Ordinary Differential Equations

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Contents

We will prove some basic existence/uniqueness results about ODEs here, with a classical/analytic viewpoint in mind. Let us first begin by stating what is meant by an *initial value problem* and what is meant by *solving an initial value problem.* A main focus will be on doing analytical proofs, which is always extremely helpful. In particular, we will see how much weird and pathological behaviors can emerge after *passing to limit*, thus justifying why commuting with limits is a sought after property in all over analysis.

1 Initial value problems

Let us begin by understanding what is meant by a differential equation. Let $D \subseteq \mathbb{R}\times\mathbb{R}^n$ be an open set. Consider a continuous function $f : D \to \mathbb{R}^n$ mapping as $(t, x) \mapsto f(t, x)$ where $t \in \mathbb{R}, x \in \mathbb{R}^n$. A fundamental goal that one wishes to achieve is to find a "nice" function $x : I \subseteq \mathbb{R} \to D$ such that the function f can be known upto first derivatives, that is, we want to construct such a function $x : I \to \mathbb{R}^n$ such that it can tell us the following about f:

1. (*Correct domain*)$\forall t \in I$, we shall have $(t, x(t)) \in D$,
2. (*Differential equation*)$\frac{dx}{dt}(t_0) = f(t_0, x(t_0)), \forall t_0 \in I$. That is, the first derivative of x can give us exactly the values that f takes on D.

To find such a function x, the main difficulty is the condition 2 above, for this requires $x : I \to \mathbb{R}^n$ to be continuously differentiable (so of class C^1) and that we necessarily have to construct a function x by the knowledge only of it's first derivative (which is $f(t, x)$).

This problem of constructing a C^1 map $x : I \subseteq \mathbb{R} \to \mathbb{R}^n$ from only the data of it's continuous first derivative is called the process of solving a differential equation.

Clearly, many C^1 maps can have same first derivative (we need only add a scalar in front). So the uniqueness of the above problem is hopeless. However, one can add an extra data to the problem above that x shall satisfy and then we do get uniqueness at times. In particular, we demand the following from x:

3. (*Initial value*) for some fixed $s_0 \in I$ and $x_0 \in \mathbb{R}^n$, we require $x(s_0) = x_0$.

We then define an initial value problem (IVP) as follows:

Definition 1.0.1. (**IVP & solutions**) Let $f : D \to \mathbb{R}^n$ be a continuous map on an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$. An IVP is a construction problem where from the tuple of data $(f, (t_0, x_0))$ for some $(t_0, x_0) \in D$, we have to construct the following:

1. an interval $I \subseteq \mathbb{R}$ containing t_0,
2. a function $x : I \to \mathbb{R}^n$.

This function x should then satisfy the following:

1. $(t, x(t)) \in D \ \forall t \in I$,
2. $\frac{dx}{dt}(t) = f(t, x) \ \forall t \in I$,
3. $x(t_0) = x_0$.

We identify the above IVP with the tuple $(f, (t_0, x_0))$. If such a function $x : I \to \mathbb{R}^n$ exists, then it is called a *solution to the IVP* $(f, (t_0, x_0))$.

1.1 Existence: Peano's theorem

We have an elementary result which tells us that, if the solution exists, then what should be its form.

Lemma 1.1.1. *Let $f : D \to \mathbb{R}^n$ be a continuous map and $(f, (t_0, x_0))$ be an IVP. Then, a continuous map $x : I \to \mathbb{R}^n$ is a solution to the IVP $(f, (t_0, x_0))$ if and only if $\forall t \in I$, $x(t)$ is the following line integral*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Proof. (L $\Rightarrow$ R) Since x is a solution, therefore $\frac{dx}{dt}(t) = f(t, x(t))\ \forall t \in I$ and $t_0 \in I$. Then use fundamental theorem of calculus to calculate the line integral of the vector field ∇x along the line joining t_0 and t.

(R $\Rightarrow$ L) By continuity of x, we get that $t \mapsto f(t, x(t))$ is continuous. Since $x(t_0) = x_0$, therefore by continuity of $t \mapsto (t, x(t))$, there exists an open interval $I \ni t_0$ of $\mathbb{R}$ such that $(t, x(t)) \in D$ for all $t \in I$. It then follows by an application of fundamental theorem of calculus that $\frac{dx}{dt}(t) = f(t, x(t))$ for each $t \in I$. $\square$

We next do Peano's theorem, which tells us that indeed solutions exists. This when combined with above tells us that solutions to IVP $(f, (t_0, x_0))$ exists and is of same "form". However, it will require a classic result in analysis called Arzela-Ascoli theorem. Let us do that first.

Theorem 1.1.2. *(Arzela-Ascoli theorem) Let $x_n : [0, 1] \to \mathbb{R}^n$ be a sequence of continuous functions such that $\{x_n\}$ is a uniformly bounded and equicontinuous family of maps. Then there exists a subsequence of $\{x_n\}$ which is uniformly convergent.*

We can now approach the existence result.

Theorem 1.1.3. *(Peano's theorem) Let $f : D \to \mathbb{R}^n$ be a continuous map where $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and let $(t_0, x_0) \in D$ so that the tuple $(f, (t_0, x_0))$ forms an IVP. Choose $r > 0$ and $c > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$[1]. Then, denoting $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$ and $h := \min\{c, \frac{r}{M}\}$, there exists a solution to the IVP $(f, (t_0, x_0))$ given by*

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

Proof. We will construct the solution x in a limiting manner. First, we may replace t_0 by 0 as we can shift the solution to t_0 thus obtained. Second, we may define x on $[0, h]$ as we may translate and scale the solution as desired. Now, consider the sequence of functions defined as follows:

$$\begin{aligned} x_n(t) : [0, h] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto \begin{cases} x_0 \text{ if } t \in [0, \frac{h}{n}], \\ x_0 + \int_0^{t - h/n} f(s, x_n(s))ds & \text{if } t \in [\frac{h}{n}, h]. \end{cases} \end{aligned}$$

[1] That is, choose a basic closed set around (t_0, x_0) in D.

So we obtain a sequence of functions $\{x_n\}$ defined over $[0,h]$. Now, in the limiting case, we will have a function exactly of the form required by Lemma 1.1.1, so we reduce to showing that a subsequence of the above converges and converges to a continuous function. We will use the Arzela-Ascoli (Theorem 1.1.2) for showing this. We thus need only show that the sequence $\{x_n\}$ is uniformly bounded and equicontinuous. For uniform boundedness, we will simply show that $x_n(t) \in \overline{B}_r(x_0)\ \forall t \in [0,h]$. This follows from the following:

$$\begin{aligned}
|x_n(t) - x_0| &\leq \left| \int_0^{t-h/n} f(s, x_n(s)) ds \right| \\
&\leq \int_0^{t-h/n} |f(s, x_n(s))|\, ds \\
&\leq M(t - \frac{h}{n}) \\
&\leq Mh \\
&\leq r.
\end{aligned}$$

Finally, to see equicontinuity, we may simply observe that for any $\epsilon > 0$ and for any $n \in \mathbb{N}$,

$$\begin{aligned}
|x_n(s) - x_n(t)| &\leq \left| \int_{t-h/n}^{s-h/n} f(u, x_n(u)) \right| du \\
&\leq \int_{t-h/n}^{s-h/n} |f(u, x_n(u))|\, du \\
&\leq M(s-t).
\end{aligned}$$

This shows equicontinuity. □

Remark 1.1.4. (*Comments on proof of Theorem 1.1.3*) The main idea of the proof was to *find the required function through a limiting procedure*, where to make sure that we do get the limit, we used Arzela-Ascoli. One of the foremost things we did as well was to *reduce to the nicest possible setting*, which will be very necessary to clear things around.

1.2 Uniqueness: Picard-Lindelöf theorem

We will now show that for an IVP $(f, (t_0, x_0))$, we may get unique solutions provided some hypotheses on f. In order to understand what this hypothesis on f is, we need to review Lipschitz and contractive functions.

Definition 1.2.1. (**(locally)Lipschitz functions**) A map $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function if $\exists L > 0$ such that $\forall\ x, y \in R$, we have

$$\|f(x) - f(y)\| < L\|x - y\|.$$

The function f is called locally Lipschitz if $\forall x \in E$, there exists $r > 0$ such that $f|_{B_r(x)}$ is a Lipschitz map.

Example 1.2.2. The map $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^{1/3}$ is not locally Lipschitz at $x = 0$. This is because if it is so, then $\exists \epsilon > 0$ such that on $B_\epsilon(0)$ the map f is Lipschitz. But for $x, y \in B_\epsilon(0)$ we get

$$\begin{aligned}|x - y| &= \left|(x^{1/3})^3 - (y^{1/3})^3\right| \\ &= \left|(x^{1/3} - y^{1/3})(x^{2/3} + y^{2/3} + (xy)^{1/3})\right| \\ &\leq 2\epsilon.\end{aligned}$$

Thus,

$$\left|x^{1/3} - y^{1/3}\right| \leq \frac{2\epsilon}{\left|x^{2/3} + y^{2/3} + (xy)^{1/3}\right|},$$

which shows that f can not be Lipschitz on $B_\epsilon(0)$.

We have that all continuously differentiable maps are locally Lipschitz.

Lemma 1.2.3. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^1-map on an open set E, then f is locally Lipschitz.*

Proof. Take $a \in E$. We reduce to showing that there exists a $\epsilon > 0$ such that $\overline{B}_\epsilon(0) \subset E$ so that the continuous map $Df : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ achieves maxima on the compact set. This follows from the fact that E is open. □

One definition that we will need is that of uniform Lipschitz.

Definition 1.2.4. (**Uniform Lipschitz**) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. Then f is called uniformly Lipschitz w.r.t. x if there exists $L > 0$ such that

$$\|f(t, x) - f(t, y)\| < L\|x - y\|$$

for all $(t, x), (t, y) \in D$.

A contraction is defined in an obvious manner.

Definition 1.2.5. (**Contractive mappings**) Let $f : X \to X$ be a continuous map of metric spaces. Then f is said to be contractive if there exists $0 < \lambda < 1$ such that

$$d(f(x), f(y)) < \lambda d(x, y)$$

for all $x, y \in X$.

Our goal is to find the conditions that one must impose on f for the IVP $(f, (t_0, x_0))$ to have a unique solution. This means we need to find a solution $x : I \to \mathbb{R}^n$ in such a manner that x is the unique solution possible on that interval I. Now a place uniqueness comes into the picture is Banach fixed point theorem. Indeed, we will use it to find such an interval I and map x so that it would be unique for the said IVP.

Theorem 1.2.6. *(Banach fixed point theorem) Let X be a complete metric space and $f : X \to X$ be a contractive mapping. Then, f has a unique fixed point.*

Proof. We will first show the existence of such a fixed point. There is an obvious process of doing so. Take any point $x_0 \in X$. We then form the sequence $\{x_n\}$ in X where $x_n = f^n(x_0)$. We claim that $\{x_n\}$ is Cauchy. Indeed, we have that for any $\epsilon > 0$ (we may take $n \geq m$):

$$\begin{aligned}
d(x_n, x_m) &= d(f^n(x_0), f^m(x_0)) \\
&< \lambda^m d(f^{n-m}(x_0), x_0) \\
&< \lambda^m \left(d(f^{n-m}(x_0), f(x_0)) + d(f(x_0), x_0)\right) \\
&< \lambda^m \left(\lambda d(f^{n-m-1}(x_0), x_0) + d(x_1, x_0)\right) \\
&= \lambda^{m+1} d(f^{n-m-1}(x_0), x_0) + \lambda^m d(x_1, x_0) \\
&< d(x_1, x_0)\left(\lambda^m + \cdots + \lambda^n\right) \\
&= \lambda^m \frac{1-\lambda^{n-m}}{1-\lambda} d(x_1, x_0) \\
&< \frac{\lambda^m}{1-\lambda} d(x_1, x_0).
\end{aligned}$$

Next, by completeness of X, we have that there exists $x = \varinjlim_n x_n$ in X. Now, $f(x) = f(\varinjlim_n x_n) = \varinjlim_n f(x_n)$ by continuity and $\varinjlim_n f(x_n) = x$ by definition of x_n. The uniqueness is simple by contractive property of f. □

We now come to the main result, the uniqueness of solutions of IVP. Before stating it, let us state how we will be proving it, using the following bijection between solutions of $(f, (t_0, x_0))$ and fixed points of certain mapping.

Construction 1.2.7. Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping where D is open and let $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Now consider the following space for some $c > 0$

$$X := C^1\left[[t_0 - c, t_0 + c], \mathbb{R}^n\right]$$

and consider the following map

$$\begin{aligned}
T : X &\longrightarrow X \\
x(t) &\longmapsto T(x)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))ds.
\end{aligned}$$

Then, by Lemma 1.1.1, we see that $x(t) \in X$ is a solution of $(f, (t_0, x_0))$ if and only if $T(x(t)) = x(t)$. Hence

$$\{\textit{Solutions of IVP } (f, (t_0, x_0))\} \cong \{\textit{Fixed points of } T : X \to X\}.$$

Theorem 1.2.8. *(Weak Picard-Lindelöf) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ such that $(f, (t_0, x_0))$ forms an IVP. Choose $c > 0$ and $r > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$. Denote $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$. If the map*

$$f : [t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \longrightarrow \mathbb{R}^n$$

is uniformly Lipschitz w.r.t. x and Lipschitz constant being L, then, denoting $h := \min\{c, \frac{r}{M}, \frac{1}{L}\}$, there exists a unique solution of IVP $(f, (t_0, x_0))$ given by

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

Proof. (Sketch) The main part of the proof will be the idea in Construction 1.2.7 and Banach fixed point theorem. Let X denote the following space

$$X := \left\{ y \in C^0\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right] \mid y(t_0) = x_0 \;\&\; \sup_{x \in [t_0 - h, t_0 + h]} \|x_0 - y(t_0)\| \leq hM \right\}.$$

Consider the following function on X

$$\begin{aligned} T : X &\longrightarrow X \\ y &\longmapsto x_0 + \int_{t_0}^{t} f(s, y(s))ds. \end{aligned}$$

By Theorem 1.2.6, we reduce to showing that function X is complete and T is a contraction mapping. Let us first show completeness of X. One then shows that $X \hookrightarrow C\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right]$ is a closed subspace and it will suffice since $C\left[[t_0 - h, t_0 + h], \mathbb{R}^n\right]$ is complete and closed subspaces of complete spaces are complete. □

We will now prove Picard-Lindelöf again but with a weakening of hypotheses as compared to Theorem 1.2.8. This is important because most of the time one doesn't has the information of Lipschitz constant L as is required in Theorem 1.2.8 while constructing the interval of the solution.

Lemma 1.2.9. *Something about Picard iterates: If f is Lipschitz with constant $L > 0$, then the Picard iterates $\{x_n(t)\}$ satisfies*

$$\|x_{n+1}(t) - x_n(t)\| \leq \frac{ML^n (t - t_0)^{n+1}}{(n+1)!}.$$

Theorem 1.2.10. *(Strong Picard-Lindelöf) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map on an open set D and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Choose $c > 0$ and $r > 0$ such that $[t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \subseteq D$. Denote $M := \max_{x \in [t_0 - c, t_0 + c] \times \overline{B}_r(x_0)} f(x)$. If the map*

$$f : [t_0 - c, t_0 + c] \times \overline{B}_r(x_0) \longrightarrow \mathbb{R}^n$$

is uniformly Lipschitz w.r.t. x, then, for any $h < \min\{c, \frac{r}{M}\}$, there exists a unique solution of IVP $(f, (t_0, x_0))$ given by

$$x : [t_0 - h, t_0 + h] \longrightarrow \overline{B}_r(x_0).$$

The following corollary tells us an alternate sufficient condition on f for the existence of unique solution to an IVP on f.

Corollary 1.2.11. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous map where D is open. If $\frac{\partial f_i}{\partial x_j} : D \to \mathbb{R}$ are continuous maps for all $1 \leq i, j \leq n$, then for each $(t_0, x_0) \in D$ there exists an open neighborhood around $(t_0, x_0) \in D$ in which there is a unique solution to IVP $(f, (t_0, x_0))$.*

Remark 1.2.12. In practice, to reduce to an open neighborhood where the solution is unique, the above corollary will be useful.

1.3 Continuation of solutions

Consider the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(t, x) \mapsto x^2$ and $(0, 1) \in \mathbb{R} \times \mathbb{R}$. One sees that the IVP $(f, (0, 1))$ has a solution given by

$$\begin{aligned} x : (-1, 1) &\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{1}{1-t}. \end{aligned}$$

However, this solution can be "extended"/"continued" to the following solution of the said IVP

$$\begin{aligned} y : (-\infty, 1) &\longrightarrow \mathbb{R} \\ t &\longmapsto \frac{1}{1-t}. \end{aligned}$$

These two are different solutions but the domain of one is inside the domain of the other. This concept of solutions extending from one domain to a larger domain will be investigated in this section.

The following definition is obvious.

Definition 1.3.1. (**Continuation of solutions**) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : I \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. Then the solution x is said to be continuable if there exists a solution y of $(f, (t_0, x_0))$ given by $y : J \to \mathbb{R}^n$ where $J \supseteq I$ and $y|_I = x$.

The following theorem tells us a sufficient criterion on the solution which would make it continuable to some larger interval.

Theorem 1.3.2. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : (a, b) \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$.*

1. *If $\varinjlim_{t \to b^-} x(t)$ exists and $(b, \varinjlim_{t \to b^-} x(t)) \in D$, then there exists $\epsilon > 0$ such that x can be continued to a solution $\tilde{x} : (a, b + \epsilon) \to \mathbb{R}^n$.*
2. *If $\varinjlim_{t \to a^+} x(t)$ exists and $(a, \varinjlim_{t \to a^+} x(t)) \in D$, then there exists $\epsilon > 0$ such that x can be continued to a solution $\tilde{x} : (a - \epsilon, b) \to \mathbb{R}^n$.*

The following lemma states that for mild conditions on f, the boundary limits might exist for a solution.

Lemma 1.3.3. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where D is open and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. If f is bounded, then for any solution $x : (a, b) \to \mathbb{R}^n$, the limits*

$$\varinjlim_{t \to b^-} x(t) \;\&\; \varinjlim_{t \to a^+} x(t) \;\textit{exist.}$$

Proof. Use Lemma 1.1.1 to get that x is uniformly continuous over (a, b), so it has unique extension to its boundary. $\square$

1.4 Maximal interval of solutions

Let $(f, (t_0, x_0))$ be an IVP and let $x : I \to \mathbb{R}^n$ be a solution. A natural question is whether there is a "maximal continuation" of x in the sense of Definition 1.3.1. This is what we investigate here. The following definition is clear.

Definition 1.4.1. (**Maximal interval of solution**) Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. The maximal interval of solution x is an interval $J \subseteq \mathbb{R}$ such that there exists a continuation of x on J and there is no continuation of $z : L \to \mathbb{R}^n$ of y where $L \supsetneq J$.

Lemma 1.4.2. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. If $x : I \to \mathbb{R}^n$ is a solution of $(f, (t_0, x_0))$, then there exists a maximal interval of solution x.*

Proof. This is a simple application of Zorn's lemma on the poset

$$P = \{y : J \to \mathbb{R}^n \mid y \text{ is a continuation of } x\}$$

where $y \leq z$ iff z is a continuation of x. $\square$

We wish to now find a characterization of maximal intervals of a solution. That is, we wish to know when can we say that a given solution is maximal.

Proposition 1.4.3. *Let $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : (a, b) \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. Then,*

1. *The interval $[t_0, b)$ is a right maximal interval of solution x if and only if for any compact subset $K \subseteq D$, there exists $t \in [t_0, b)$ such that $(t, x(t)) \notin K$.*
2. *The interval $(a, t_0]$ is a left maximal interval of solution x if and only if for any compact subset $K \subseteq D$, there exists $t \in (a, t_0]$ such that $(t, x(t)) \notin K$.*

Proof. (Sketch) By symmetry, we reduce to showing 1. The main idea is to use the maximality and the results of previous section. $\square$

1.5 Solution on boundary

In this section, we investigate the limiting cases of solutions of ODEs on a maximal interval (see Lemma 1.4.2). We see that if the one-sided limit of a maximal solution exists, then it's graph has to lie on the boundary of the domain.

Theorem 1.5.1. *Let $f : D \to \mathbb{R}^n$ be a continuous map where $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and let $(t_0, x_0) \in D$ so to make $(f, (t_0, x_0))$ an IVP. If $x : I \to \mathbb{R}^n$ is a solution to $(f, (t_0, x_0))$ and $I = (a, b)$ is a maximal interval of solution, then*

1. *If $\partial D \neq \emptyset$, $b < \infty$ and $\varinjlim_{t \to b^-} x(t)$ exists, then*

$$\left(b, \varinjlim_{t \to b^-} x(t) \right) \in \partial D.$$

2. If $\partial D = \emptyset$, $b < \infty$ then

$$\limsup_{t \to b^-} x(t) = \infty.$$

A similar statement holds for left sided limit towards a.

Proof. 1. Suppose not. Then $\left(b, \lim_{t\to b^-} x(t)\right) \in D$ as D is open. It follows from Lemma 1.3.2 that $[t_0, b)$ is not maximal.

2. Suppose not. Then $\limsup_{t\to b^-} x(t) \neq \infty$. Hence, there exists $M > 0$ such that $\|x(t)\| < M$ for all $t \in [t_0, b)$. Now, construct $K = [t_0, b] \times C$ where C is a compact disc such that $\forall t \in [t_0, b)$, $x(t) \in C$, which can be chosen as an appropriate disc in $B_M(x_0)$. Since $K \subseteq D$, therefore by Proposition 1.4.3 we get a contradiction to maximality of $[t_0, b)$. □

That's all we have to say here, so far.

1.6 Global solutions

So far we have studied solutions $x(t)$ to IVP defined only on some small enough intervals I such that $(t, x(t)) \in D$. However, we defined $D \subseteq \mathbb{R} \times \mathbb{R}^n$ as an arbitrary open set. In this section we would restrict to certain type of domains D, namely of the form $D = I \times \mathbb{R}^n$ and will try to study whether we can obtain a solution $x(t) : I \to \mathbb{R}^n$ to an IVP $(f, (t_0, x_0))$. If they exists, we call such a solution to be a *global solution* of the IVP $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ with initial values $(t_0, x_0) \in I \times \mathbb{R}^n$.

Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where $I \subseteq \mathbb{R}$ is an open interval and choose $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Let $x : J \to \mathbb{R}^n$ be a solution of $(f, (t_0, x_0))$. The main result of this section says that every such solution $x(t)$ can be extended to a global solution on I given some regularity conditions of values of f.

Theorem 1.6.1. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map where $I \subseteq \mathbb{R}$ is an open interval and choose $(t_0, x_0) \in D$ so that $(f, (t_0, x_0))$ forms an IVP. Suppose*

$$\|f(t, x)\| \leq M(t) + \|x\| N(t)$$

where $M, N : I \to \mathbb{R}$ are non-negative continuous maps, $\forall (t, x) \in I \times \mathbb{R}^n$. Then any solution $x : J \to \mathbb{R}^n$ of $(f, (t_0, x_0))$ can be continued to a solution $\tilde{x} : I \to \mathbb{R}^n$.

We are more interested in the applications of the above theorem, which we now present.

Corollary 1.6.2. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be uniformly Lipschitz w.r.t. x. Then, there exists a unique global solution $x : I \to \mathbb{R}^n$ of the IVP $(f, (t_0, x_0))$.*

Proof. We get that $\exists L > 0$ such that

$$\|f(t, x) - f(t, y)\| < L \|x - y\|$$

for all $t \in I$ and $x, y \in \mathbb{R}^n$. In particular for $y = 0$, we get

$$\begin{aligned} \|f(t, x)\| &\leq \|f(t, x) - f(t, 0)\| + \|f(t, 0)\| \\ &\leq L\|x\| + \|f(t, 0)\| \end{aligned}$$

where $N(t) = L$ and $M(t) = \|f(t,0)\|$ in the notation of Theorem 1.6.1. Hence, by the same theorem, if there exists a solution of $(f,(t_0,x_0))$, say x on $J \subseteq I$, then it extends to a solution on I. Now by Strong Picard-Lindelöf (Theorem 1.2.10), we conclude that there is a unique solution on I; if there are two solutions on I, then by restriction on the interval obtained from Picard-Lindelöf, we would get a contradiction to it's uniqueness. $\square$

For a system of equations linear in x, for $x \in \mathbb{R}^n$, we have the following result.

Corollary 1.6.3. *Let $f(t,x) = A(t)x + b(t)$ be a map from $I \times \mathbb{R}^n$ to $\mathbb{R}^n$ where $A(t) \in C(I, \mathbb{R}^{n\times n})$ and $b \in C(I,\mathbb{R}^n)$ for an open interval $I \subseteq \mathbb{R}$ and $x = (x_1,\ldots,x_n)$. For $(t_0,x_0) \in I \times \mathbb{R}^n$, consider the IVP $(f,(t_0,x_0))$. Then there exists a unique solution*

$$x : I \times \mathbb{R}^n \to \mathbb{R}^n.$$

Proof. Using triangle inequality, we obtain

$$\|f(t,x)\| \leq \|A(t)\|\|x\| + \|b\|.$$

The result follows by an application of Theorem 1.6.1 and Corollary 1.6.2. $\square$

2 Linear systems

So far, we covered solutions of ODE of the form

$$\frac{dx}{dt} = f(t,x(t))$$

where $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $x : I \to \mathbb{R}^n$. In particular, $\frac{dx}{dt}$ is given as

$$\frac{dx}{dt}(t) = \begin{bmatrix} \frac{dx_1}{dt} & \frac{dx_2}{dt} & \cdots & \frac{dx_n}{dt} \end{bmatrix}$$

where each $x_i : I \to \mathbb{R}$. On the other hand, the right side consists of $f(t,x)$, which is a continuous function from a subset of $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$.

In this section, we would now study in detail a particular type of IVP in which the aforementioned function $f(t,x)$ is a linear map. In particular, the mapping f is given by

$$\begin{aligned} f : D \subseteq \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (t,x) &\longmapsto Ax \end{aligned}$$

for a real matrix A.

Remark 2.0.1. One should keep in mind that these are not new ODEs; a linear system is same as $\frac{dx}{dt} = f(t,x)$ where $f(t,x) = Ax$, so they are special cases of general ODEs and have special properties like uniqueness of solutions. In particular, all the results of the previous section on general ODEs will obviously hold in the linear case.

Remark 2.0.2. By Lemma 1.1.1, we know that a solution of $\frac{dx}{dt} = Ax$ is necessarily of the form

$$x(t) = x_0 + A\int_0^t x(s)ds.$$

2.1 Some properties of matrices

Let us begin by stating some of the properties of matrix algebra, especially of exponential of matrices as it will be used in Theorem 2.2.1. Since these are not fancy results so we omit the proof of all except the main observations required in each.

Theorem 2.1.1. *Let $A, B \in M_n(\mathbb{R})$. Then,*

1. $\|A+B\| \leq \|A\| + \|B\|$.
2. $\|AB\| \leq \|A\|\|B\|$.
3. *The series e^X defined by*

$$e^X := \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

converges for all $X \in M_n(\mathbb{R})$.

4. $e^0 = I$.
5. $(e^A)^T = e^{A^T}$.
6. *e^X is invertible and $(e^X)^{-1} = e^{-X}$ for all $X \in M_n(\mathbb{R})$.*
7. *If $AB = BA$, then $e^{A+B} = e^A e^B = e^B e^A$.*
8. *If $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then $e^A = \operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.*
9. *If P is invertible, then $e^{PAP^{-1}} = Pe^A P^{-1}$.*

Proof. We omit the proof of all but the 3. To show that the series converges, by M-test, we reduce to showing that $\sum_n \frac{\|X\|^n}{n!}$ converges as

$$\|\frac{X^n}{n!}\| \leq \frac{\|X\|^n}{n!}.$$

Indeed, it converges to $e^{\|X\|}$. □

Out of the above, perhaps the most important is the last one, as it tells us that if we have a diagonalizable matrix $A = PDP^{-1}$, then knowing its eigenvalues (that is, knowing D) and the matrix P is enough for us to calculate the e^A. Indeed, one should note that the exponent of a matrix is not easy to compute all the time!

We now give the lemma which will be quite useful for our goals, that the derivative of exponential of matrices is the obvious one.

Lemma 2.1.2. *Let $X \in M_n(\mathbb{R})$. Then,*

$$\frac{d}{dt} e^{At} = Ae^{At}.$$

Proof. One would need to interchange two limits at one point, which could only be done if the convergences are uniform. This could be shown by M-test. □

2.2 Fundamental theorem of linear systems

The most important theorem for linear systems of the form $\frac{dx}{dt} = Ax$ is that that they have a unique solution.

Theorem 2.2.1. *Let $A \in M_n(\mathbb{R})$. Then for any $x_0 \in \mathbb{R}^n$, the IVP*

$$\frac{dx}{dt} = Ax(t)$$

with $x(0) = x_0$ has a unique solution given by

$$x(t) = e^{At}x_0.$$

Proof. Suppose $y(t)$ is another solution. Then, define $z(t) = e^{-At}y(t)$. Differentiating this, we get

$$\frac{d}{dt}z(t) = -Ae^{-At}y(t) + e^{-At}\frac{dy}{dt}(t).$$

Since $\frac{dy}{dt} = Ay$, thus the above equation gives $\frac{d}{dt}z(t) = -Ae^{-At}y + e^{-At}Ay = 0$. Hence $z(t) = c$ is constant, therefore $y(t) = ce^{At}$. Since $y(0) = x_0 = c$, therefore $y = x$. ☐

2.2.1 Non-homogeneous linear systems

A non-homogeneous linear system is a linear IVP with an *offset*; they are of the form:

$$\frac{dx}{dt} = Ax(t) + b(t)$$

with $x(0) = x_0$. Their solution have a peculiar form.

Lemma 2.2.2. *Let $\frac{dx}{dt} = Ax(t) + b(t)$ with $x(0) = x_0$ be a non-homogeneous IVP for $A \in M_n(\mathbb{R})$. Then x is a solution if and only if*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds.$$

Proof. We can multiply the IVP by e^{-At} to obtain

$$\begin{aligned} e^{-At}\frac{dx}{dt} &= Ae^{-At}x + e^{-At}b(t) \\ e^{-At}\frac{dx}{dt} - Ae^{-At}x &= e^{-At}b(t) \\ \frac{d}{dt}[e^{-At}x] &= e^{-At}b(t) \\ x(t) &= e^{At}x_0 + e^{At}\int_0^t e^{-As}b(s)ds. \end{aligned}$$

One can easily check that the given form satisfies the IVP, by an application of fundamental theorem of calculus. ☐

3 Stability of linear systems in $\mathbb{R}^2$

Consider the linear IVP given by

$$\frac{dx}{dt} = Ax(t)$$

with $x(0) = x_0$ where $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ and $A \in M_2(\mathbb{R})$. From the fundamental theorem, we know that the solution is of the form $x(t) = e^{At}x_0$. By Jordan form, we know that there exists base change matrix $P \in GL_2(\mathbb{R})$ such that $A = P^{-1}BP$ where B is in Jordan form and hence it is of either of the three forms:

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \;,\; B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \;,\; B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

By Theorem 2.1.1, 9, we get

$$x(t) = e^{At}x_0 = e^{P^{-1}BPt}x_0 = P^{-1}e^{Bt}Px_0$$

so we reduce to understanding the plots of $e^{Bt}x_0$ for the aforementioned three cases, in order to understand the plot of $e^{At}x_0$ as both are related by coordinate transformation by P.

A phase portrait of a linear system

$$\frac{dx}{dt} = Ax(t)$$

is a plot of $x_1(t)$ vs $x_2(t)$ for various choices of initial points. Indeed, the choice of initial points is paramount if one ought to find the behavior of solutions. On the basis of the analysis of the three cases for B, we make the following definitions.

Definition 3.0.1. Let $\frac{dx}{dt} = Ax$ be a linear system where $\det A \neq 0$ and $A \in M_2(\mathbb{R})$. Then, the system is said to have

1. **saddle at origin** if $A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ where $\lambda < 0 < \mu$,
2. **node at origin** if
   (a) $A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ where λ, μ have same sign,
   (b) $A \sim \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$,
3. **focus at origin** if $A \sim \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$,
4. **center at origin** if $A \sim \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$.

4 Autonomous systems

An IVP is said to be *autonomous* if the governing equation

$$\frac{dx}{dt} = f(x(t))$$

is such that the continuous map $f : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is independent of time parameter t and we further assume that $f \in C^1$. In such a case we write $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, and for a fixed initial datum, the maximal interval of existence is unique as well (see Corollary 1.2.11)

One calls a point $x_0 \in D$ to be an *equilibrium point* of the $\frac{dx}{dt} = f(x(t))$ if $f(x_0) = 0$.

4.1 Flows and Liapunov stability theorem

In our attempt at a better understanding of the autonomous system's dependence on initial point, we develop a basic machinery to handle it. The phase plots were a tool only available for linear systems, but we are not dealing with then in this section. Note however that a linear system is also autonomous.

The first tool we want to make is the notion of flows.

Definition 4.1.1. (**Flows**) Consider the following autonomous ODE

$$\frac{dx}{dt} = f(x(t))$$

where $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map. Denote by $\varphi(-, y) : I_y \to E$ to be a solution of the IVP $(f, (0, y))$ defined on **the** maximal interval of existence I_y for $\varphi(-, y)$ (Lemma 1.4.2). The map

$$\begin{aligned} \varphi : I \times E &\longrightarrow E \\ (t, y) &\longmapsto \varphi(t, y) \end{aligned}$$

is called the flow of the system and the map $\varphi(t, -) : E \to E$ is called the *flow of the system at time t*. As we argued in the beginning, there is only one maximal interval of existence for each initial datum.

Remark 4.1.2. For a pair $(t, y) \in I \times E$, the value of the flow $\varphi(t, y) \in E$ tells us where the solution $\varphi(-, y)$ takes the initial point y at time t.

We have some obvious observations.

Lemma 4.1.3. *Consider the following autonomous ODE*

$$\frac{dx}{dt} = f(x(t)).$$

Let $\varphi : I \times E \to E$ be the flow of the system. Then,

1. $\varphi(0, y) = y$.
2. $\varphi(s, \varphi(t, y)) = \varphi(s + t, y)$.
3. $\varphi(-t, \varphi(t, y)) = y$.

Proof. Trivial. □

We now define the important notions surrounding stability.

Definition 4.1.4. (**Stability**) Consider the following autonomous ODE

$$\frac{dx}{dt} = f(x(t)).$$

Let $\varphi : I \times E \to E$ be the flow of the system.

1. An equilibrium point $x_0 \in E$ is said to be *(Liapunov)stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in B_\delta(x_0) \implies \varphi(t, x) \in B_\epsilon(x_0)\ \forall t \geq 0$.
2. An equilibrium point $x_0 \in E$ is said to be *unstable* if it is not stable.
3. An equilibrium point $x_0 \in E$ is said to be *asymptotically stable* if it is stable and $\exists r > 0$ such that

$$x \in B_r(x_0) \implies \varinjlim_{t\to\infty} \varphi(t, x) = x_0.$$

We are now ready to state one of the most important results in stability theory, the Liapunov stability theorem. This result gives a sufficient condition for stability of a given point in the domain of $f : E \to \mathbb{R}^n$ of an autonomous system.

Theorem 4.1.5. *(Liapunov stability theorem) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and*

$$\frac{dx}{dt} = f(x(t))$$

be a given autonomous system with $x_0 \in E$ being an equilibrium point. If there exists a map of class C^1

$$V : E \to \mathbb{R}$$

such that $V(x_0) = 0$ and $V(x) > 0$ for all $x \in E \setminus \{x_0\}$, then

1. *if $V'(x) \leq 0$ for all $x \in E \setminus \{x_0\}$, then x_0 is stable,*
2. *if $V'(x) < 0$ for all $x \in E \setminus \{x_0\}$, then x_0 is asymptotically stable,*
3. *if $V'(x) > 0$ for all $x \in E \setminus \{x_0\}$, then x_0 is unstable.*

Remark 4.1.6. It is important to note that for most of the autonomous systems in nature, the function V as above which will do the job will be the energy functional of the physical system, that is, sum of kinetic and potential energy.

5 Linearization and flow analysis

Consider the following *system*:

$$x' = f(x)$$

where $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map and E is an open set. In the terminology of what we have covered so far, we have an autonomous system. In general, the above system may not be linear, as we studied previously. However, we can *linearize* the system at an equilibrium point x_0, as we shall show below. Indeed, this allows us to analyze the general autonomous system around each point as if it were linear.

Construction 5.0.1. (*Linearization of system at a point*) Let $E \subseteq \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}^n$ be a C^1 map. Let $x_0 \in E$ be an equilibrium point. For any $x \in E$, by Taylor's theorem, we get

$$\begin{aligned} f(x) &= f(x_0) + Df(x_0)(x - x_0) + \text{higher order terms} \\ &= Df(x_0)(x - x_0) + \text{higher order terms} \\ &= A(x - x_0) + \text{higher order terms.} \end{aligned}$$

We thus call the $x' = Df(x_0)x$ to be the *linearization of the system f at point x_0.*

Few definitions are in order.

Definition 5.0.2. (**Hyperbolic, sink, source & saddle points**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. An equilibrium point $x_0 \in E$ is said to be:

1. *hyperbolic* if all eigenvalues of $Df(x_0)$ has non-zero real part,
2. *sink* if all eigenvalues of $Df(x_0)$ has negative real part,
3. *source* if all eigenvalues of $Df(x_0)$ has positive real part,
4. *saddle* if there exists eigenvalues λ, μ of $Df(x_0)$ such that real part of λ is > 0 and real part of μ is < 0.

5.1 Stable manifold theorem

"*For a non-linear system, there are stable and unstable submanifolds, so that once you are in either of them, the flow will constrain you to remain there.*"

We will do an important theorem in the theory of linearization of autonomous systems. We shall avoid the proof this theorem. A reference is pp 107, [cite Perko]. Let us first define three important subspaces corresponding to a linear system.

Definition 5.1.1. (**Stable, unstable & center subspaces**) Let

$$x' = Ax$$

be a linear system where $A \in M_n(\mathbb{R})$. Let $\lambda_j = a_j + ib_j$ be eigenvalues of A and $w_j = u_j + iv_j$ be a generalized eigenvector of λ_j. Then,

1. the stable subspace E^s is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \dots, n$ such that $a_j < 0$,
2. the unstable subspace E^u is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \dots, n$ such that $a_j > 0$,
3. the center subspace E^c is defined to be the span of all u_j, v_j in $\mathbb{R}^n$ for those $j = 1, \dots, n$ such that $a_j = 0$.

Lemma 5.1.2. *Let $x' = Ax$ be a linear system for $A \in M_n(\mathbb{R})$. Then,*

1. $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$,
2. *E^s, E^u and E^c are invariant under the flow $\varphi(t, x)$ of the linear system, which as we know is given by $e^{At}x$.*

Proof. 1. This is easy, as generalized eigenvectors always span the whole space.
2. We need only show that for a generalized eigenvector w_j corresponding to $\lambda_j = a_j + ib_j$ with $a_j < 0$, the vector $A^k w_j$ is again a genralized eigenvector. Indeed, this follows from definition of a generalized eigenvector as $(A - \lambda_j I)w_j$ is again a generalized eigenvector. □

We now come to the real deal.

Theorem 5.1.3. *(Stable manifold theorem) Let $E \subseteq \mathbb{R}^n$ be an open subset with $0 \in E$, consider $f : E \to \mathbb{R}^n$ to be a C^1-map and consider the system that it defines. Denote E^s and E^u to be the stable and unstable subspaces of the system $x' = Df(0)x$. If,*

- *$f(0) = 0$,*
- *$Df(0) : \mathbb{R}^n \to \mathbb{R}^n$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part,*

then:

1. *There exists a k-dimensional differentiable manifold S inside E such that*
   (a) *$T_0 S = E^s$,*
   (b) *for all $t \geq 0$ and for all $x \in S$, we have*
   $$\varphi(t, x) \in S,$$
   (c) *for all $x \in S$, we have*
   $$\lim_{\overrightarrow{t \to \infty}} \varphi(t, x) = 0.$$
2. *There exists an $n-k$-dimensional differentiable manifold inside E such that*
   (a) *$T_0 U = E^u$,*
   (b) *for all $t \leq 0$ and for all $x \in U$, we have*
   $$\varphi(t, x) \in S,$$
   (c) *for all $x \in U$, we have*
   $$\lim_{\overrightarrow{t \to -\infty}} \varphi(t, x) = 0.$$

Let us explain via an example

Example 5.1.4. Consider the system

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}.$$

This is not a linear system as for $f((x_1, x_2, x_3)) = (-x_1, -x_2 + x_1^2, x_3 + x_1^2)$, the above system is given by

$$x' = f(x) \tag{1}$$

and $f(x)$ is clearly not linear in x. However, note that $f(0) = 0$. Thus, linearizing the system (1) at 0, we obtain the linear system

$$x' = Ax \tag{2}$$

where

$$A := Df(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So A has two eigenvalues with negative real part, namely -1 and -1 and one eigenvalue with positive real part, namely 1. In particular A is diagonalizable, hence E^s and E^u are just span of the eigenvectors (as all generalized eigenvectors in this case are just your regular eigenvectors). Hence we see

$$E^s = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$
$$E^u = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence $E^s = x$-y plane and E^u is the z-axis of $\mathbb{R}^3$.

By an application of stable manifold theorem on this system, the stable manifold S is of dimension 2 and unstable manifold U is of dimension 1. Now, by elementary calculations, we can actually solve the linear system (2) and we thus obtain the following solution

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ x_3(t) &= c_3 e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}). \end{aligned}$$

Hence, the flow of the system is given by

$$\begin{aligned} \varphi : \mathbb{R} \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (t, (c_1, c_2, c_3)) &\longmapsto \begin{pmatrix} cc_1 e^{-t} \\ c_2 e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ c_3 e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}) \end{pmatrix}. \end{aligned}$$

Now, notice the following for any $c = (c_1, c_2, c_3) \in \mathbb{R}^3$

$$\begin{aligned} \varinjlim_{t \to \infty} \varphi(t, c) = 0 &\iff c_3 + \frac{c_1^2}{3} = 0 \\ \varinjlim_{t \to -\infty} \varphi(t, c) = 0 &\iff c_1 = c_2 = 0. \end{aligned}$$

Notice the fact that the above equivalence is very particular to this example. But this leads us to the following conclusions

$$\begin{aligned} S &= \{(c_1, c_2, c_3) \in \mathbb{R}^3 \mid c_3 + c_1^2/3\} \\ U &= \{(c_1, c_2, c_3) \in \mathbb{R}^3 \mid c_1 = c_2 = 0\} \cong z\text{-axis}. \end{aligned}$$

Note that it is indeed true that for all $c \in S$ and any $t \geq 0$, $\varphi(t, c) \in S$. Similarly for U. Finally, one can check that $T_0 S = E^s$ and $T_0 U = E^u$, where the latter is immediate.

5.2 Poincaré-Bendixon theorem

So far, for a system we have defined its flow. Flow or integral curves of the system holds important information about the system at hand. However, we have not done any serious analysis with them. We shall begin the analysis of flows of a system now and prove the aforementioned theorem. It's use is predominantly to find closed trajectories of a system, which most of the times appears as a boundary of two differing phenomenon of the system, hence the importance of closed trajectories and of the theorem.

We first set up the terminology to be used in order to define basic objects of analysis of flow of a system.

Definition 5.2.1. (ω & α -limit set) Let $E \subseteq \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}^n$ be a C^1 map. Let $\varphi : \mathbb{R} \times E \to \mathbb{R}^n$ be the flow of the system. Then,

1. a point $y \in E$ is said to be a ω-limit point of $x \in E$ if there exists a sequence $t_1 < t_2 < \cdots < t_n < \ldots$ in $\mathbb{R}$ such that $\varinjlim_{n\to\infty} t_n = \infty$ and $\varinjlim_{n\to\infty} \varphi(t_n, x) = y$.
2. a point $y \in E$ is said to be an α-limit point of $x \in E$ if there exists a sequence $t_1 > t_2 > \cdots > t_n > \ldots$ in $\mathbb{R}$ such that $\varinjlim_{n\to\infty} t_n = -\infty$ and $\varinjlim_{n\to\infty} \varphi(t_n, x) = y$.

Let $x \in E$, the set of all ω and α limit points of x are denoted $L_\omega(x)$ and $L_\alpha(x)$ respectively.

The following are some simple observations from the definition

Lemma 5.2.2. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system given by it.*

1. *If $y \in L_\omega(x)$ and $z \in L_\omega(y)$ then $z \in L_\omega(x)$.*
2. *If $y \in L_\omega(x)$ and $z \in L_\alpha(y)$ then $z \in L_\omega(x)$.*
3. *For any $x \in E$, the limit sets $L_\omega(x)$ and $L_\alpha(x)$ are closed in E.*

Using the concept of limit points, we can define certain nice subspaces of E conducive to them.

Definition 5.2.3. (**Positively invariant set**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. A region $D \subseteq E$ is said to be positively invariant if for all $x \in D$, $\varphi(t, x) \in D$ for all $t \geq 0$ where $\varphi : \mathbb{R} \times E \to E$ is the flow.

We then have the following simple result.

Lemma 5.2.4. *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it.*

1. *If x, z are on same flow line/trajectory, then $L_\omega(x) = L_\omega(z)$.*
2. *For any $x \in E$, the limit set $L_\omega(x)$ is positively invariant.*
3. *If $D \subseteq E$ is a closed positively invariant set, then for all $x \in D$, $L_\omega(x) \subseteq D$.*

We now define another set of tools helpful in doing flow analysis. First is a notion which will come in handy while trying to discuss both the topology of underlying space and the flow together. A hyperplane in $\mathbb{R}^n$ is a codimension 1 linear subspace.

Definition 5.2.5. (**Local sections**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. Let $0 \in E$. A local section S of f is an open connected subset of a linear hyperplane $H \subseteq \mathbb{R}^n$ such that $0 \in S$ and H is transverse to f, that is, $f(x) \notin H$ for all $x \in S$.

The next tool helps to "straighten" out flow around a local section.

Definition 5.2.6. (**Flow box**) Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map on an open set E and consider the system defined by it. Let $0 \in E$ and S be a local section of f. A flow box around S is a diffeomorphism Φ between $(-\epsilon, \epsilon) \times S \subseteq \mathbb{R} \times E$ and $V_\epsilon \subseteq E$ given by $V_\epsilon := \{\varphi(t, x) \mid t \in (-\epsilon, \epsilon), x \in S\}$:

$$\begin{aligned}\Phi : (-\epsilon, \epsilon) \times S &\longrightarrow V_\epsilon \\ (t, x) &\longmapsto \varphi(t, x).\end{aligned}$$

We identify $(-\epsilon, \epsilon) \times S$ as the flow box around S.

For a flow box, the diffeomorphism is important as it tells us that we can assume WLOG in a flow box that flow line are identical to the orthogonal coordinate system of $(-\epsilon, \epsilon) \times S \subseteq \mathbb{R}^{n+1}$.

We would now like to do flow analysis for the special case of planar systems. Indeed, the main theorem of this section is about the behaviour of certain limit sets of planar systems.

Let us first observe that for a planar system, any local section intersects a flow line at only discretely many points.

Lemma 5.2.7. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ and consider a local section S around x. Let*

$$\Sigma := \{\varphi(t, x) \in E \mid t \in [-l, l]\}.$$

Then $\Sigma \cap S$ is discrete.

Next, we see that if a sequence of points in a local section S of a planar system is monotonous in S and those same points appear in a trajectory, then it is monotonous in that trajectory as well. Indeed, a sequence of points $\{\varphi(t_n, x)\}$ along a trajectory is said to be *monotonous* if $\varinjlim_{n\to\infty} t_n = \infty$. Note that for a planar system, a codimension 1 linear subspace is a line, hence it has an inherent order and thus we can talk about monotonous sequences in a local section.

Proposition 5.2.8. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let S be a local section of the system. If $x_n = \varphi(t_n, x)$ is a sequence of points monotonous along the trajectory and $x_n \in S$, then $\{x_n\}$ are monotonous in S as well.*

One can use the above proposition to deduce some "eventual" properties of points in a local section by observing their intersection points with a flow line (which are discrete). Further it can be used for replacing a sequence along a trajectory to a sequence along a local section, which might be easier to analyze (as it's behaviour will just be that of monotonous sequences in $\mathbb{R}$).

Next we see an important observation, that trajectories of some special points cannot intersect a local section at more than one point(!)

Lemma 5.2.9. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. For some $x \in E$, let $y \in L_\omega(x) \cup L_\alpha(x)$. Then the trajectory of y intersects any local section at not more than single point.*

The next result is interesting, for it says that if the trajectory of a point intersects a local section, then there is a whole neighborhood worth of point around it, each of whose trajectories will intersect the local section(!) In some sense, this corresponds to the continuity of flow.

Proposition 5.2.10. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $\varphi : \mathbb{R} \times E \to \mathbb{R}^2$ denote the flow of the system. Let S be a local section around $y \in E$. If there exists $z_0 \in E$ such that for some $t_0 > 0$ we have $\varphi(t_0, z_0) = y$, then*

1. *there exists an open set $U \ni z_0$,*
2. *there exists a unique C^1-map $\tau : U \to \mathbb{R}$,*

where τ has the property that $\tau(z_0) = t_0$ and

$$\varphi(\tau(z), z) \in S \; \forall z \in U.$$

With this, we define the main object of study, a closed orbit.

Definition 5.2.11. (**Closed orbits**) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. A closed orbit is a periodic trajectory which doesn't contain an equilibrium point.

Note that if a trajectory contains an equilibrium point, then it will terminate after some finite time, hence the above requirement.

We now come to the main theorem of this section, which tells us a sufficient condition to find a closed orbits of a planar system.

Theorem 5.2.12. *(Poincaré-Bendixon theorem) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ be such that $L_\omega(x)$ ($L_\alpha(x)$) is a non-empty compact limit set which doesn't contain an equilibrium point. Then $L_\omega(x)$ ($L_\alpha(x)$) is a closed orbit.*

Let us now give some applications of the above theorem. First, we can classify limit sets $L_\omega(x)$ completely.

Theorem 5.2.13. *(Classification of limit sets) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. Let $x \in E$ be such that $L_\omega(x)$*

- *is connected,*
- *is compact,*
- *has finitely many equilibrium points.*

Then one of the following holds

1. *$L_\omega(x)$ is a singleton.*
2. *$L_\omega(x)$ is periodic trajectory with no equilibrium points.*
3. *$L_\omega(x)$ consists of equilibrium points $\{x_j\}$ and a set of non-periodic trajectories $\{\gamma_i\}$ such that for all i, the trajectory γ_i tends to some x_j as $t \to \pm\infty$.*

The main use of Poincaré-Bendixon is to find limit cycles.

Definition 5.2.14. (**Limit cycles**) Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. A limit cycle is a periodic trajectory γ such that there exists $x \in E$ for which $\gamma \subseteq L_\omega(x)$ or $\gamma \subseteq L_\alpha(x)$.

We now state the corollary of Poincaré-Bendixon which allows us to find the existence of limit cycles.

Corollary 5.2.15. *Let $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map on an open set E and consider the planar system defined by it. If there exists a subseteq $D \subseteq E$ such that D*

1. *is compact,*
2. *is positively invariant,*
3. *has no equilibrium points,*

then there exists a limit cycle in D.

Proof. By Poincaré-Bendixon, we need only find $x \in D$ such that $L_\omega(x)$ is compact, as then $L_\omega(x)$ itself will be the limit cycle. This is straightforward, as D is positively invariant and compact, so $L_\omega(x)$ is inside D and is closed (hence compact). $\square$

6 Second order ODE

We now discuss some basic theory of second order ordinary differential equations.

Definition 6.0.1. (**Second order system and solutions**) Let $I \subseteq \mathbb{R}$ be an interval of $\mathbb{R}$ and consider $a_0, a_1, a_2, g \in C(I)$ to be four continuous maps $I \to \mathbb{R}$ such that $a_0(x) > 0\ \forall x \in I$. Then, a second order system with parameters a_0, a_1, a_2, g is given by

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x). \tag{3}$$

Note that $y' := \frac{dy}{dx}$. A *solution* of a second order system (q, r, f) is a $C^2(I)$ map $y(x)$ such that it satisfies (3).

Remark 6.0.2. A second order ODE can be written in the form

$$y'' + q(x)y' + r(x)y = f(x)$$

where $q, r, f \in C(I)$. This form is the one that we shall use and will identify a second order system by the tuple (q, r, f).

Remark 6.0.3. On the $\mathbb{R}$-vector space $C^2(I)$ of twice continuously differentiable functions, every 2nd order system (q, r, f) defines a linear transformation

$$\begin{aligned} L : C^2(I) &\longrightarrow C(I) \\ y(x) &\longmapsto (D^2 + q(x)D + r(x))y \end{aligned}$$

where $D : C^2(I) \to C(I)$ is the derivative transformation $y \mapsto y'$, which is evidently linear. In this notation, we can write a second order system (q, r, f) as

$$Ly = f$$

where $L = D^2 + qD + r$. We call this linear transformation L the *transform associated to* (q, r, f).

Definition 6.0.4. (**Solution space**) Let (q, r, f) be a 2nd order system and $L : C^2(I) \to C(I)$ be the associated transform. The solution space of (q, r, f) is defined as the $\operatorname{Ker}(L) \subseteq C^2(I)$. Note that the set of all solutions of (q, r, f) in $C^2(I)$ is given by $L^{-1}(f) \subseteq C^2(I)$.

Lemma 6.0.5. *Let (q, r, f) be a 2nd order system and L be the associated transform. Then* $\dim_{\mathbb{R}}(\mathrm{Ker}\,(L)) = 2$.

We now observe that one can obtain all solutions of the 2nd order system $S := (q, r, f)$ by obtaining a basis of the solution space of S and one solution of S.

Lemma 6.0.6. *Let $S = (q, r, f)$ be a 2nd order system and L be the associated transform. Then, for any $y_p \in L^{-1}(f)$*

$$L^{-1}(f) = y_p + \mathrm{Ker}\,(L).$$

Proof. Observe that $y - y_p \in \mathrm{Ker}\,(L)$ and a linear transformation has all fibers of same size. □

We define a tool which helps in distinguishing independent or dependent solutions of a homogeneous system.

Definition 6.0.7. (**Wronskian**) Let $f, g \in C^1(I)$. The Wronskian of f and g is given by

$$W(f, g) : I \to \mathbb{R}$$

where for any $x \in I$, we have

$$\begin{aligned} W(f, g)(x) :&= \det \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} \\ &= f(x)g'(x) - g(x)f'(x). \end{aligned}$$

Lemma 6.0.8. *Let $(q, r, 0)$ be a homogeneous system and let $y_1, y_2 \in C^2(I)$ be two solutions. Then,*

1. *$W(y_1, y_2)$ is either constant 0 for all $x \in I$ or $W(y_1, y_2)(x) \neq 0$ for all $x \in I$.*
2. *y_1, y_2 are linearly independent if and only if $W(y_1, y_2) \neq 0\ \forall x \in I$.*

6.1 Zero set of homogeneous systems

Let (q, r, f) be a 2nd order system and let y be a solution. There are some peculiar properties of the zero set $Z(y) := \{x \in I \mid y(x) = 0\} \subseteq \mathbb{R}$. We first show that the set $Z(y)$ is discrete if the system is homogeneous.

Lemma 6.1.1. *Let $(q, r, 0)$ be a 2nd order homogeneous system and let y be a solution. The zeroes of $y(x)$ are isolated, that is, $Z(y)$ is discrete.*

6.1.1 Strum separation and comparison theorems

These theorems are at the heart of the analysis of zeros of homogeneous systems.

Theorem 6.1.2. *(Strum separation theorem) Let $(q, r, 0)$ be a 2nd order homogeneous system. Let y_1, y_2 be two distinct linearly independent solutions of the system. Then,*

1. *$Z(y_1)$ and $Z(y_2)$ are disjoint.*
2. *$Z(y_1)$ and $Z(y_2)$ are braided, that is, for any two x_1^1 and x_2^1 in $Z(y_1)$, there exists $x_1^2 \in Z(y_2)$ between them, and vice versa.*

Theorem 6.1.3. *(Strum comparison test) Consider two homogeneous 2nd order systems* $(0, r_1, 0)$ *and* $(0, r_2, 0)$. *Let* y *be a solution of* $(0, r_1, 0)$ *and* u *be a solution of* $(0, r_2, 0)$, *both non-trivial. Let* $x_1, x_2 \in Z(u)$ *such that*

1. $r_1(x) \geq r_2(x)$ *for all* $x \in (x_1, x_2)$,
2. $\exists\, x_k \in (x_1, x_2)$ *such that* $r_1(x_k) > r_2(x_k)$.

Then, there exists $z \in Z(y)$ *such that* $z \in (x_1, x_2)$.

6.2 Boundary value problems

A *boundary value problem* (BVP) is a second order system on an interval $I = [a, b]$ given by

$$y'' + qy' + ry = f$$

for $q, r, f \in C(I)$ such that its solutions has to satisfy certain conditions on the boundary given by

$$\begin{aligned} B_a(y) &:= \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ B_b(y) &:= \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{aligned}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$. This is clearly a different problem than that of IVP. However, with some construction, we can convert this problem into a pair of 2nd order IVPs. It will turn out that the solution of this pair has important consequences for the original IVP at hand.

6.2.1 Reduction to a pair of 2nd order IVPs and criterion for uniqueness of BVP solution

Theorem 6.2.1. *Let* $I = [a, b]$ *and* $q, r, f \in C(I)$. *Consider the 2nd order system* (q, r, f) *and denote the associated transform as* $L : C^2(I) \to C^2(I)$. *From the system* (q, r, f) *consider the BVP given explicitly by*

$$\begin{aligned} Ly &:= y'' + qy' + ry = f \\ B_a(y) &:= \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ B_b(y) &:= \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{aligned} \tag{4}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$. *Construct the following two 2nd order IVPs*

$$\begin{aligned} Ly &:= y'' + qy' + ry = 0 \\ y(a) &= \beta_1 \\ y'(a) &= -\alpha_1 \end{aligned} \tag{5}$$

and

$$\begin{aligned} Ly &:= y'' + qy' + ry = 0 \\ y(b) &= \beta_2 \\ y'(b) &= -\alpha_2. \end{aligned} \tag{6}$$

Then the following are equivalent

1. *Let* y_1 *be a solution of* (5) *and* y_2 *be a solution of* (6)*. Then* y_1 *and* y_2 *are linearly independent in the solution space* $\mathrm{Ker}\,(L)$*.*
2. *The homogeneous BVP*

$$\begin{aligned} Ly := y'' + qy' + ry = 0 \\ B_a(y) = 0 \\ B_b(y) = 0 \end{aligned} \tag{7}$$

   has only 0 as solution.
3. *The BVP* (4) *has a unique solution.*

6.2.2 Variation of parameters

Variation of parameters can give us a general form of a particular solution of $Ly = f$, in terms of the solutions of IVPs (5) and (6). Indeed, we have the following theorem.

Theorem 6.2.2. *Let* y_1 *be a solution of* (5) *and* y_2 *be a solution of* (6)*. Let*

$$\begin{aligned} c_1(x) &= \int_a^x \frac{-f(s)y_2(s)}{W(y_1, y_2)(s)} ds \\ c_2(x) &= \int_a^x \frac{f(s)y_1(s)}{W(y_1, y_2)(s)} ds. \end{aligned} \tag{8}$$

Then,

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \tag{9}$$

is a particular solution of $Ly = f$ *with* $y_p(a) = 0$*.*

Further, we obtain a general form of solution of BVP (4).

Theorem 6.2.3. *Consider the notations of Theorems 6.2.1 and 6.2.2.*

1. *Any solution* y *of BVP* (4) *is*

$$y = y_p - c_1(b)y_1$$

   where y_1 *is a solution of* (5) *and* $c_1(x)$ *is defined in* (8)*.*
2. *Any solution of the BVP* (4) *is given by the integral*

$$y(x) = \int_a^b G(x,s)f(s)ds \tag{10}$$

   for all $x \in I$*, where*

$$G(x,s) = \begin{cases} \frac{y_1(x)y_2(s)}{W(y_1,y_2)(s)} & \text{if } x \le s \le b \\ \frac{y_1(s)y_2(x)}{W(y_1,y_2)(s)} & \text{if } a \le s \le x. \end{cases} \tag{11}$$

   This map G *is called the Green's function for the transformation* $L : C^2(I) \to C(I)$*.*

6.2.3 Strum-Liouville system

Let $p, q \in C^2(I)$ and $f \in C(I)$ with $p > 0$. Define the 2nd order system

$$py'' + p'y' + qy = f.$$

We can write it in neater terms as follows

$$(py')' + qy = f. \tag{12}$$

We will call this the *Strum-Liouville system*, denoted by (p, q, f), and the associated transform as $L : C^2(I) \to C(I)$ mapping $y \mapsto (py')' + qy$. Consequently, (12) can be written as

$$Ly := (py')' + qy = f.$$

We have some basic results about the associated transform L.

Lemma 6.2.4. *Let (p, q, f) be a Strum-Liouville system and L be the associated transform.*

1. *(Lagrange's identity) If $y_1, y_2 \in C^2(I)$, then*

$$y_1 Ly_2 - y_2 Ly_1 = (pW(y_1, y_2))'.$$

2. *(Abel's formula) If y_1, y_2 are solutions of $Ly = 0$, that is, they are solutions of the Strum-Liouville system defined by $(p, q, 0)$, then*

$$W(y_1, y_2) = c/p$$

for some constant $c \in \mathbb{R}$.

6.2.4 Strum-Liouville Boundary Value Problems (SL-BVPs)

Consider a homogeneous Strum-Liouville system $(p, q, 0)$ and let L be the associated transform. Consider $r \in C(I)$ and $\lambda \in \mathbb{C}$. Then, a *Strum-Liouville boundary value problem* is a following type of 2nd order BVP

$$Ly + \lambda ry = 0 \tag{13}$$

with

$$B_a(y) = 0$$
$$B_b(y) = 0.$$

6.2.5 Strum-Liouville EigenValue Problems (SL-EVPs)

An SL-EVP consists of an SL-BVP (13) and the following question: find $\lambda \in \mathbb{C}$ such that the SL-BVP (13) admits a non-zero solution $y_\lambda \in C^2(I)$. In such a case λ is called the *eigenvalue* and y_λ the *eigenfunction* of the corresponding SL-EVP. We then call the tuple (p, q, r) as the SL-EVP.

6.2.6 Types of SL-EVPs

We further classify an SL-EVP (p, q, r) based on the properties of the underlying functions.

1. **regular** if $p > 0$ and $r > 0$ on $[a, b]$,
2. **singular** if $p > 0$ on (a, b), $p(a) = 0 = p(b)$ and $r \geq 0$ on $[a, b]$,
3. **periodic** if $p > 0$ on $[a, b]$, $p(a) = p(b)$ and $r > 0$ on $[a, b]$.

We next see that any eigenvalue of SL-EVP is always real.

Lemma 6.2.5. *Let (p, q, r) be a regular SL-EVP. Then all eigenvalues of (p, q, r) are real.*