(x)$ where $s(x) \in \bigoplus_{i \in I} \mathcal{F}_{i,x}$ and $\bigoplus_{i \in I} \mathcal{F}_{i,x}$ is an $\mathcal{O}_{X,x}$-module. By exactly same techniques employed in proving them in earlier cases, it can be observed that the above defines a map $\eta : \mathcal{O}_X \times \oplus_{i \in I} \mathcal{F}_i \to \oplus_{i \in I} \mathcal{F}_i$ which is a sheaf map. □

We now cover the other construction we know from algebra.

2.1.8 Direct product of modules

Definition 2.1.24. (**Direct product of modules**) Let $\{\mathcal{F}\}_{i\in I}$ be a family of $\mathcal{O}_X$-modules. The direct product of them is defined to be the sheaf

$$U \mapsto \prod_{i\in I} \mathcal{F}_i(U)$$

with product of restrictions as its restriction. Indeed, it is immediate it is a sheaf and that the canonical map $\eta_U : \mathcal{O}_X(U) \times \prod_{i\in I} \mathcal{F}_i(U) \to \prod_{i\in I} \mathcal{F}_i(U)$ mapping as $(c, (s_i)_{i\in I}) \mapsto (c \cdot s_i)_{i\in I}$ makes $\prod_{i\in I} \mathcal{F}_i$ an $\mathcal{O}_X$-module. If $\mathcal{F}_i = \mathcal{F}$ for all $i \in I$, then we denote

$$\prod_{i\in I} \mathcal{F} = \mathcal{F}^{\prod I} = \mathcal{F}^I$$

as is usually done in algebra.

We now define tensor product of two $\mathcal{O}_X$-modules.

2.1.9 Tensor product of modules

Definition 2.1.25. (**Tensor product of modules**) Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. The tensor product of $\mathcal{F}$ and $\mathcal{G}$ is given by the sheafification of the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U),$$

denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, as the following lemma shows.

Lemma 2.1.26. *$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is an $\mathcal{O}_X$-module and $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ for each $x \in X$.*

Proof. The second statement is immediate from Lemma **??**. The $\mathcal{O}_X$-module structure is the obvious one: pick any open $U \subseteq X$ and then consider the map

$$\begin{aligned} \eta_U : \mathcal{O}_X(U) \times (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) &\longrightarrow (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) \\ (a, s : U \to \amalg_{x\in U} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x) &\longmapsto as \end{aligned}$$

where $as(x) = a_x s(x)$. One easily checks that this defines a well-defined natural sheaf map. □

A simple observation also yields the usual identity we know from modules.

Lemma 2.1.27. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,*

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}.$$

Proof. Consider the map

$$\eta : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \longrightarrow \mathcal{F}$$

given on an open $U \subseteq X$ by the map corresponding to the following natural isomorphism (Theorem **??**)

$$\eta_U : \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \xrightarrow{\cong} \mathcal{F}(U).$$

This yields the similar isomorphic map on stalks via Lemma 2.1.26 to yield the result via Theorem **??**, 3. □

Tensor product of modules is obviously commutative.

Lemma 2.1.28. *Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. Then, $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{G} \otimes \mathcal{F}$.*

Proof. Construct the map $\tilde{\eta} : \mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{F}$ as the unique map corresponding to the following

$$\begin{tikzcd} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \arrow[r, "\eta_U", "\cong"'] \arrow[dr, "j_U \eta_U"'] & \mathcal{G}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \arrow[d, "j_U"] \\ & (\mathcal{G} \otimes \mathcal{F})(U) \end{tikzcd} .$$

This map on the stalks gives the usual twist isomorphism $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x \cong \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$. □

2.1.10 Free, locally free & finite locally free $\mathcal{O}_X$-modules

Definition 2.1.29. (**Free, locally free and finite locally free modules**) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then,

1. $\mathcal{F}$ is called *free* if $\mathcal{F} \cong \mathcal{O}_X^{(I)}$ for some index set I,
2. $\mathcal{F}$ is called *locally free* if for all $x \in X$, there exists open $U \ni x$ such that $\mathcal{F}_{|U} \cong \mathcal{O}_{X|U}^{(I_x)}$ where I_x is an indexing set depending on x,
3. $\mathcal{F}$ is called *finite locally free* if $\mathcal{F}$ is locally free and the indexing set I_x is finite for each $x \in X$. If $I_x = I$ and I has size n, then we say that $\mathcal{F}$ is *locally free of rank n*.

We now observe that the hom sheaf of two locally free modules of finite rank is again locally free of finite rank.

Lemma 2.1.30. *Let $\mathcal{F}, \mathcal{E}$ be two locally free $\mathcal{O}_X$-modules of ranks n and m respectively. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}$ are both locally free module of rank nm.*

Proof. For each $x \in X$, there exists an open set $U \ni x$ such that $\mathcal{F}_{|U} \cong \mathcal{O}_{X|U}^n$ and $\mathcal{E}_{|U} \cong \mathcal{O}_{X|U}^m$. We then observe the following

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E})(U) = \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{F}_{|U}, \mathcal{E}_{|U}\right) \cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}^n, \mathcal{O}_{X|U}^m\right) \cong \mathcal{O}_{X|U}^{nm}$$

where the last isomorphism can be established easily by reducing to the usual module case ($\mathrm{Hom}_R(R^n, R^m) \cong R^{nm}$).

For tensor, we proceed similarly as above. By replacing X by U, we need only show that $\mathcal{O}_X^n \otimes \mathcal{O}_X^m \cong \mathcal{O}_X^{nm}$. Indeed, by universal property of sheafification, it is sufficient to describe a map of presheaves $(\mathcal{O}_X^n \otimes \mathcal{O}_X^m)^- \to \mathcal{O}_X^{nm}$ which is an isomorphism on stalks. The usual isomorphism $R^n \otimes R^m \to R^{nm}$ gives such a map of presheaves, as required. □

An important corollary of the above lemma is as follows.

Corollary 2.1.31. *Let $\mathcal{F}$ be be a locally free module of rank n. Then the dual $\mathcal{F}^\vee$ is locally free of rank n.*

Proof. By Lemma 2.1.30, $\mathcal{F}^\vee$ is locally free of rank n. □

One may think of finite locally free modules as those modules which are locally free in the usual sense. Consequently, these modules satisfy global version of the properties enjoyed by the usual notion of free modules, as the following result shows.

Proposition 2.1.32. [5] *Let $\mathcal{E}$ be a finite locally free of rank n. Then,*

1. $\mathcal{E}^{\vee\vee} \cong \mathcal{E}$.
2. *For any $\mathcal{O}_X$-module $\mathcal{F}$, we have*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Proof. As $\mathcal{E}$ is locally of free of rank n, therefore there is an open cover $\{U_i\}$ of X such that $\mathcal{E}_{|U_i} \cong \mathcal{O}^n_{X|U_i}$. Let $\{B_j\}$ be a basis of X where each B_j is in some U_i. Consequently, we reduce to constructing an isomorphism in each case only as sheaves over the basis $\{B_j\}$.

1. Indeed, as each B_j is in some U_i, therefore $\mathcal{E}_{|B_j} \cong \mathcal{O}^n_{X|B_j}$. Consequently, we get the following isomorphisms for any $U \in \{B_j\}$

$$\begin{aligned}
\mathcal{E}^{\vee\vee}(U) &= \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)\big|_U, \mathcal{O}_{X|U}\right) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{O}_X)\big|_U, \mathcal{O}_{X|U}\right) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left((\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X))^n\big|_U, \mathcal{O}_{X|U}\right) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{O}^n_{X|U}, \mathcal{O}_{X|U}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{O}_{X|U}\right)^n \\
&\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)(U)^n \\
&\cong \mathcal{O}_X(U)^n \\
&\cong \mathcal{E}(U),
\end{aligned}$$

and its naturality with resepect to restrictions is evident.

2. Pick any $U \in \{B_j\}$. We then have

$$\begin{aligned}
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})(U) &\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{E}_{|U}, \mathcal{F}_{|U}\right) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}^n_{X|U}, \mathcal{F}_{|U}\right) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X|U}}\left(\mathcal{O}_{X|U}, \mathcal{F}_{|U}\right)^n \\
&\cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})(U)^n \\
&\cong \mathcal{F}(U)^n \\
&\cong (\mathcal{O}^n_X \otimes_{\mathcal{O}_X\mathcal{F}})(U)
\end{aligned}$$

by Lemma 2.1.27. The fact that this isomorphism is natural with respect to restrictions is immediate. ∎

[5] Exercise II.5.1 of Hartshorne.

2.1.11 Invertible modules and the Picard group

Definition 2.1.33. (**Invertible modules**) An $\mathcal{O}_X$-module $\mathcal{L}$ is said to be invertible if it is locally free of rank 1.

The name is justified by the fact that the set of all invertible modules upto isomorphism forms a group under tensor product and is one of the important invariants of a (ringed) space amongst many others. We now show that indeed this forms a group. We will drop the subscript $\mathcal{O}_X$ from the tensor product, for clarity, in the following.

Proposition 2.1.34. *Let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ be invertible $\mathcal{O}_X$-modules. Then,*

1. *$\mathcal{L}_1 \otimes \mathcal{L}_2$ is invertible,*
2. *$(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{L}_3 \cong \mathcal{L}_1 \otimes (\mathcal{L}_2 \otimes \mathcal{L}_3)$,*
3. *$\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{O}_X$.*

Proof. 1. This is a local question, so pick $x \in X$ and an open set $U \ni x$ such that $\mathcal{L}_{1|U} \cong \mathcal{O}_{X|U} \cong \mathcal{L}_{2|U}$. We wish to construct a natural map $(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2)(U) \to \mathcal{O}_X(U)$ which is an isomorphism. By Theorem **??**, it suffices to show a natural isomorphism $\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U) \to \mathcal{O}_X(U)$. This is constructed quite easily as $\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U) \cong \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \cong \mathcal{O}_X(U)$. Thus we just need to consider $\mathrm{id}_{\mathcal{O}_X(U)}$.
2. This is again a local question, which can be answered by establishing an isomorphism (by using Theorem **??**)

$$(\mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_2(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_3(U) \cong \mathcal{L}_1(U) \otimes_{\mathcal{O}_X(U)} (\mathcal{L}_2(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}_3(U))$$

for any open $U \subseteq X$, but that is an immediate observation from algebra.
3. By Corollary 2.1.31, we have that $\mathcal{L}^\vee$ is invertible. By Theorem **??**, 3, the result would follow if we can show that there is a natural $\mathcal{O}_X$-linear map $\varphi : \mathcal{L}^\vee \otimes \mathcal{L} \to \mathcal{O}_X$ such that for each point $x \in X$ there exists an open set $x \in U \subseteq X$ such that on U, φ yields an $\mathcal{O}_X(U)$-linear isomorphism $(\mathcal{L}^\vee \otimes \mathcal{L})(U) \cong \mathcal{O}_X(U)$. We may take U small enough so that $\mathcal{L}^\vee_{|U} \cong \mathcal{O}_{X|U} \cong \mathcal{L}_{|U}$. Thus, after replacing X by U, we may assume $\mathcal{L} = \mathcal{O}_X = \mathcal{L}^\vee$. By Lemmas 2.1.20 and 2.1.27, we obtain the following isomorphisms

$$\mathcal{L}^\vee \otimes \mathcal{L} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \otimes \mathcal{O}_X \cong \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathcal{O}_X \cong \mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X.$$

This can easily be promoted to a sheaf map. □

Definition 2.1.35. (**Picard group of X**) The Picard group of X is defined to be the set of all isomorphism classes of invertible modules with the operation of tensor product. We denote this by

$$\mathrm{Pic}(X)$$

The Proposition 2.1.34 and Lemma 2.1.28 shows that $\mathrm{Pic}(X)$ is indeed an abelian group.

2.1.12 Direct and inverse image modules

In this and the next sections, we show how the modules behave under map of ringed spaces.

Definition 2.1.36. (**Direct image**) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the direct image of $\mathcal{F}$ under f is the direct image sheaf $f_*\mathcal{F}$ which is again an $\mathcal{O}_Y$-module given by the following composition

$$\mathcal{O}_Y \times f_*\mathcal{F} \xrightarrow{f^\flat \times \mathrm{id}} f_*\mathcal{O}_X \times f_*\mathcal{F} \xrightarrow{f_*m} f_*\mathcal{F}$$

where $m : \mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$ is the $\mathcal{O}_X$-module structure on $\mathcal{F}$. Note that f_* commutes with products as f_* is a right-adjoint.

The inverse image of a module, on the other hand, is an involved construction.

Definition 2.1.37. (**Inverse image**) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. The inverse image of $\mathcal{G}$ is defined to be the map

$$f^*\mathcal{G} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

which is indeed an $\mathcal{O}_X$-module as the following lemma shows.

Lemma 2.1.38. *The sheaf $f^*\mathcal{G}$ is an $\mathcal{O}_Y$-module.*

Proof. We need to show three statements:

1. $\mathcal{O}_X$ is an $f^{-1}\mathcal{O}_Y$-module.
2. $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module.
3. $f^*\mathcal{G}$ is an $\mathcal{O}_X$-module.

Statement 1 follows from the following composition

$$f^{-1}\mathcal{O}_Y \times \mathcal{O}_X \xrightarrow{f^\sharp \times \mathrm{id}} \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

where the latter is just the multiplication structure on $\mathcal{O}_X$. Statement 2 follows from $\mathcal{O}_Y$-module structure on $\mathcal{G}$ and the fact that $f^{-1}(\mathcal{G} \times \mathcal{G}') = f^{-1}\mathcal{G} \times f^{-1}\mathcal{G}'$ for two sheaves $\mathcal{G}, \mathcal{G}'$ over Y. Indeed, the latter follows from the fact that $f^+(\mathcal{G} \times \mathcal{G}') = f^+\mathcal{G} \times f^+\mathcal{G}'$, which in turn follows from the fact that filtered colimit commutes with finite limits. Statement 3 now follows immediately. □

We now state an important result, that is $f_* \vdash f^*$.

Proposition 2.1.39. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces. Then,*

$$\begin{tikzcd} \mathbf{Mod}(\mathcal{O}_Y) \arrow[r, shift left=2, "f^*"] \arrow[r, phantom, "\perp"] & \mathbf{Mod}(\mathcal{O}_X) \arrow[l, shift left=2, "f_*"] \end{tikzcd} .$$

In other words, we have a natural isomorphism of groups

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Proof. Omitted. □

2.1.13 Sums & intersections of submodules

2.1.14 Modules generated by sections

2.1.15 Inverse limit

Do Hartshorne Exercise 1.12 as well.

2.1.16 Direct limit

Do Hartshorne Exercise 1.11 as well.

2.1.17 Tensor, symmetric & exterior powers

We now define $T(\mathcal{F})$, $S(\mathcal{F})$ and $\wedge(\mathcal{F})$ for a module $\mathcal{F}$.

Definition 2.1.40 ($T(\mathcal{F}), \mathrm{Sym}(\mathcal{F})$ **and** $\wedge(\mathcal{F})$)**.** Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The sheafification of presheaf $U \mapsto T(\mathcal{F}(U))$ or $\mathrm{Sym}(\mathcal{F}(U))$ or $\wedge(\mathcal{F}(U))$ is denoted to be $T(\mathcal{F})$ or $S(\mathcal{F})$ or $\wedge(\mathcal{F})$ called the tensor or symmetric or exterior algebra, respectively. This is an $\mathcal{O}_X$-algebra, i.e. a sheaf of rings which is an $\mathcal{O}_X$-module. Moreover, we have

$$T(\mathcal{F}) = \bigoplus_{n\geq 0} T^n(\mathcal{F})$$

where $T^n(\mathcal{F})$ is the sheafification of $U \mapsto T^n(\mathcal{F}(U))$. Note that this makes sense as sheafification is a left adjoint, so it commutes with all colimits. We call $T^n(\mathcal{F})$ the n^{th}-tensor power of $\mathcal{F}$. Similarly, we define $\mathrm{Sym}^n(\mathcal{F})$ and $\wedge^n(\mathcal{F})$.

Lemma 2.1.41. *If $\mathcal{F} = \mathcal{O}_X^n$ is a free $\mathcal{O}_X$-module of rank n, then*
1. *$T^r(\mathcal{F}) \cong \mathcal{O}_X^{n^r}$,*
2. *$\mathrm{Sym}^r(\mathcal{F}) \cong \mathcal{O}_X^{{}^{n+r-1}C_{n-1}}$,*
3. *$\wedge^r(\mathcal{F}) \cong \mathcal{O}_X^{{}^nC_r}$.*

Proof. All three isomorphisms are obtained by defining a corresponding map of presheaves which is an isomorphism on stalks, where this map is induced from the usual map in algebra:

$$\begin{aligned} R^n \otimes R^m &\cong R^{nm} \\ \mathrm{Sym}^r(R^n) &\cong R^{{}^{n+r-1}C_r} \\ \wedge^r(R^n) &\cong R^{{}^nC_r}. \end{aligned}$$

Then the corresponding map on sheaves induced by universal property of sheafification is an isomorphism as it is so on stalks. ☐

We now indulge in generalizing some local properties of tensor algebra to this global case. We first have the standard observation of instantiating these definitions on the finite locally free case, which generalizes the usual tensor calculations of free modules.

Lemma 2.1.42. [6] *Let $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-module of rank n. Then, $T^r(\mathcal{F}), \mathrm{Sym}^r(\mathcal{F})$ and $\wedge^r(\mathcal{F})$ is a finite locally free $\mathcal{O}_X$-module of rank n^r, ${}^{n+r-1}C_{n-1}$ and ${}^nC_r$ respectively.*

Proof. Let $\{U_\alpha\}$ be an open cover of X where $\mathcal{F}$ is $\mathcal{O}^n_{X|U_\alpha}$ for each α. Let $\mathcal{B}$ be a basis of X such that for any $B \in \mathcal{B}$, we have $B \subseteq U_\alpha$ for some α. Observe that $\mathcal{F}_{|B} \cong \mathcal{O}^n_{X|B}$. Hence, we may replace X by B to assume that $\mathcal{F}$ is free of rank n. The result now follows from Lemma 2.1.41. ☐

Another global phenomenon that is borrowed by tensor calculation of free modules is the perfect pairing of wedge product.

[6] Exercise II.5.16 of Hartshorne.

2.1.18 $\mathcal{E}xt$ module

2.1.19 $\mathcal{T}or$ module

2.2 The abelian category of $\mathcal{O}_X$-modules

We now show an important result that category of $\mathcal{O}_X$-modules over any ringed space is an abelian category (thus we can do whole of homological algebra over it!). We have essentially done everything, but we write it here for clear reference.

Theorem 2.2.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathbf{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules is an abelian category.*

Proof. For any two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, we have $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an abelian group where for any two $f, g \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, the sum $h = f + g$ is defined to as follows: pick any open $U \subseteq X$ and define $h_U = f_U + g_U$. This is an $\mathcal{O}_X$-linear sheaf map because f and g are. Hence $\mathbf{Mod}(\mathcal{O}_X)$ is preadditive. Moreover $\mathbf{Mod}(\mathcal{O}_X)$ is additive. This is what we did in the preceding section while defining finite products of $\mathcal{O}_X$-modules. The preceding section also shows that $\mathbf{Mod}(\mathcal{O}_X)$ has all kernels and cokernels. Consequently, we need only show that the for any $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Mod}(\mathcal{O}_X)$, $\operatorname{CoIm}(f) \cong \operatorname{Im}(f)$. Indeed, this is a local question and can be thus immediately seen by first isomorphism theorem. More precisely, we need only construct this isomorphism on a basis of X, where the canonical map $\operatorname{CoIm}(f) \to \operatorname{Im}(f)$ is an isomorphism by first isomorphism theorem. This completes the proof. □

Theorem 2.2.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the abelian category $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We wish to find an injective $\mathcal{O}_X$-module $\mathcal{I}$ such that $\mathcal{F} \hookrightarrow \mathcal{I}$. First note that for each $x \in X$, we have an injective $\mathcal{O}_{X,x}$-module I_x such that $\mathcal{F}_x \hookrightarrow I_x$ by Theorem **??**. Observe that I_x is a sheaf over $i : \{x\} \hookrightarrow X$. Let $\mathcal{I} = \prod_{x \in X} i_* I_x$ be the corresponing $\mathcal{O}_X$-module. We claim that $\mathcal{I}$ is an injective $\mathcal{O}_X$-module and there is an injective map $\mathcal{F} \hookrightarrow \mathcal{I}$.

To see that there is an injective map $\mathcal{F} \hookrightarrow \mathcal{I}$, we claim the following three isomorphisms

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) \cong \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_* I_x) \cong \prod_{x \in X} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x).$$

The first isomorphism is immediate from limit preserving property of covariant hom. The second isomorphism is obtained by the following isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_* I_x) \cong \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) \tag{$*$}$$

for each $x \in X$. Indeed, this follows from the maps $f \mapsto f_x$ and $(\tilde{\kappa} : \mathcal{F} \to i_* I_x) \leftarrow\!\shortmid (\kappa : \mathcal{F}_x \to I_x)$ where $\tilde{\kappa}$ is defined on an open set $U \subseteq X$ as $\tilde{\kappa}_U : \mathcal{F}(U) \to I_x$ mapping as $s \mapsto \kappa((U, s)_x)$. These are clearly inverses of each other. It then follows that a map $\mathcal{F} \to \mathcal{I}$ is equivalent to a collection of maps $\mathcal{F}_x \to I_x$ and since we have $\mathcal{F}_x \hookrightarrow I_x$, therefore we obtain a unique injective map $\mathcal{F} \hookrightarrow \mathcal{I}$.

Finally, we claim that $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact as a functor into the category of abelian groups. To this end, by left exactness of hom, we need only show that this is right exact. This immediately follows from isomorphism $(*)$ and I_x being injective and that product of surjective homomorphisms is surjective. This completes the proof. □

3 Bundles

We give here the general theory of fiber, principal and vector bundles. When the need arises, we will instantiate this into different areas (like in the chapter on differential geometry). The material in previous chapter will allow a very united way of looking at the notion of bundles, and will start portraying the intimate connection that bundles and cohomology has.

3.1 Generalities on twisting atlases

Let $p : E \to B$ be a map of topological spaces/manifolds together with a specified subsheaf of groups $\mathcal{G} \subseteq \mathcal{A}_B(E) \in \mathbf{Sh}(B)$ where $\mathcal{A}_B(E)$ is the sheaf of homeomorphisms/isomorphisms over B; for any open $U \subseteq B$, the group $\mathcal{A}_B(E)(U)$ consists of all homeomorphisms/isomorphisms $\varphi : p^{-1}(U) \to p^{-1}(U)$ such that $p \circ \varphi = p$.

The tuple $(p : E \to B, \mathcal{G})$ is the pre-datum for defining $(p, \mathcal{G})$-twisting atlas for a map $\pi : X \to B$.

Definition 3.1.1 (**$(p, \mathcal{G})$-twisting atlas for a map**)**.** Let $p : E \to B$ be a map and $\mathcal{G}$ be a subsheaf of groups $\mathcal{G} \subseteq \mathcal{A}_B(E)$. Let $\pi : X \to B$ be a map. Then, a $(p, \mathcal{G})$-twisting atlas for π is a family $(U_i, h_i)_{i \in I}$ where $\{U_i\}_{i \in I}$ is an open cover of B and $h_i : \pi^{-1}(U_i) \xrightarrow{\cong} p^{-1}(U_i)$ is an isomorphism over U_i such that for any $i, j \in I$, denoting $U_{ij} = U_i \cap U_j$, we have

$$\begin{tikzcd} p^{-1}(U_{ij}) \arrow[rr, shift right, "h_j^{-1}\big|_{p^{-1}(U_{ij})}"'] \arrow[dr, "p"'] & & \pi^{-1}(U_{ij}) \arrow[ll, shift right, "h_i|_{\pi^{-1}(U_{ij})}"'] \arrow[dl, "\pi"] \\ & U_{ij} & \end{tikzcd}$$

and from which we require that

$$h_{ij} = h_i|_{\pi^{-1}(U_{ij})} \circ h_j^{-1}\Big|_{p^{-1}(U_{ij})}$$

is a section in $\mathcal{G}(U_{ij})$. We then call $\pi : X \to B$ together with (U_i, h_i) *a twist of $p : E \to B$ with structure sheaf $\mathcal{G}$.*

Using this, we may define a general notion of a bundle.

Definition 3.1.2 (**Bundles**)**.** Let $\pi : X \to B$ be a map, F a space/manifold and $p : B \times F \to B$ be the projection map onto first coordinate. Then π is a bundle with fiber F if there is a $(p, \mathcal{A}_B(B \times F))$-twisting atlas for π. Equivalently, π is a bundle with fiber F if it is a twist of $p : B \times F \to B$ with full structure sheaf $\mathcal{A}_B(B \times F)$.

Remark 3.1.3. Let $\pi : X \to B$ be a bundle with fiber F. Consequently we have a $\mathcal{A}_B(B \times F)$-twisting atlas of $p : B \times F \to B$ denoted (U_i, h_i), where $h_i : \pi^{-1}(U_i) \to p^{-1}(U_i)$ is an isomorphism over U_i such that the transition maps $h_{ij} : p^{-1}(U_{ij}) = U_{ij} \times F \to U_{ij} \times F = p^{-1}(U_{ij})$ is just an isomorphism over U_{ij} (i.e. $h_{ij} \in \mathcal{A}_B(B \times F)(U_{ij})$).

4 Differential forms and de-Rham cohomology

Do this from Section 8.6 and Section 10.4 of Wedhorn, via sheaf cohomology. Add motivation from courses.

4.1 Differential forms on $\mathbb{R}^n$

We first discuss differential forms on $\mathbb{R}^n$ as this provides clear and sufficient motivation for the abstract treatment of differential forms in all other places where it is used. We begin by defining the main ingredients. The material of Section **??** is used in the following.

Definition 4.1.1. (**Coordinate forms on** $\mathbb{R}^n$) Fix $n \in \mathbb{N}$. Let $V = \mathbb{R}^n$ be the n-dimensional $\mathbb{R}$-module. The functional

$$\begin{aligned} dx_i : V &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\longmapsto x_i \end{aligned}$$

is called the i^{th}-coordinate form on V, for each $i = 1, \dots, n$. Note that dx_i is a 1-form/1-tensor, i.e. $dx_i \in M^1(V) = V^*$. Observe that dx_i is the dual basis of V^* corresponding to standard basis e_i of V.

Next, we define a multilinear map which for each choices of axes, gives the volume of the parallelopiped obtained by the projection along those axes, given a parallelopiped spanned by some vectors.

Definition 4.1.2. (**Projection forms on** $\mathbb{R}^n$) Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Let $V = \mathbb{R}^n$ be the n-dimensional $\mathbb{R}$-module. Let $I = (i_1, \dots, i_k)$ be an ordered k-tuple where $1 \leq i_j \leq n$ for each $j = 1, \dots, k$. Then, we define the I-projection form as

$$dx_I := \pi_k(dx_{i_1} \otimes \ldots \otimes dx_{i_k}) = D_I$$

which is an alternating k-form on V, that is $dx_I \in \Lambda^k(V)$ (see Example **??**). More explicitly, it is given by the following k-linear form on V

$$\begin{aligned} dx_I : V \times \cdots \times V &\longrightarrow \mathbb{R} \\ (v_1, \dots, v_k) &\longmapsto \det \begin{bmatrix} dx_{i_1}(v_1) & dx_{i_2}(v_1) & \dots & dx_{i_k}(v_1) \\ dx_{i_1}(v_2) & dx_{i_2}(v_2) & \dots & dx_{i_k}(v_2) \\ \vdots & \vdots & \dots & \vdots \\ dx_{i_1}(v_k) & dx_{i_2}(v_k) & \dots & dx_{i_k}(v_k) \end{bmatrix}. \end{aligned}$$

Remark 4.1.3. Recall from Theorem **??** that $\Lambda^k(V)$ has basis given by dx_I for distinct increasing k-tuples from $1, \dots, n$. Thus, $\{dx_I\}_I$ forms an $\mathbb{R}$-basis of $\Lambda^k(V)$ of size ${}^nC_k$.

Remark 4.1.4. Recall that wedge product of forms is given by the following (where one defines them only on the basis elements)

$$\begin{aligned} \Lambda^k(V) \times \Lambda^l(V) &\longrightarrow \Lambda^{k+l}(V) \\ (dx_I, dx_J) &\longmapsto dx_I \wedge dx_J := dx_{(I,J)} \end{aligned}$$

where recall that $dx_{(I,J)}$ will be zero if there is any index common in I and J (see Definition **??**), where I, J are increasing tuples of indices from $\{1, \ldots, n\}$ of lengths k and l respectively. From the above, we see that for any alternating k-form $\omega = \sum_I a_I dx_I$ and alternating l-form $\eta = \sum_J b_j dx_J$, their wedge product is defined as

$$\omega \wedge \eta = \sum_J \sum_I a_I b_J (dx_I \wedge dx_J).$$

Remark 4.1.5. Let $U \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. Observe that $\mathcal{C}^\infty(U)$, the ring of smooth $\mathbb{R}$-valued functions on U, is an $\mathbb{R}$-algebra. In the same vein, we know that alternating k-forms $\Lambda^k(\mathbb{R}^n)$ forms an $\mathbb{R}$-vector space of dimension ${}^nC_k$ (see Theorem **??**).

Definition 4.1.6. (**Differential k-forms**) Let $U \subseteq \mathbb{R}^n$ be an open set and $0 \leq k \leq n$. The module of differential k-forms is defined to be the following $\mathbb{R}$-vector space

$$\Omega_U^k = \Lambda^k(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathcal{C}^\infty(U).$$

As $\Lambda^k(\mathbb{R}^n)$ is a free $\mathbb{R}$-module with rank ${}^nC_k$, therefore Ω_U^k is a free $\mathcal{C}^\infty(U)$-module of rank ${}^nC_k$.

Remark 4.1.7. Observe that $\{\Omega_U^k\}$ obtains the wedge product structure from the wedge product on $\{\Lambda^k(\mathbb{R}^n)\}$ as we may define for $\omega = \sum_I f_I dx_I \in \Lambda^k(\mathbb{R}^n)$ and $\eta = \sum_J g_J dx_J$ the following

$$\begin{aligned}\omega \wedge \eta &:= \left(\sum_I f_I dx_I\right) \wedge \left(\sum_J g_J dx_J\right) \\ &= \sum_I \sum_J f_I g_J dx_I \wedge dx_J.\end{aligned}$$

Thus, $\bigoplus_{k \geq 0} \Omega_U^k$ forms a graded $\mathcal{C}^\infty(U)$-algebra.

Remark 4.1.8. An arbitrary element $\omega \in \Omega_U^k$ is called a differential k-form over U and is written as

$$\omega = \sum_{I \in X_k} f_I(x_1, \ldots, x_n) dx_I$$

where X_k is the set of size ${}^nC_k$ of all k-combinations in increasing order of $\{1, \ldots, n\}$ and $f_I \in \mathcal{C}^\infty(U)$ is a smooth function. Observe that $\Omega_U^0 = \mathcal{C}^\infty(U)$.

We now construct the exterior derivative which will be a differential over the chain complex Ω_U^k, as we will see soon.

Definition 4.1.9. (**Exterior derivative**) Let $U \subseteq \mathbb{R}^n$ be an open subset and $\{\Omega_U^k\}_{k \in \mathbb{N}}$ be the modules of differential k-forms. For each $k \in \mathbb{N} \cup \{0\}$, we define a map $d : \Omega_U^k \to \Omega^{k+1_U}$ as follows. Define for $k = 0$ the following

$$\begin{aligned}d : \Omega_U^0 &\longrightarrow \Omega_U^1 \\ f &\longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\end{aligned}$$

where since $f \in \mathcal{C}^\infty(U)$ is smooth, therefore so is $\partial f/\partial x_i$. Further, since $dx_i \in \Lambda^1(\mathbb{R}^n)$, therefore the above is well-defined. For $k \geq 1$, we define d as follows

$$\begin{aligned} d : \Omega_U^k &\longrightarrow \Omega_U^{k+1} \\ \omega = \sum_{I \in X_k} f_I dx_I &\longmapsto d\omega = \sum_{I \in X_k} df_I \wedge dx_I \end{aligned}$$

where $dx_I \in \lambda^k(\mathbb{R}^n)$. Observe that $df_I \in \Omega_U^1$, thus indeed $df_I \wedge dx_I \in \Omega_U^{k+1}$. This map d is called the exterior derivative of differential forms.