Homological Methods

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1 The setup : abelian categories

Methods employed in homological algebra comes in handy to attack certain type of local-global problems in geometry. We would like to discuss some foundational homological algebra in this chapter in the setting of additive and abelian categories. The main goal is not to illuminate foundations but to quickly get to the working theory which can allow us to develop deeper results elsewhere in this notebook. Using the Freyd-Mitchell embedding theorem, we can always assume that any (small) abelian category $\mathbf{A}$ is a full subcategory of $\mathbf{Mod}(R)$ over some ring R. Thus we will freely do the technique of diagram chasing in the following, implicitly assuming $\mathbf{A}$ to be embedded in a module category. Consequently, the main example to keep in mind throughout this chapter is of-course the category of R-modules, $\mathbf{Mod}(R)$.

Let us begin with the basic definitions. Let $\mathbf{A}$ be a category. Then $\mathbf{A}$ is said to be *preadditive* if for any $x, y \in \mathbf{A}$, the homset $\mathrm{Hom}\,(x, y)$ is an abelian group and the composition $\mathrm{Hom}\,(x, y) \times \mathrm{Hom}\,(y, z) \to \mathrm{Hom}\,(x, z)$ is a bilinear map. For two preadditive categories $\mathbf{A}, \mathbf{B}$ a functor $F : \mathbf{A} \to \mathbf{B}$ is called *additive* if for all $x, y \in \mathbf{A}$, the function $\mathrm{Hom}_{\mathbf{A}}\,(x, y) \to \mathrm{Hom}_{\mathbf{B}}\,(Fx, Fy)$ is a group homomorphism.

Let $\mathbf{A}$ be a preadditive category and $f : x \to y$ be an arrow. This mean for any two object $w, z \in \mathbf{A}$, there is a zero arrow $0 \in \mathrm{Hom}\,(w, z)$. Then, we can define the usual notions of algebra as follows.

1. $i : \mathrm{Ker}\,(f) \to x$ is defined by the following universal property w.r.t. $fi = 0$:

$$\begin{tikzcd} x' \arrow[d, dashed, "\exists!"'] \arrow[dr, "\forall g \text{ s.t. } fg=0"] & & \\ \mathrm{Ker}\,(f) \arrow[r, "i"] & x \arrow[r, "f"] & y \end{tikzcd}$$

2. $j : y \to \mathrm{CoKer}\,(f)$ defined by the following universal property w.r.t. $jf = 0$:

$$\begin{tikzcd} & & y' \\ x \arrow[r, "f"] & y \arrow[ur, "\forall g \text{ s.t. } gf=0"] \arrow[r, "j"] & \mathrm{CoKer}\,(f) \arrow[u, dashed, "\exists!"'] \end{tikzcd} .$$

3. $k : x \to \mathrm{CoIm}\,(f)$ is defined to be the cokernel of the kernel map $i : \mathrm{Ker}\,(f) \to x$.
4. $l : \mathrm{Im}\,(f) \to y$ is defined to be the kernel of the cokernel map $j : y \to \mathrm{CoKer}\,(f)$.

Hence, for each $f : x \to y$ in a preadditive category $\mathbf{A}$, we can contemplate the following four type of maps:

$$\begin{tikzcd} \mathrm{Ker}\,(f) \arrow[d] & \mathrm{CoKer}\,(f) \\ x \arrow[r, "f"] \arrow[d] & y \arrow[u] \\ \mathrm{CoIm}\,(f) & \mathrm{Im}\,(f) \arrow[u] \end{tikzcd}$$

Lemma 1.0.1. *In a preadditive category, if a coproduct $x \oplus y$ exists, then so does the product $x \times y$ and vice versa. In such a case, $x \oplus y \cong x \times y$.* □

A preadditive category $\mathbf{A}$ is said to be *additive* if it contains all finite products, including the empty ones. By the above lemma, we require zero objects and sums of objects to exist.

An additive category $\mathbf{A}$ is said to be *abelian* if all kernels and cokernels exist and the natural map for each $f : x \to y$ in $\mathbf{A}$

$$\mathrm{CoIm}\,(f) \to \mathrm{Im}\,(f)$$

is an isomorphism. This intuitively means that the first isomorphism theorem holds in abelian categories by definition.

2 Homology, resolutions and derived functors

In this section, we shall discuss basic topics of homological algebra in abelian categories, which we shall need to setup the sheaf cohomology in geometry and Lie group cohomology in algebra and etcetera, etcetera.

2.1 Homology

We first define cochain complexes and maps, cohomology and homotopy of such. Since this section is mostly filled with *trivial matters*, therefore we shall allow ourselves to be a bit sketchy with proofs.

Definition 2.1.1. (**Cochain complexes, maps and cohomology**) A cochain complex $A^\bullet$ is a sequence of object $\{A^i\}_{i\in\mathbb{Z}}$ with a map $d^i : A^i \to A^{i+1}$ called the coboundary maps which satisfies $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. A map $f : A^\bullet \to B^\bullet$ of cochain complexes is defined as a collection of maps $f^i : A^i \to B^i$ such that the following commutes

$$\begin{tikzcd} A^i \arrow[r, "d^i"] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d^i"'] & B^{i+1} \end{tikzcd}.$$

That is, $d^i f^i = f^{i+1} d^i$ for each $i \in \mathbb{Z}$. For a cochain complex $A^\bullet$, we define the i^{th} cohomology object as the quotient

$$h^i(A^\bullet) := \operatorname{Ker}\left(d^i\right)/\operatorname{Im}\left(d^{i-1}\right).$$

With the obvious notion of composition, we thus obtain a category of cochain complexes $\operatorname{coCh}(\mathbf{A})$ over the abelian category $\mathbf{A}$.

We now show that h^i forms a functor over $\operatorname{coCh}(\mathbf{A}\ )$.

Lemma 2.1.2. *Let $\mathbf{A}$ be an abelian category. The i^{th}-cohomology assignment is a functor*

$$\begin{aligned} h^i : \operatorname{coCh}(\mathbf{A}\ ) &\longrightarrow \mathbf{A} \\ A^\bullet &\longmapsto h^i(A^\bullet). \end{aligned}$$

Proof. For a map of complexes $f : A^\bullet \to B^\bullet$, we first define the map $h^i(f)$

$$\begin{aligned} h^i(f) : h^i(A^\bullet) &\longrightarrow h^i(B^\bullet) \\ a + \operatorname{Im}\left(d^{i-1}\right) &\longmapsto f^i(a) + \operatorname{Im}\left(d^{i-1}\right). \end{aligned}$$

This is well defined group homomorphism. Further, it is clear that this is functorial. $\square$

With this, we obtain the cohomology long-exact sequence.

Lemma 2.1.3. *Let $\mathbf{A}$ be an abelian category and $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ a short-exact sequence in $\operatorname{coCh}(\mathbf{A})$. Then there is a map $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ for each $i \in \mathbb{Z}$ such that the following is a long exact sequence*

$$\begin{tikzcd} & & {} \arrow[dll, dotted] \\ h^i(A^\bullet) \arrow[r] & h^i(B^\bullet) \arrow[r] & h^i(C^\bullet) \arrow[dll, "\delta^i"] \\ h^{i+1}(A^\bullet) \arrow[r] & h^{i+1}(B^\bullet) \arrow[r] & h^{i+1}(C^\bullet) \arrow[dll, dotted] \\ {} & & \end{tikzcd}.$$

Proof. (*Sketch*) The proof relies on chasing an element $c \in \operatorname{Ker}(d)$ of C^i till we obtain an element $a \in A^{i+1}$ in $\operatorname{Ker}(d)$, in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "d"'] & B^{i-1} \arrow[r] \arrow[d, "d"'] & C^{i-1} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{i} \arrow[r] \arrow[d, "d"'] & B^{i} \arrow[r] \arrow[d, "d"'] & C^{i} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] & B^{i+1} \arrow[r] & C^{i+1} \arrow[r] & 0
\end{tikzcd}$$

The chase is straightforward and is thus omitted. The resultant map is indeed a well-defined group homomorphism. □

We now define homotopy of maps of complexes

Definition 2.1.4. (**Homotopy between maps**) Let **A** be an abelian category and $f, g : A^\bullet \to B^\bullet$ be two maps of cochain complexes. Then a homotopy between f and g is defined to be a collection of maps $k := \{k^i : A^i \to B^{i-1}\}_{i\in\mathbb{Z}}$ such that $f^i - g^i = dk^i + k^{i+1}d$ for each $i \in \mathbb{Z}$:

$$\begin{tikzcd}
 & A^i \arrow[r, "d"'] \arrow[dl, "k^i"'] \arrow[d, shift right, "g^i"'] \arrow[d, shift left, "f^i"] & A^{i+1} \arrow[dl, "k^{i+1}"] \\
B^{i-1} \arrow[r, "d"'] & B^i &
\end{tikzcd}.$$

As one might expect, homotopic maps induces *same* (not isomorphic, but actually same) maps on cohomology.

Lemma 2.1.5. *Let* **A** *be an abelian category and* $f, g : A^\bullet \to B^\bullet$ *be two maps of cochain complexes. If* $k : f \sim g$ *is a homotopy between* f *and* g*, then* $h^i(f) = h^i(g)$ *as maps* $h^i(A^\bullet) \to h^i(B^\bullet)$ *for all* $i \in \mathbb{Z}$.

Proof. (*Sketch*) Pick any $a \in \operatorname{Ker}(d)$ in A^i. We wish to show that $f^i(a) - g^i(a) \in \operatorname{Im}(d)$. This follows from unravelling the definition of homotopy $k : f \sim g$. □

We now define the notion of exact functors between two abelian categories.

Definition 2.1.6. (**Exactness of functors**) Let **A** and **B** be abelian categories. A functor $F : \mathbf{A} \to \mathbf{B}$ is said to be

1. *additive* if the map $\operatorname{Hom}_{\mathbf{A}}(A, B) \to \operatorname{Hom}_{\mathbf{B}}(FA, FB)$ is a group homomorphism,
2. *left exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $0 \to FA' \to FA \to FA''$ is exact,
3. *right exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $FA' \to FA \to FA'' \to 0$ is exact,
4. *exact* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $0 \to FA' \to FA \to FA'' \to 0$ is exact,
5. *exact at middle* if it is additive and for every short exact sequence $0 \to A' \to A \to A'' \to 0$ the sequence $FA' \to FA \to FA''$ is exact.

Remark 2.1.7. It is important to keep in mind that all the above definitions are made for *short* exact sequences; a left exact A functor may not map a long exact sequence $0 \to A_1 \to \dots$ to a long exact sequence $0 \to FA_1 \to \dots$.

There are two prototypical examples of such functors in the category of R-modules.

Example 2.1.8. ($- \otimes_R M$ and $\mathrm{Hom}_R(M, -)$) Let R be a commutative ring and M be an R-module. It is a trivial matter to see that the functor $- \otimes_R M : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$ is right exact but not left exact as applying $- \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ on the following shows where $\gcd(n, m) = 1$:

$$0 \to n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Indeed, $n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is not injective as the former is an infinite ring whereas the latter is finite.

Consider the covariant hom-functor $\mathrm{Hom}_R(M, -) : \mathbf{Mod}(R) \to \mathbf{Mod}(R)$. This can easily be seen to be left exact. This is not right exact as applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ to the above exact sequence would yield (note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$).

We next dualize the above theory study the dual notion of homology, without much change. **TODO.**

2.2 Resolutions

We begin with injective objects, resolutions and having enough injectives.

Definition 2.2.1. (**Injective objects and resolutions**) Let $\mathbf{A}$ be an abelian category. An object $I \in \mathbf{A}$ is said to be injective if the functor thus represented, $\mathrm{Hom}_{\mathbf{A}}(-, I) : \mathbf{A}^{\mathrm{op}} \to \mathbf{AbGrp}$ is exact. An injective resolution of an object $A \in \mathbf{A}$ is an exact cochain complex

$$A \xrightarrow{\epsilon} I^0 \to I^1 \to \dots$$

where each I^i is an injective object. We denote an injective resolution of A by $\epsilon : A \to I^\bullet$.

The following are equivalent characterizations of injective objects.

Proposition 2.2.2. *Let $\mathbf{A}$ be an abelian category and $I \in \mathbf{A}$. Then the following are equivalent*

1. *The functor $\mathrm{Hom}_{\mathbf{A}}(-, I)$ is exact.*
2. *For any monomorphism $i : A \to B$ and any map $f : A \to I$, there is an extension $\tilde{f} : B \to I$ to make following commute*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\tilde{f}"] \\ & I & \end{tikzcd}.$$

3. *Any exact sequence*

$$0 \to I \to A \to B \to 0$$

splits.

Proof. 1. $\Rightarrow$ 2. is immediate from definition. 2. $\Rightarrow$ 3. follows from using the universal property of item 2 on $\mathrm{id} : I \to I$ and monomorphism $0 \to I \to A$. For 3. $\Rightarrow$ 1., we need only check right exactness of $\mathrm{Hom}_{\mathbf{A}}(-, I)$, which follows immediately from item 3. □

The following are some properties of injective objects.

Proposition 2.2.3. *Let* $\mathbf{A}$ *be an abelian category. If* $\{I_i\}_i$ *is a collection of injective objects of* $\mathbf{A}$ *and* $\prod_i I_i$ *exists, then it is injective.*

Proof. As $\mathrm{Hom}_{\mathbf{A}}(-, \prod_i I_i) \cong \prod_i \mathrm{Hom}_{\mathbf{A}}(-, I_i)$ and arbitrary product of surjective maps is surjective, therefore the claim follows. □

We see that any two injective resolutions of an object are homotopy equivalent.

Lemma 2.2.4. *Let* $\mathbf{A}$ *be an abelian category and* $A \in \mathbf{A}$ *be an object with two injective resolutions* $\epsilon : A \to I^\bullet$ *and* $\eta : A \to J^\bullet$. *Then there exists a homotopy* $k : \epsilon \sim \eta$.

Proof. Comparison Theorem 2.3.7, pp 40, [cite Weibel Homological Algebra]. □

We then define when an abelian category has enough injectives.

Definition 2.2.5. (**Enough injectives**) An abelian category $\mathbf{A}$ is said to have enough injectives if for each object $A \in \mathbf{A}$, there is an injective object $I \in \mathbf{A}$ such that A is a subobject of I, $A \leq I$.

In such abelian categories, all objects have injective resolutions.

Lemma 2.2.6. *Let* $\mathbf{A}$ *be an abelian category with enough injectives. Then all objects* $A \in \mathbf{A}$ *admit injective resolutions* $\epsilon : A \to I^\bullet$.

Proof. Pick any object $A \in \mathbf{A}$. As $\mathbf{A}$ has enough injectives, therefore we have $0 \to A \xrightarrow{\epsilon} I^0$. Consider $\mathrm{CoKer}(\epsilon)$ and let it be embedded in some injective object I^1, which yields the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "\epsilon"] & I^0 \arrow[r, "d"] \arrow[d, two heads] & I^1 \\ & & \mathrm{CoKer}(\epsilon) \arrow[ur, hook] & \end{tikzcd}.$$

Continue this diagram by considering $\mathrm{CoKer}(d)$ which embeds in some other injective I^2 to further yield the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "\epsilon"] & I^0 \arrow[rr, "d"] \arrow[d, two heads] & & I^1 \arrow[r, "d"] \arrow[d, two heads] & I^2 \\ & & \mathrm{CoKer}(\epsilon) \arrow[urr, hook] & & \mathrm{CoKer}(d) \arrow[ur, hook] & \end{tikzcd}.$$

This builds the required injective resolution. □

We now give examples of abelian categories with enough injectives. Recall that a *divisible group* G is an abelian group such that for any $g \in G$ and ay $n \in \mathbb{Z}$ there exists $h \in G$ such that $g = nh$ (see Definition **??**).

Theorem 2.2.7. *Let R be a commutative ring with 1. Then,*

1. *Any divisible group in* **AbGrp** *is an injective object.*
2. *If G is an injective abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is an injective R-module.*
3. **AbGrp** *is an abelian category which has enough injectives.*
4. $\mathbf{Mod}(R)$ *is an abelian category which has enough injectives.*

Proof. The main idea of the proofs of the later parts is to use injective objects constructed in a bigger category and an adjunction to a lower category to construct injectives in the smaller subcategory. Further, embedding each object in a large enough product of injectives (which would remain injective by Proposition 2.2.3) would show enough injectivity.

1. By Corollary **??**, the statement follows.

2. Recall that $F(-) : \mathbf{Mod}(R) \rightleftarrows \mathbf{AbGrp} : \mathrm{Hom}_{\mathbb{Z}}(R, -)$ is an adjunction, where F is the forgetful functor. Consequently $\mathrm{Hom}_{\mathbb{Z}}(F(M), G) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, G))$. It then follows that $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is injective.

3. Observe that $\mathbb{Q}/\mathbb{Z}$ is a divisible, thus injective abelian group by item 1. Let G be an abelian group. Consider the abelian group

$$I = \prod_{\mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}.$$

By Proposition 2.2.3, I is an injective abelian group. We now construct an injection $\varphi : G \to I$, which would complete the proof. We have the canonical map

$$\begin{aligned} \theta : G &\longrightarrow I \\ g &\longmapsto (\varphi(g))_{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Q}/\mathbb{Z})}. \end{aligned}$$

For this to be well-defined, we need to show that $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ is non-zero. Indeed, we claim that for any element $g \in G$, there is a $\mathbb{Z}$-linear map $\varphi_g : G \to \mathbb{Q}/\mathbb{Z}$ such that $\varphi_g(g) \neq 0$. This would suffice as if $\theta(g) = 0$ for some $g \in G$, then $\varphi(g) = 0$ for all $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. Consequently, $\varphi_g(g) = 0$, which cannot happen, hence θ is injective. So we need only show the existence of φ_g. Indeed, if $|g| = \infty$, then we have an injection $\mathbb{Z} \hookrightarrow G$ taking $1 \mapsto g$. Pick any non-zero map $f : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. By injectivity of $\mathbb{Q}/\mathbb{Z}$, f extends to $\varphi_g : G \to \mathbb{Q}/\mathbb{Z}$ which is non-zero at g. On the other hand, if $|g| = k < \infty$, then consider the inclusion $\mathbb{Z}/k\mathbb{Z} \hookrightarrow G$ taking $\bar{1} \mapsto g$. Then, consider the map $f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ taking $\bar{1} \mapsto \frac{1}{k}$. Then, by injectivity of $\mathbb{Q}/\mathbb{Z}$, it extends to $\varphi_g : G \to \mathbb{Q}/\mathbb{Z}$ which is non-zero at g.

4. Pick any R-module M. We wish to find an injective R-module I such that $M \leq I$. By items 1 and 2, we know that $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective R-module. By the proof of item 2, we also know that

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})).$$

Conseuqently, by Proposition 2.2.3, we have an injective module

$$I = \prod_{\mathrm{Hom}_{\mathbb{Z}}(M,\mathrm{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z}))} \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}),$$

We claim that the following map

$$\begin{aligned}\theta : M &\longrightarrow I\\ m &\longmapsto (\varphi(m))_{\varphi\in\mathrm{Hom}_{\mathbb{Z}}(M,\mathrm{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z}))}\end{aligned}$$

is injective. Indeed, we claim that for each $m \in M$, there exists $\varphi_m \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$ such that $\varphi_m(m) \neq 0$. By the above isomorphism, we equivalently wish to show the existence of $g_m \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ such that $g_m(m) \neq 0$. This is immediate from the proof of item 3. □

We next dualize the above theory and study projective objects, projective resolutions and having enough projectives to define homology. **TODO**.

2.3 Derived functors and general properties

First, for each covariant left exact functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories, we will produce a sequence of functors R^iF for each $i \geq 0$. We will then dualize it.

Definition 2.3.1. (**Right derived functors of a left-exact functor**) Let $F : \mathbf{A} \to \mathbf{B}$ be a left exact functor of abelian categories where $\mathbf{A}$ has enough injectives. Then, define for each $i \geq 0$ the following

$$\begin{aligned}R^iF : \mathbf{A} &\longrightarrow \mathbf{B}\\ A &\longmapsto h^i(F(I^\bullet))\end{aligned}$$

where $\epsilon : A \to I^\bullet$ is any injective resolution of A. We call R^iF the i^{th} right derived functor of the left exact functor F.

Remark 2.3.2. Indeed the above definition is well-defined, by Lemmas 2.1.5 and 2.2.4. Further, keep in mind the Remark 2.1.7.

Some of the basic properties of right derived functors are as follows. First, the 0^{th}-right derived functor of F is canonically isomorphic to F.

Lemma 2.3.3. *Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ has enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. Then, there is a natural isomorphism*

$$R^0F \cong F.$$

Proof. Pick any object $A \in \mathbf{A}$ with an injective resolution $0 \to A \xrightarrow{\epsilon} I^\bullet$. Consequently, $R^0F(A)$ is the cohomology of

$$0 \to F(I^0) \xrightarrow{Fd^0} F(I^1),$$

that is, $R^0F(A) = \mathrm{Ker}\left(Fd^0\right)$. But since F is left-exact and we have the following exact sequence

$$0 \to A \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} \mathrm{Im}\left(d^0\right),$$

therefore we get that $\mathrm{Ker}\left(Fd^0\right) = \mathrm{Im}\,(F\epsilon)$. This also needs the observation that if F is left-exact, then for any map $f \in \mathbf{A}$, we have $F(\mathrm{Im}\,(f)) \cong \mathrm{Im}\,(Ff)$. Since ϵ is injective, then so is $F\epsilon$ and thus $\mathrm{Im}\,(F\epsilon) \cong FA$. □

Remark 2.3.4. Let $I \in \mathbf{A}$ be an injective object. Then we claim that $R^iF(I) = 0$ for all $i \geq 1$. Indeed, this follows immediately because we have $0 \to I \xrightarrow{\text{id}} I \to 0$ as a trivial injective resolution of I.

The following is an important property of right derived functors which makes them ideal for defining the general notion of cohomology, because they always have long exact sequene in cohomology.

Theorem 2.3.5. *Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ has enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. If*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathbf{A}$, then we have a long exact sequence in right derived functors of F as in

$$\begin{tikzcd}
0 \arrow[r] & R^0FA \arrow[r] & R^0FB \arrow[r] & R^0FC \arrow[dll, "\delta_0"'] \\
 & R^1FA \arrow[r] & R^1FB \arrow[r] & R^1FC \arrow[dll, dotted, "\delta_1"'] \\
 & & &
\end{tikzcd}.$$

It follows from above theorem that if F is exact, then R^iF are trivial for $i \geq 1$.

Corollary 2.3.6. *Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ has enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be an exact functor. Then,*

$$R^iF = 0$$

for all $i \geq 1$.

Proof. Pick any object $A \in \mathbf{A}$ and let $I \in A$ be an an injective object such that $0 \to A \to I$ is an injective map. Then we have a short-exact sequence

$$0 \to A \to I \to B \to 0$$

where $B = I/A$. By Theorem 2.3.5, Lemma 2.3.3 and Remark 2.3.4, it follows that we have a long exact sequence in right derived functors of F as in

$$\begin{tikzcd}
R^iFA \arrow[r] & 0 \arrow[r] & R^iFB \arrow[dll, "\delta_i"'] \\
R^{i+1}FA \arrow[r] & 0 \arrow[r] & R^{i+1}FB
\end{tikzcd}.$$

It follows from exactness of the above sequence that $R^iFB \cong R^{i+1}FA$ for all $i \geq 1$. Repeating the same process for B (embedding B into an injective object and observing the resultant long exact sequence), we obtain that

$$R^{i+1}FA \cong R^1FC$$

for some object $C \in \mathbf{A}$. Replacing A by C, it thus suffices to show that $R^1FA = 0$.

In the beginning of the above long exact sequence we have

$$\begin{tikzcd}
0 \arrow[r] & FA \arrow[r] & FI \arrow[r, two heads] & FB \arrow[dll, "\delta_0"] \\
& R^1FA \arrow[r] & 0 \arrow[r] & R^1FB \arrow[dll, dotted, dash] \\
& {} & &
\end{tikzcd}$$

from which it follows via exactness that δ_0 is surjective and $\operatorname{Ker}(\delta_0) = FB$. We then deduce that $R^1FA = 0$, as required. $\square$

Injective resolutions might be hard to find in general, but given a left exact functor F, it would be somewhat easier to find objects J such that $R^iF(J) = 0$ for all $i \geq 1$. The remarkable property of such objects is that it can help to calculate the value of right derived functors of F for objects admitting resolutions by them.

Definition 2.3.7 (**Acyclic resolution**)**.** Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ have enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. An object $J \in \mathbf{A}$ is said to be acyclic if $R^iF(J) = 0$ for all $i \geq 1$. An acyclic resolution of $A \in \mathbf{A}$ is an exact sequence of the form

$$0 \to A \xrightarrow{\epsilon} J^0 \to J^1 \dots$$

where each J^i is acyclic.

The name "acyclic" is justified since they have zero cohomology, so all cocycles are coboundaries, so there are no cycles for that object.

Remark 2.3.8. Note that for an acyclic resolution $0 \to A \xrightarrow{\epsilon} J^\bullet$, we have $h^0(F(J^\bullet)) \cong FA$ by following the steps as in the proof of Lemma 2.3.3.

We then have the following useful theorem.

Proposition 2.3.9. *Let $\mathbf{A}, \mathbf{B}$ be two abelian categories where $\mathbf{A}$ have enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor. For $A \in \mathbf{A}$, let $0 \to A \xrightarrow{\epsilon} J^\bullet$ be an acyclic resolution. Then for all $i \geq 0$, there is a natural isomorphism*

$$R^iF(A) \cong h^i(F(J^\bullet)).$$

Derived functors are equivalent to datum of what is defined to be a universal δ-functor. In the rest of this section we setup the definitions and only state the result.

TODO : Universal δ-functors.

3 Results for $\mathbf{Mod}(R)$

When the abelian category is that of modules over a commutative ring R, then we have some special results which is very useful in homotopy theory.

3.1 Universal coefficients

3.2 Künneth theorem

3.3 $\otimes$-Hom adjunction