Sheaf Theory

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Contents

1 Recollections

The notion of sheaves plays perhaps the most important role in modern viewpoint of geometry. It is thus important to understand the various constructions that one can make on them. We assume the reader knows the definition of a sheaf on a space X and morphism of sheaves. We begin with some recollections.

Remark 1.0.1 (*Map on stalks*)**.** Recall that a map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X defines for each point $x \in X$ a map of stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ given by $s_x \mapsto \varphi_U(s)_x$ where s is a section of $\mathcal{F}$ over $U \subseteq X$. One can check quite easily that this is well-defined and that this map φ_x is in-fact the unique map given by the universal property of the colimit in the diagram below:

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r] & \varinjlim_{V \ni x} \mathcal{G}(V) \\ \mathcal{F}(U) \arrow[r] \arrow[u, "\varphi_U"] & \varinjlim_{V \ni x} \mathcal{F}(V) \arrow[u, dashed, "\varphi_x"'] \end{tikzcd} .$$

Hence φ_x is the unique map which makes the above diagram commute.

Remark 1.0.2 (*Subsheaves*)**.** Recall that $\mathcal{F} \hookrightarrow \mathcal{G}$ is a subsheaf if $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ such that for $U \hookrightarrow V$, the restriction map $\rho_{V,U} : \mathcal{G}(V) \to \mathcal{G}(U)$ restricts to $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$.

Remark 1.0.3 (*Constant sheaves*)**.** For an abelian group A and a space X, one defines the constant sheaf $\mathcal{A}$ as the sheaf which for each open set $U \subseteq X$ assigns $\mathcal{A}(U) = \{s : U \to A \mid s \text{ is continuous}\}$, where A is given the discrete topology. One sees instantly that this is a sheaf. Further one observes that if $U = U_1 \amalg \cdots \amalg U_k$ where U_i are components of open set U and U_i are open, then $\mathcal{A}(U) \cong A \oplus \cdots \oplus A$ k-times. In particular, for any open connected subset U, we get $\mathcal{A}(U) = A$.

We now begin by showing how to construct a sheaf out of a presheaf over X.

2 The sheafification functor

Let X be a topological space, denote the category of presheaves on X by $\mathbf{PSh}(X)$ and denote the category of sheaves over X by $\mathbf{Sh}(X)$. We have a canonical inclusion functor $i : \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$. We construct it's left adjoint commonly known as the process of sheafifying a presheaf.

Theorem 2.0.1. *(Sheafification) Let X be a topological space and let F be a presheaf over X. Then there exists a pair $(\mathcal{F}, i_F)$ of a sheaf $\mathcal{F}$ and a map $i_F : F \to \mathcal{F}$ such that for any sheaf $\mathcal{G}$ and a morphism of presheaves $\varphi : F \to \mathcal{G}$, there exists a unique morphism of sheaves $\tilde{\varphi} : \mathcal{F} \to \mathcal{G}$ such that the following commutes*

$$\begin{tikzcd} \mathcal{F} \arrow[r, dashed, "\tilde{\varphi}"] & \mathcal{G} \\ F \arrow[u, "i_F"] \arrow[ur, "\varphi"'] & \end{tikzcd} ,$$

that is, we have a natural bijection

$$\mathrm{Hom}_{\mathbf{PSh}(X)}(F, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, \mathcal{G}).$$

Moreover:

1. *$(\mathcal{F}, i_F)$ is unique upto unique isomorphism.*
2. *For every $x \in X$, the map on stalks $i_{F,x} : F_x \to \mathcal{F}_x$ is bijective.*

3. *For any map of presheaves $\varphi : F \to G$, we get a map of sheaves $\tilde{\varphi} : \mathcal{F} \to \mathcal{G}$ which is unique w.r.t. the commuting of the following natural square:*

$$\begin{tikzcd} \mathcal{F} \arrow[r, dashed, "\tilde{\varphi}"] & \mathcal{G} \\ F \arrow[u, "i_F"] \arrow[r, "\varphi"'] & G \arrow[u, "i_G"'] \end{tikzcd} .$$

Hence we have a functor

$$\begin{aligned} (-)^{++} : \mathbf{PSh}(X) &\longrightarrow \mathbf{Sh}(X) \\ F &\longmapsto F^{++} := \mathcal{F}. \end{aligned}$$

Proof. We explicitly construct the sheaf $\mathcal{F}$ out of F. We define the local sections of $\mathcal{F}$ by using germs and turning the gluing condition of sheaf definition onto itself. In particular, define

$$\mathcal{F}(U) := \left\{ ((s_{i_x})_x) \in \prod_{x \in U} F_x \mid \forall x \in U,\ \exists \text{ open } W \ni x \ \&\ t \in F(W) \text{ s.t. } \forall p \in W,\ t_p = (s_{i_p})_p \right\}.$$

The restriction map for $U \hookrightarrow V$ of $\mathcal{F}$ is given by $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, $((s_{i_x})_x) \mapsto ((s_{i_x})_x)_{x \in U}$, that is, $\rho_{V,U}$ is just the projection map. Next, we show that $\mathcal{F}$ satisfies the gluing criterion and that is where we will see how the above definition of sections of $\mathcal{F}$ came about. Take an open set $U \subseteq X$ and an open cover $U = \bigcup_{i \in I} U_i$. Let $s_i \in \mathcal{F}(U_i)$ be a corresponding collection of sections such that for all $i, j \in I$, we have $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$. We wish to thus construct a section $t \in \mathcal{F}(U)$ such that $\rho_{U,U_i}(t) = s_i$ for all $i \in I$. Indeed let $((t_{i_x})_x) \in \prod_{x \in U} F_x$ where $t := (t_{i_x})_x = (s_i)_x$ if $x \in U_i$. Then since for any $x \in U$, there exists $U \supseteq U_i \ni x$ and $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U,U_i}(t) = ((t_{i_x})_x)_{x \in U_i} = ((s_i)_x)_{x \in U_i}$, we thus conclude that $t \in \mathcal{F}(U)$. So $\mathcal{F}$ satisfies the gluing condition. The locality is quite simple. Next the map i_F is given as follows on sections:

$$\begin{aligned} i_{F,U} : F(U) &\longrightarrow \mathcal{F}(U) \\ s &\longmapsto (s_x). \end{aligned}$$

Now, it can be seen by definition of colimits that $\mathcal{F}_x = F_x$. Finally, let $\mathcal{G}$ be a sheaf and let $\varphi : F \to \mathcal{G}$ be a morphism of presheaves, then we can define $\tilde{\varphi}$ by gluing the germs as follows:

$$\begin{aligned} \tilde{\varphi}_U : \mathcal{F}(U) &\longrightarrow \mathcal{G}(U) \\ ((s_{i_x})_x) &\longmapsto [\varphi_{W_x}(s_{i_x})] \end{aligned}$$

where $[\varphi_{W_x}(s_{i_x})]$ denotes the unique section in $\mathcal{G}(U)$ that one gets by considering the open cover $\bigcup_{x \in U} W_x$ where $s_{i_x} \in \mathcal{F}(W_x)$ and considering the gluing of corresponding sections $\varphi_{W_x}(s_{i_x}) \in \mathcal{G}(W_x)$. These sections agree on intersections because φ is a natural transformation and (s_{i_x}) agree on intersections as sections of $\mathcal{F}(U)$. Hence we have the unique map $\tilde{\varphi}$. Moreover, it is clear that $\tilde{\varphi} \circ i_F = \varphi$. $\square$

Corollary 2.0.2. *Let F be a presheaf over a topological space X, then for all $x \in X$, $F_x = (F^{++})_x$.*

Proof. By construction of F^{++}. $\square$

Corollary 2.0.3. *If $\mathcal{F}$ is a sheaf over a topological space X, then $\mathcal{F}^{++} = \mathcal{F}$.*

Proof. Follows immediately from the universal property of the sheafification, Theorem 2.0.1. □

Remark 2.0.4. The sections of sheaf $\mathcal{F}$ in an open set U containing x is defined in such a manner so that $f \in \mathcal{F}(U)$ can be constructed locally out of sections of F. In particular, we can write $\mathcal{F}(U)$ more clearly as follows

$$\mathcal{F}(U) = \left\{ s : U \to \coprod_{x \in U} F_x \mid \forall x \in U,\ s(x) \in F_x \ \&\ \exists \text{ open } x \in V \subseteq U \ \&\ \exists t \in F(V) \text{ s.t. } s(y) = t_y\ \forall y \in V \right\}.$$

Note that this is exactly the realization that $\mathcal{F}(U)$ is the set of section of the étale space of the sheaf $\mathcal{F}$ (see Section 4). Most of the time in practice, we would work with the universal property of $\mathcal{F}$ in Theorem 2.0.1 as it is much more amenable, but the above must be kept in mind as it is used, for example, to make sure that certain algebraic constructions of $\mathcal{O}_X$-modules remains $\mathcal{O}_X$-modules (no matter how trivial they may sound).

We note that sheafification and restrictions to open sets commute.

Lemma 2.0.5. *Let X be a space, $U \subseteq X$ be an open subset and F be a presheaf over X. Then,*

$$(F|_U)^{++} \cong (F^{++})\big|_U.$$

Proof. Immediate from universal property of sheafification (Theorem 2.0.1). □

3 Morphisms of sheaves

All sheaves are abelian sheaves in this section. One of the most important aspects of using sheaves is that the injectivity and bijectivity of φ_x can be checked on sections. We first show that taking stalks is functorial

Lemma 3.0.1. *Let X be a topological space, $\mathcal{F}, \mathcal{G}$ be two sheaves over X and $x \in X$ be a point. Then the following mapping is functorial:*

$$\begin{aligned} \mathbf{Sh}(X) &\longrightarrow \mathbf{AbGrp} \\ \mathcal{F} &\longmapsto \mathcal{F}_x \\ \mathcal{F} \xrightarrow{f} \mathcal{G} &\longmapsto \mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x. \end{aligned}$$

Proof. Immediate, just remember how composition of two natural transforms is defined. □

Another simple lemma about sheaves and stalks is that equality of two sections can be checked at the stalk level.

Lemma 3.0.2. *Let X be a topological space and $\mathcal{F}$ be a sheaf over X. If $s, t \in \mathcal{F}(U)$ for some open $U \subseteq X$ such that $(U, s)_x = (U, t)_x\ \forall x \in U$, then $s = t$ in $\mathcal{F}(U)$.*

Proof. By equality on stalks, it follows that we have an open set $W_x \ni x$ in U for all $x \in U$ such that $\rho_{U,W_x}(s) = \rho_{U,W_x}(t)$. The result follows from the unique gluing property of sheaf $\mathcal{F}$. □

The above result therefore show why almost all the time it is enough to work with stalks in geometry. Let us now define an injective and surjective map of sheaves.

Definition 3.0.3. (**Injective & surjective maps**) Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves on X. A map of sheaves $f : \mathcal{F} \to \mathcal{G}$ is said to be

1. *injective* if for all opens $U \subseteq X$, the local homomorphism $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective,
2. *surjective* if for all opens $U \subseteq X$ and all $s \in \mathcal{G}(U)$, there exists an open covering $\{U_i\}_{i\in I}$ such that $\rho_{U,U_i}(s) \in \mathrm{Im}\,(f_{U_i})$,
3. *bijective* if f is injective and surjective.

Heuristically, one may understand the notion of f being surjection by saying that every local section of $\mathcal{G}$ is locally constructible by the image of $\mathcal{F}$ under the map f.

For each map of sheaves, we can also define two corresponding sheaves which are global algebraic analogues of the local algebraic constructions.

Definition 3.0.4. (**Quotient sheaf**) Let X be a topological space and $\mathcal{F}$ be a sheaf on X. For a subsheaf $\mathcal{S} \subseteq \mathcal{F}$, one defines the quotient sheaf $\mathcal{F}/\mathcal{S}$ as the sheafification of the presheaf F/S defined on open sets $U \subseteq X$ by

$$F/S(U) := \mathcal{F}(U)/\mathcal{S}(U).$$

Definition 3.0.5. (**Image & kernel sheaves**) Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X and $f : \mathcal{F} \to \mathcal{G}$ be a morphism. Then,

1. *image sheaf* is the sheafification of the presheaf $\mathrm{Im}\,(f)$ defined on open sets $U \subseteq X$ by
$$(\mathrm{Im}\,(f))(U) := \mathrm{Im}\,(f_U),$$
and we denote it by the same symbol, $\mathrm{Im}\,(f)$,
2. *kernel sheaf* is the sheafification of the presheaf $\mathrm{Ker}\,(f)$ defined on open sets $U \subseteq X$ by
$$(\mathrm{Ker}\,(f))(U) := \mathrm{Ker}\,(f_U)$$
and we denote it by the same symbol, $\mathrm{Ker}\,(f)$.

In both the above definitions, the important aspect is the sheafification of the canonical presheaves.

The main point is that one can check all the three notions introduced in Definition 3.0.3 for $f : \mathcal{F} \to \mathcal{G}$ by checking on stalks $f_x : \mathcal{F}_x \to \mathcal{G}_x$ for all $x \in X$.

Theorem 3.0.6. [1] *Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X. Then, a map $f : \mathcal{F} \to \mathcal{G}$ is*

1. *injective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,*
2. *surjective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$,*
3. *bijective if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$,*
4. *an isomorphism if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$*[2].

[1] Exercise II.1.2, II.1.3 and II.1.5 of Hartshorne.

[2] In general, we should write "... *if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism*", but since we are in the setting of abelian sheaves and bijective homomorphism of abelian groups is an isomorphism, so we can get away with this.

5. *an isomorphism if and only if* $f : \mathcal{F} \to \mathcal{G}$ *is bijective.*

Proof. The proof is more of an exercise to get a familiarity with the flexibility of sheaf language. The main idea almost everywhere is to do some local calculations and use sheaf axioms to construct a unique section out of local sections.

1. (L $\Rightarrow$ R) We wish to show that f_x is injective. Suppose for two $(U,s)_x, (V,t)_y \in \mathcal{F}_x$ we have $f_x((U,s)_x) = f_x((V,t)_x) \in \mathcal{G}_x$, which translates to $(U, f_U(s))_x = (V, f_U(t))_x$. We wish to show that $(U,s)_x = (V,t)_y$. By definition of equality on stalks, we obtain open $W \subseteq U \cap V$ containing x such that

$$\rho_{U,W}(f_U(s)) = \rho_{V,W}(f_V(t)).$$

   By the fact that f is a natural transformation, we further translate the above equality to

$$f_W(\rho_{U,W}(s)) = f_W(\rho_{V,W}(t)).$$

   By injectivity of homomorphism f_W, we obtain

$$\rho_{U,W}(s) = \rho_{V,W}(t)$$

   in $\mathcal{F}(W)$. Hence by the definition of equality on stalks, we obtain $(U,s)_x = (V,t)_x$.

   (R $\Rightarrow$ L) Pick any open $U \subseteq X$. We wish to show that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. Let $s \in \mathcal{F}(U)$ be such that $f_U(s) = 0$. Thus for all $x \in U$, we have $(U, f_U(s))_x = 0$. Further, by definition of the map f_x, we obtain $f_x((U,s))_x = (U, f_U(s))_x = 0$. By injectivity of f_x, we obtain $(U,s)_x = 0$ for all $x \in U$[3]. By definition of equality on stalks, we obtain an open cover $\{W_x\}_{x\in U}$ such that $x \in W_x$ and $s|_{W_x} := \rho_{U,W_x}(s) = 0$. Since f is a natural transformation, we therefore obtain that $\{s|_{W_x}\}_{x\in U}$ is a matching family, i.e. on intersections of W_x, W_y, the corresponding sections agree. Hence, there is a unique glue of $\{s|_{W_x}\}_{x\in U}$ denote $t \in \mathcal{F}(U)$. Since each $s|_{W_x} = 0$, therefore we have two glues of the family over U, one is 0 and the other is s. By uniqueness of the glue, it follows that $s = 0$.
2. (L $\Rightarrow$ R) Pick any $x \in X$. We wish to show that $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective. Pick any $(V,t)_x \in \mathcal{G}_x$. We wish to show that for some open $U \ni x$, we have $(U,s)_x \in \mathcal{F}_x$ such that

$$(V,t)_x = (U, f_U(s))_x.$$

   Since $t \in \mathcal{G}(V)$, therefore by surjectivity of f that there exists an open cover $\{V_i\}_{i\in I}$ of V such that

$$\rho_{V,V_i}(t) \in \operatorname{Im}(f_{V_i}).$$

   Therefore we may pick $s_i \in \mathcal{F}(V_i)$ such that

$$\begin{aligned}\rho_{V,V_i}(t) &= f_{V_i}(s_i)\\ &= f_{V_i}(\rho_{V_i,V_i}(s_i))\\ &= \rho_{V_i,V_i}(f_{V_i}(s_i)).\end{aligned}$$

[3] We could be done right here by Lemma 3.0.2.

Thus, $(V,t)_x$ and $(V_i, f_{V_i}(s_i))_x$ are same.

(R $\Rightarrow$ L) We wish to show that $f : \mathcal{F} \to \mathcal{G}$ is surjective. Pick any open set $V \subseteq X$ and $t \in \mathcal{G}(V)$. We wish to find an open cover $\{W_i\}$ of V such that $s_i \in \mathcal{F}(V_i)$ and $f_{V_i}(s_i) = \rho_{V,V_i}(t)$. Since we have $(V,t)_x \in \mathcal{G}_x$ for all $x \in V$, therefore by surjectivity of each $f_x : \mathcal{F}_x \to \mathcal{G}_x$, we obtain germs $(W_x, s_x)_x \in \mathcal{F}_x$ such that $(W_x, f_{W_x}(s_x))_x = (V,t)_x$ for all $x \in V$. By shrinking W_x and restricting s_x, we may assume $\{W_x\}$ covers V. Thus we have an open cover of V such that for all $s_x \in \mathcal{F}(W_x)$, we have $f_{W_x}(s_x) = \rho_{V,W_x}(t)$.

3. Trivially follows from 1. and 2.
4. (L $\Rightarrow$ R) Use the fact that taking stalks is a functor (Lemma 3.0.1).
   (R $\Rightarrow$ L) Let $g_x : \mathcal{G}_x \to \mathcal{F}_x$ be the inverse homomorphism of f_x for each $x \in X$. Using g_x, we can easily construct a sheaf homorphism $g : \mathcal{G} \to \mathcal{F}$ which will be the inverse of f. Indeed, consider the following map for any open $U \subseteq X$
   $$\begin{aligned} g_U : \mathcal{G}(U) &\longrightarrow \mathcal{F}(U) \\ t &\longmapsto s \end{aligned}$$
   where $s \in \mathcal{F}(U)$ is formed as the unique glue of the matching family
   $$\{s_x \in \mathcal{F}(U_x)\}_{x \in U}$$
   where $(U,t)_x = (U_x, f_{U_x}(s_x))_x$ for each $x \in U$ and $U_x \subseteq U$. In particular, $s_x = g_x((U_x, \rho_{U,U_x}(t))_x)$. This is obtained via the bijectivity of f_x. Consequently, g is a sheaf homomorphism, which is naturally the inverse of f.
5. Follows from 3. and 4.

$\square$

The following theorem further tells us that our intuition about algebra can be globalized, and equality of sheaf morphisms can be checked on each stalk.

Theorem 3.0.7. *Let X be a topological space and $\mathcal{F}, \mathcal{G}$ be two sheaves over X. Then, a map $f : \mathcal{F} \to \mathcal{G}$*

1. *is injective if and only if the kernel sheaf* Ker (f) *is the zero sheaf,*
2. *is surjective if and only if the image sheaf* Im (f) *is $\mathcal{G}$,*
3. *is equal to another map $g : \mathcal{F} \to \mathcal{G}$ if and only if $f_x = g_x$ for all $x \in X$.*

Proof. The main idea in most of the proofs below is to either use the definition or the universal property of sheafification.

1. (L $\Rightarrow$ R) Let $f : \mathcal{F} \to \mathcal{G}$ be injective. We wish to show that $\mathrm{Ker}\,(f) = 0$. Since the kernel presheaf $\ker f = 0$, therefore its sheafification $\mathrm{Ker}\,(f) = 0$.
   (R $\Rightarrow$ L) Let $\mathrm{Ker}\,(f) = 0$. We wish to show that f is injective. Suppose to the contrary that f is not injective. We have that $(\mathrm{Ker}\,(f))_x = 0$ for all $x \in X$. Thus there exists an open set $U \subseteq X$ such that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is not injective. Hence, there exists, $0 \neq s \in \mathcal{F}(U)$ such that $f_U(s) = 0$. Thus, we have an element $(U,s)_x \in (\ker f)_x = (\mathrm{Ker}\,(f))_x = 0$ for all $x \in U$. Hence $s = 0$ by Lemma 3.0.2, which is a contradiction.
2. (L $\Rightarrow$ R) Let $f : \mathcal{F} \to \mathcal{G}$ be a surjective map. In order to show that $\mathrm{Im}\,(f) = \mathcal{G}$, we will show that $\mathcal{G}$ satisfies the universal property of sheafification (Theorem 2.0.1). For this, consider a

sheaf $\mathcal{H}$ and a presheaf map $h : \operatorname{im}(f) \to \mathcal{H}$. Consider the inclusion map $\iota : \operatorname{im}(f) \hookrightarrow \mathcal{G}$. We will construct a unique sheaf map $\tilde{h} : \mathcal{G} \to \mathcal{H}$ which will be natural such that $\tilde{h} \circ \iota = h$. Pick any open set $U \subseteq X$. We wish to define the map

$$\tilde{h}_U : \mathcal{G}(U) \longrightarrow \mathcal{H}(U).$$

Take $t \in \mathcal{G}(U)$. By surjectivity of f, there exists a covering $\{U_i\}$ of U and matching family $s_i \in \mathcal{F}(U_i)$ for all i such that

$$f_{U_i}(s_i) = \rho_{U,U_i}(t) =: t_i.$$

We shall construct an element $\tilde{h}_U(t) \in \mathcal{H}(U)$. Indeed, we first claim that

$$\{h_{U_i}(t_i) \in \mathcal{H}(U_i)\}_i$$

is a matching family. This can be shown by keeping the following diagram in mind and the fact that $\{s_i\}$ is a matching family:

$$\begin{tikzcd}
\mathcal{H}(U_i) \arrow[d, "\rho_{U_i,U_i\cap U_j}"'] & \operatorname{im}(f_{U_i}) \arrow[l, "h_{U_i}"'] \arrow[d, "\rho_{U_i,U_i\cap U_j}"] & \mathcal{F}(U_i) \arrow[l, "f_{U_i}"'] \arrow[d, "\rho_{U_i,U_i\cap U_j}"] \\
\mathcal{H}(U_i \cap U_j) & \operatorname{im}\left(f_{U_i\cap U_j}\right) \arrow[l, "\tilde{h}_{U_i\cap U_j}"] & \mathcal{F}(U_i \cap U_j) \arrow[l, "f_{U_i\cap U_j}"]
\end{tikzcd}.$$

Thus we get a unique glue which we define to be the image of $\tilde{h}_U$ for the section $t \in \mathcal{G}(U)$, denoted $\tilde{h}_U(t) \in \mathcal{H}(U)$. Uniqueness and naturality follows from construction.
(R $\Rightarrow$ L) We have that $(\operatorname{im}(f))^{++} = \mathcal{G}$. Pick any open set $U \subseteq X$ and a section $t \in \mathcal{G}$. We wish to find an open cover $\{U_i\}_{i\in I}$ of U and $s_i \in \mathcal{F}(U_i)$ such that $f_{U_i}(s_i) = \rho_{U,U_i}(t)$ for all $i \in I$. Indeed, by Corollary 2.0.2, we obtain that $\mathcal{G}_x = \operatorname{im}(f)_x$ for all $x \in X$. Hence for the chosen (U, t), we obtain for each $x \in U$, by appropriately shrinking and restricting, an open set $W_x \subseteq U$ containing x and a section $s_x \in \mathcal{F}(W_x)$ satisfying $\rho_{U,W_x}(t) = f_{W_x}(s_x)$.

3. (L $\Rightarrow$ R) Trivial.
   (R $\Rightarrow$ L) Suppose for all $x \in X$ we have $f_x = g_x : \mathcal{F}_x \to \mathcal{G}_x$. We wish to show that $f = g$. Pick an open set $U \subseteq X$ and consider $s \in \mathcal{F}(U)$. We wish to show that $f_U(s) = g_U(s)$. For each $x \in U$, we have $(U, s)_x \in \mathcal{F}_x$ and by the fact that $f_x = g_x$, we further have

$$(U, f_U(s))_x = (U, g_U(s))_x.$$

Hence for all $x \in U$, there exists open $x \in W_x \subseteq U$ such that

$$\rho_{U,W_x}(f_U(s)) = \rho_{U,W_x}(g_U(s)).$$

It is then an easy observation that both $\{\rho_{U,W_x}(g_U(s))\}_{x\in U}$ and $\{\rho_{U,W_x}(f_U(s))\}_{x\in U}$ forms the same matching family. Hence we have a unique glue by sheaf axiom of $\mathcal{G}$ to obtain $f_U(s) = g_U(s)$ in $\mathcal{G}(U)$.

□

Lemma 3.0.8. *Let X be a topological space. Then, the following are equivalent:*

1. *The following is an exact sequence of sheaves over X*

$$\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'',$$

that is, $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$.

2. *The following is an exact sequence of stalks for each $x \in X$*

$$\mathcal{F}'_x \xrightarrow{f_x} \mathcal{F}_x \xrightarrow{g_x} \mathcal{F}''_x.$$

Proof. (1. $\Rightarrow$ 2.) Pick any $(U, s)_x \in \mathcal{F}_x$ which is in $\ker g_x$. Thus, there exists $V \subseteq U$ open such that $\rho_{U,V}(g_U(s)) = g_V(\rho_{U,V}(s)) = 0$. Thus $\rho_{U,V}(s) \in \mathcal{F}(V)$ is in $\mathrm{Ker}\,(g) = \mathrm{Im}\,(f)$ and thus $(V, \rho_{U,V}(s))_x = (U, s)_x \in \mathcal{F}_x$ is in $\mathrm{im}\,(f_x)$. Conversely, for $(U, f_x(t))_x \in \mathrm{im}\,(f_x)$, we see that since $g \circ f = 0$, then $(U, g_x(f_x(t)))_x = 0$.

(2. $\Rightarrow$ 1.) This is immediate, by looking at a section of $\mathcal{F}$ at any open set (use Remark 2.0.4). $\square$

Given an open subset U of X and a sheaf over U, we can extend it to a sheaf over X by zeros. This in particular means extending a sheaf from a subspace in such a way so that stalks outside of the subspace are always zero. This operation would be fundamental in cohomology and other places as it yields a nice exact sequence corresponding to any closed or open subset of X.

Definition 3.0.9 (Extending a sheaf by zeros). Let X be a space and $i : Z \hookrightarrow X$ be an inclusion of a closed set and $j : U \hookrightarrow X$ be an inclusion of an open set.

1. If $\mathcal{F}$ is a sheaf over Z, then $i_*\mathcal{F}$ is a sheaf over X called the extension of $\mathcal{F}$ to X by zeros.
2. If $\mathcal{F}$ is a sheaf over U, then the extension of $\mathcal{F}$ to X by zeroes, denoted $j_!\mathcal{F}$ is the sheafification of the presheaf over X given by

$$V \longmapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U \\ 0 & \text{else.} \end{cases}$$

The main result is as follows.

Proposition 3.0.10. [4] *Let X be a space, $i : Z \hookrightarrow X$ be closed and $j : U \hookrightarrow X$ be open. Then,*

1. *If $\mathcal{F}$ is a sheaf over Z, then for any $p \in X$, we have*

$$(i_*\mathcal{F})_p = \begin{cases} \mathcal{F}_p & \textit{if } p \in Z \\ 0 & \textit{if } p \notin Z. \end{cases}$$

2. *If $\mathcal{F}$ is a sheaf over U, then for any $p \in X$, we have*

$$(j_!\mathcal{F})_p = \begin{cases} \mathcal{F}_p & \textit{if } p \in U \\ 0 & \textit{if } p \notin U. \end{cases}$$

Moreover, $(j_!\mathcal{F})_{|U} = \mathcal{F}$ and $j_!\mathcal{F}$ is unique w.r.t these two properties.

[4] Exercise II.1.19 of Hartshorne.

Proof. The first item follows immediately from the fact that $Z \subseteq X$ is a closed subset. In particular, if $p \notin Z$, then there is a cofinal collection of open sets containing p on which $i_*\mathcal{F}$ is 0.

For the second item, we proceed as follows. Let G be the presheaf as in Definition 3.0.9, 2. Note that

$$G_p = \begin{cases} \mathcal{F}_p & \text{if } p \in V \subseteq U \text{ for some open } V \subset X, \\ 0 & \text{else.} \end{cases}$$

In particular, if $p \in U$, then $G_p = \mathcal{F}_p$ and if $p \notin U$, then $G_p = 0$. Since stalks before and after sheafification are same, therefore we have our result for stalks. Next, $(j_!\mathcal{F})_{|U} = \mathcal{F}$ because over U, the presheaf $G_{|U}$ itself is a sheaf, so sheafification of G will yield a sheaf equal to $\mathcal{F}$ over U. Further $j_!\mathcal{F}$ is unique with the two properties as if for any other sheaf $\mathcal{G}$ which satisfies that $\mathcal{G}_{|U} = \mathcal{F}$, then we get an map of presheaves $G \to \mathcal{G}$ which induces an isomorphism on stalks. By universal property of sheafification (Theorem 2.0.1), we deduce that $j_!\mathcal{F} \cong \mathcal{G}$. $\square$

With the above result, we have a useful short exact sequence.

Corollary 3.0.11. *Let X be a space and $\mathcal{F}$ be a sheaf over X. Let $i : Z \hookrightarrow X$ be a closed subspace and $j : U = X \setminus Z \hookrightarrow X$ be the corresponding open subspace. Then there is a short exact sequence*

$$0 \longrightarrow j_!\mathcal{F}_{|U} \longrightarrow \mathcal{F} \longrightarrow i_*\mathcal{F}_{|Z} \longrightarrow 0$$

where $\mathcal{F}_{|Z} = i^{-1}\mathcal{F}$. We call this the extension by zero short exact sequence.

Proof. Following the notation of proof of Proposition 3.0.10, we see that we have an injective map $G \to \mathcal{F}$, which then by universal property and local nature of injectivity gives an injective map $j_!\mathcal{F}_{|U} \to \mathcal{F}$. The map $\mathcal{F} \to i_*\mathcal{F}_{|Z}$ is obtained by considering the unit map of the adjunction $i_* \vdash i^{-1}$. This is surjective because on the stalks, we obtain $(i_*\mathcal{F}_{|Z})_p = \mathcal{F}_p$ if $p \in Z$ or 0 otherwise by above result. To show exactness at middle, we again go to stalks (Lemma 3.0.8) and observe that if $p \in U$, then we get exact sequence $0 \to \mathcal{F}_p \stackrel{\text{id}}{\to} \mathcal{F}_p \to 0 \to 0$ and if $p \in Z$, then we get the exact sequence $0 \to 0 \to \mathcal{F}_p \stackrel{\text{id}}{\to} \mathcal{F}_p \to 0$. $\square$

4 Sheaves are étale spaces

Another important and in some sense dual viewpoint of sheaves over X is that they can be equivalently defined as a certain type of bundle over X and all such bundles arises only from a sheaf. This is important because this viewpoint naturally extends the usual concepts of covering spaces, bundles and vector bundles to that of sheaves. In particular, a lot of classical constructs in algebraic topology can be equivalently be seen as specific instantiates of the notion of étale space of the sheaf.

Definition 4.0.1. (**Étale space**) Let X be a topological space and let $\pi : E \to X$ be a bundle over X. Then (E, π, X) is said to be étale over X or just étale if for all $e \in E$, there exists an open set $V \ni e$ of E such that $p(V)$ is open and $p|_V : V \to p(V)$ is a homeomorphism, that is, if p is a local homeomorphism. A morphism of étale spaces $(E_1, \pi_1, X), (E_2, \pi_2, X)$ over X is given by a continuous map $f : E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$. Denote the category of étale spaces over X by $\mathbf{Et}/X$.

Clearly, covering spaces over X are étale spaces over X, but not all étale spaces over X are covering, of-course. We now wish to show that the sheafification functor factors through a functor mapping a presheaf to an étale space. In particular, we want to show the existence of functor F, G so that the following commutes

$$\begin{tikzcd} & \mathbf{Et}/X \arrow[dr, "F"] & \\ \mathbf{PSh}(X) \arrow[ur, "G"] \arrow[rr, "(-)^{++}"] & & \mathbf{Sh}(X) \arrow[ll, hook] \end{tikzcd}.$$

Construction 4.0.2. (*Étale space of a sheaf*) Let us now show the construction of the above functors:

1. (*The functor G*) Let P be a presheaf over X. The étale space $E := G(P)$ is given by the disjoint union of all stalks:

$$E := \coprod_{x \in X} P_x.$$

The topology on E is given by the initial topology of the map

$$\begin{aligned} \pi : E &\longrightarrow X \\ s_x &\longmapsto x. \end{aligned}$$

In particular, E has a basis given by sets of the form $B_{U,s} \subseteq E$ where $B_{U,s} = \{s_x \in E \mid x \in U\}$ and $s \in P(U)$. Next, we wish to establish that π is a local homeomorphism. So take any $s_x \in E$ and consider the basic open set $B_{U,s} \ni s_x$. The map $\pi|_{B_{U,s}} : B_{U,s} \to \pi(B_{U,s})$ takes $s_x \mapsto x$. This is a homeomorphism because we can construct an inverse given by $x \mapsto s_x$. A simple calculation checks that this is continuous. Hence indeed, (E, π, X) is an étale space over X.

Next consider a map of presheaves $\varphi : F \to G$. We can construct a map of corresponding étale spaces as

$$\begin{aligned} \hat{\varphi} : (E_F, \pi_F, X) &\longrightarrow (E_G, \pi_G, X) \\ s_x &\longmapsto \varphi_x(s_x). \end{aligned}$$

This map is continuous and a valid bundle map over X. This defines the functor G.

2. (*The functor F*) Let $\pi : E \to X$ be an étale space over X. Then, we can construct a sheaf $\mathcal{E}$ over X out of it. This is done in a very natural way by considering the set of sections over U of $\mathcal{E}$ to be quite literally the set of *cross-sections*[5] of map π on U. That is, define:

$$\mathcal{E}(U) := \{s : U \to E \mid \pi \circ s = \mathrm{id}_U\}.$$

The fact that this is indeed a sheaf can be seen by a general phenomenon that for any continuous map $f : X \to Y$, the set of all cross-sections of f over open subsets of Y assembles

[5]In-fact, historically the notion of sheaf was really that of this étale space, and that is why to this day, we still use the terminology of "sections" of a sheaf over an open subset.

itself into a sheaf. Hence, we have constructed a sheaf $\mathcal{E}$ out of an étale space E over X.

Next consider a map of étale spaces $\xi : (E_1, \pi_1, X) \to (E_2, \pi_2, X)$. we can construct a map of corresponding sheaves $\tilde{\xi} : \mathcal{E}_1 \to \mathcal{E}_2$ by defining the following for open $U \subseteq X$:

$$\begin{aligned} \tilde{\xi}_U : \mathcal{E}_1(U) &\longrightarrow \mathcal{E}_2(U) \\ s &\longmapsto \xi \circ s. \end{aligned}$$

One can check that this is indeed a valid sheaf morphism. This defines the functor F.

We then see that the categories $\mathbf{Et}/X$ and $\mathbf{Sh}(X)$ are equivalent.

Theorem 4.0.3. [6] *(The étale viewpoint of sheaves) Let X be a topological space. The functors F and G as defined in Construction 4.0.2 defines an equivalence of categories*

$$\mathbf{Sh}(X) \equiv \mathbf{Et}/X.$$

We will prove this result in many small lemmas below. We would first like to observe that for any étalé bundle E over X yields a sheaf by $F(E)$ whose stalks are bijective to fibres of E.

Lemma 4.0.4. *Let (E, π, X) be an étalé bundle over X and let $\mathcal{E}$ be the sheaf obtained by $F((E, \pi, X))$. Then, for any $x \in X$, the following is a bijection*

$$\begin{aligned} \tau_x : \mathcal{E}_x &\longrightarrow E_x := \pi^{-1}(x) \\ (U, s)_x &\longmapsto s(x). \end{aligned}$$

Proof. We first show that τ_x is injective. Let $(U, s)_x, (V, t)_x$ be two germs such that $p = s(x) = t(x)$. We wish to show that s and t are equal on an open subset in $U \cap V$. As E is étaleé, therefore we have an open $A \subseteq E$ with $p \in A$ such that $\pi|_A : A \to \pi(A)$ is a homeomorphism. Consequently, we see that the open set $W = \pi(A) \cap U \cap V$ would do just fine.

We now show surjectivity. Pick $e \in E_x$. As E is étalé, we thus get an open set $A \ni e$ in E such that $\pi|_A : A \to \pi(A)$ is a homeomorphism. Denote the inverse of this homeomorphism by $g : \pi(A) \to A$. This is therefore a section of E over $\pi(A)$ where $x \in \pi(A)$. Consequently, $(\pi(A), g)_x \in \mathcal{E}_x$ is such that τ_x maps it to e. $\square$

Proof of Theorem 4.0.3. We first show that $F \circ G$ is naturally isomorphic to sheafification functor. Let $\mathcal{E}$ be a presheaf, $(E, \pi) = G(\mathcal{E})$ and $F(E, \pi) = \mathcal{E}'$. We wish to show that there is a natural isomorphism $\mathcal{E}^{++} \to \mathcal{E}'$. By Theorem 2.0.1 and 3.0.6, 3, it suffices to show that there is a map of presheaves $\mathcal{E} \to \mathcal{E}'$ which is isomorphism on stalks.

Consider the map $\varphi : \mathcal{E} \to \mathcal{E}'$ which on an open set $U \subseteq X$ gives the following map

$$\begin{aligned} \varphi_U : \mathcal{E}(U) &\longrightarrow \mathcal{E}'(U) \\ s &\longmapsto U \overset{f_{U,s}}{\to} E \end{aligned}$$

[6] Exercise II.1.13 of Hartshorne.

where $f_{U,s} : U \to E$ maps as $x \mapsto (U,s)_x$. Since $f_{U,s}$ is just the stalk map, then as in Construction 4.0.2, $f_{U,s}$ is continuous. Now on the stalks, we get the following commutative diagram by Lemma 4.0.4:

$$\begin{tikzcd} \mathcal{E}_x \arrow[r, "\varphi_x"] \arrow[d] & \mathcal{E}'_x \\ E_x \arrow[ur, dash, "\cong"'] & \end{tikzcd},$$

where the vertical map takes a germ $(U,s)_x$ and maps it to the element represented in $E_x = \mathcal{E}_x$, as $E_x = \pi^{-1}(x) = \{x \in E \mid \pi(e) = x\} = \{(U,s)_y \in E \mid \pi((U,s)_y) = y = x\}$. Consequently, the vertical map is a bijection and thus φ_x is a bijection. The naturality of this isomorphism can be checked trivially.

We now wish to show that $G \circ F$ is naturally isomorphic to the identity functor on $\mathbf{Et}/X$. Pick an étalé bundle (E,π) over X, denote $F(E,\pi) = \mathcal{E}$ and $G(\mathcal{E}) = (E', \pi')$. We wish to find a homeomorphism φ so that the following commutes:

$$\begin{tikzcd} E' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & E \arrow[dl, "\pi"] \\ X & \end{tikzcd}.$$

Consider the following map

$$\begin{aligned} \varphi : E' &\longrightarrow E \\ (U,s)_x &\longmapsto s(x). \end{aligned}$$

By Lemma 4.0.4, φ is a bijective map. We thus reduce to showing that φ is a continuous open map.

To show continuity, consider an open set $A \subseteq E$ and then observe that

$$\begin{aligned} \varphi^{-1}(A) &= \{(U,s)_x \in E' \mid s(x) \in A\} \\ &= \{(U,s)_x \in E' \mid x \in s^{-1}(A)\} \\ &= \bigcup_{U \ni x, s: U \to E} B_{U,s} \end{aligned}$$

and since $B_{U,s} \subseteq E'$ is a basic open, therefore φ is continuous.

Finally, to show that φ is open, one reduces to showing that if $s : U \to E$ is a continuous section of bundle (E,π) and $U \subseteq X$ is an open set, then $s(U)$ is an open set in E (by working with a basic open $B_{U,s} \subseteq E'$). This follows from the fact that since π is a local homeomorphism, therefore for each $e \in s(U)$, there exists an open set $A \ni e$ in E such that $s(U) \cap A \ni e$ and since $\pi : s(U) \cap A \to \pi(s(U) \cap A) = U \cap \pi(A)$ is a homeomorphism, we further get that $s(U) \cap A$ is open (as $U \cap \pi(A)$ is open). Consequently, $s(U)$ is open. $\square$

Remark 4.0.5. (*The sheaf associated to a covering space*) By the above equivalence, each covering space space over X, which is an étale map, determines a unique sheaf (upto isomorphism). We analyze this sheaf. Recall that a local system is just a name for locally constant sheaf. We write $\mathrm{LocSys}(X)$ to denote the category of all local systems.

Proposition 4.0.6. *Let X be a connected and locally path-connected space. The there is an equivalence of categories*

$$\mathbf{Cov}(X) \equiv \mathrm{LocSys}(X)$$

where $\mathbf{Cov}(X)$ is the category of covering spaces over X and $\mathrm{LocSys}(X)$ is the category of locally constant sheaves of sets over X.

Proof. We will show that this equivalence is induced from the equivalence of Theorem 4.0.3. It is sufficient to show that F maps covering spaces to locally constant sheaves and vice-versa for G. Indeed, if (E, p, X) is a covering space and $\mathcal{E}$ is the associated sheaf, then for a connected evenly covered neighborhood $U \subseteq X$ for which $p^{-1}(U) = \amalg_{\alpha \in A_U} V_\alpha$ where $p : V_\alpha \to U$ is a homeomorphism, we get that the set of sections $\mathcal{E}(U)$ is just A_U by connectedness. Moreover, it is clear that $\mathcal{E}(V) = A_U$ again for any connected $V \subseteq U$. This shows that $\mathcal{E}_{|U} = \underline{A_U}$. Hence $\mathcal{E}$ is a local system.

Conversely, if $\mathcal{E}$ is a local system with (E, p, X) its associated étale space, then for $U \subseteq X$ such that $\mathcal{E}_{|U} = \underline{A}$, we get that $p^{-1}(U) = \amalg_{x \in U} \mathcal{E}_x = \amalg_{x \in U} A = \amalg_{\alpha \in A} V_\alpha$ where $V_\alpha = \{\alpha \in A_x \mid x \in U\}$. We first claim that V_α is open. Indeed, it is the basic open set $B_{U,\alpha}$. Next, $V_\alpha \cap V_\beta =$ is clear. Finally, $p : V_\alpha \to U$ being a homeomorphism is also clear as this is a bijection and p is an open map as it is étale. $\square$

5 Direct and inverse image

Let $f : X \to Y$ be a continuous map of topological spaces. Then one can derive two functors $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and $f^{-1} : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ which are adjoint of each other, called direct and inverse image functors respectively. While f_* is easy to define, it is usually the inverse image of a sheaf that causes trouble for its obscurity if one works with the definition that inverse image functor is left-adjoint to direct image functor. This is resolved by working with the corresponding étale spaces (Theorem 4.0.3). In this section we will show how to construct them.

Let us first define the direct image functor.

Definition 5.0.1. (**Direct image**) Let $f : X \to Y$ be a continuous map. Then, for any sheaf $\mathcal{F}$ on X, we can define its direct image under f as $f_*\mathcal{F}$ whose sections on open $V \subseteq Y$ are given by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)).$$

This can easily be seen to be a sheaf. For any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X, we can define the map of direct image sheaves as

$$\begin{aligned} (f_*\varphi)_V : f_*\mathcal{F}(V) &\longrightarrow f_*\mathcal{G}(V) \\ s &\longmapsto \varphi_{f^{-1}(V)}(s). \end{aligned}$$

This defines a functor

$$f_* : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y).$$

One defines the inverse image of a sheaf as follows:

Definition 5.0.2. (**Inverse image**) Let $f : X \to Y$ be a continuous map and let $\mathcal{G}$ be a sheaf over Y. Consider a presheaf F over X constructed by the data of $\mathcal{G}$ as follows. Let $U \subseteq X$ be open, then define

$$f^{+}\mathcal{G}(U) := \varinjlim_{\text{open } V \supseteq f(U)} \mathcal{G}(V),$$

where restriction maps of $f^{+}\mathcal{G}$ is given by the unique map obtained by universality of colimits. Then, $f^{+}\mathcal{G}$ is a presheaf over X and this construction is functorial again by universal property of colimits:

$$f^{+} : \mathbf{PSh}(Y) \longrightarrow \mathbf{PSh}(X).$$

Let $f^{-1}\mathcal{G} = (f^{+}\mathcal{G})^{++}$ denote the sheafification of $f^{+}\mathcal{G}$. This sheaf is called the inverse sheaf of $\mathcal{G}$ under f. Now for any map of sheaves $\varphi : \mathcal{G} \to \mathcal{H}$ over Y, we get a corresponding map of inverse image sheaves $f^{-1}\varphi : f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{H}$ by composition of two functors. This yields a functor

$$f^{-1} : \mathbf{Sh}(Y) \longrightarrow \mathbf{Sh}(X).$$

As is visible, this definition is quite obscure if one likes elemental definitions. We thus give some general properties enjoyed by inverse sheaf.

Lemma 5.0.3. *Let $f : X \to Y$ be a continuous map and $\mathcal{G}$ be a sheaf over Y.*

1. *If f is open, then $f^{-1}\mathcal{G} = \mathcal{G}(f(-))$.*
2. *If f is constant to $y \in Y$, then $f^{-1}\mathcal{G}$ is the constant sheaf on X with sections $\mathcal{G}_y$.*
3. *If $X = \{x\}$ is a singleton space, then $f^{-1}\mathcal{G}$ is the constant sheaf on X with sections $\mathcal{G}_{f(x)}$.*
4. *If $x \in X$, then*

$$(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}.$$

Proof. 1. One notes that $f^{+}\mathcal{G}(U) := \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \mathcal{G}(f(U))$. The mapping $\mathcal{G}(f(-))$ is a sheaf, hence sheafifying it will yield the same sheaf.
2. We see that $f^{+}\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \varinjlim_{V \ni y} \mathcal{G}(V) = \mathcal{G}_y$ and presheaves with constant values are sheaves, as restrictions are identity.
3. We see that $f^{+}\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) = \varinjlim_{V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}$ and presheaves with constant values are sheaves, as restrictions are identity.
4. By passing to the right adjoint, one observes that for $f : X \to Y$ and $g : Y \to Z$ continuous maps, one can obtain the following natural isomorphism of functors

$$(g \circ f)^{-1} \cong f^{-1} \circ g^{-1}.$$

Consider the composite $f \circ \iota$ where $\iota : \{x\} \hookrightarrow X$ is the inclusion map. Consequently, by 3. above, we obtain the following

$$\begin{aligned} \mathcal{G}_{f(x)} &\cong (f \circ \iota)^{-1}(\mathcal{G})(\{x\}) \\ &\cong (\iota^{-1} \circ f^{-1})(\mathcal{G})(\{x\}) \\ &\cong \iota^{-1}(f^{-1}\mathcal{G})(\{x\}) \\ &\cong (f^{-1}\mathcal{G})_{f(x)}. \end{aligned}$$

$\square$

The following is a fundamental duality between inverse and direct image functors.

Theorem 5.0.4. [7] *(Direct and inverse image adjunction) Let $f : X \to Y$ be a continuous map. Then the the inverse image functor is the left adjoint of direct image functor* [8]

$$\mathbf{Sh}(Y) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathbf{Sh}(X) \ .$$

In particular, we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}\left(f^{-1}\mathcal{F}, \mathcal{G}\right) \cong \mathrm{Hom}_{\mathbf{Sh}(Y)}\left(\mathcal{F}, f_*\mathcal{G}\right).$$

One situation that we will find ourselves a lot in algebraic geometry is when $f : X \to Y$ will be a closed immersion of topological spaces ($f : X \to f(X)$ is homeomorphism and $f(X) \subseteq Y$ is closed) and for a sheaf $\mathcal{F}$ over X, we would like to find $(f_*\mathcal{F})_{f(x)}$ for each point $x \in X$. This is a situation where the stalk of direct image can be calculated quite easily.

Lemma 5.0.5. *Let $f : X \to Y$ will be a closed immersion of topological spaces and $\mathcal{F}$ a sheaf over X. Then, there is a natural isomorphism*

$$(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x.$$

Proof. From a straightforward unravelling of definitions of the two stalks, the result follows from the observation that each open set $U \ni x$ in X is in one-to-one correspondence with open set $f(U) \ni f(x)$ in Y. □

Remark 5.0.6. We wish to know how the inverse image of sheaves changes the stalk. Let $f : X \to Y$ be a continuous map and let $\mathcal{F}$ be a sheaf on Y. Consider the inverse sheaf $f^{-1}\mathcal{F}$ on X. Let $x \in X$. Then we have that (Lemma 5.0.3, 4)

$$(f^{-1}\mathcal{F})_x \cong \mathcal{F}_{f(x)}.$$

The importance of this is that, suppose $f : X \to Y$ is given together with $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X and Y respectively and a map $\varphi^\flat : \mathcal{G} \to f_*\mathcal{F}$ over Y, which is equivalent to $\varphi^\sharp : f^{-1}\mathcal{G} \to \mathcal{F}$ over X. Now, most of the time, our interest in a sheaf is only limited to stalks (functions defined in *some* open subset around a point), therefore we are mostly interested in considering only the map induced at the level of stalks at a point $f(x) \in Y$:

$$\varphi^\flat_{f(x)} : \mathcal{G}_{f(x)} \longrightarrow (f_*\mathcal{F})_{f(x)}.$$

But the description of the stalk $(f_*\mathcal{F})_{f(x)}$ is usually not simple to derive. But dually, we may ask the map of stalks of the other map at $x \in X$, and we directly land into the stalks

$$\varphi^\sharp_x : \mathcal{G}_{f(x)} \cong (f^{-1}\mathcal{G})_x \longrightarrow \mathcal{F}_x.$$

[7] Exercise II.1.18 of Hartshorne.

[8] admirers of topoi may see this as a quintessential example of geometric map of topoi.

However, this is a strange map as the stalks are of sheaves which are not on same space. In particular, this map is given as follows. For any open $V \ni f(x)$ in Y, we have the following maps:

$$\mathcal{G}(V) \xrightarrow{\varphi_V^\flat} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}_x \ .$$

Passing to colimits ($\varphi_V^\flat$ commutes with restrictions), one can see that we get the map $\varphi_x^\sharp : \mathcal{G}_{f(x)} \to \mathcal{F}_x$ back.

It is a good principle to keep in mind that if we wish to work with explicit local sections, then we should look for the "flat" map and if it is enough to work with germs, then we should look for the "sharp" map, even though the above remark telling us how to construct the map of stalks from the "flat" maps on each open set.

This map $\varphi_x^\sharp$ can be heuristically be defined as the map which on sections which makes sure that a non-invertible section remains non-invertible after going through the map. Hence we mostly work only with maps $f^{-1}\mathcal{G} \to \mathcal{F}$ if we are interested only at the stalk level (which is more than enough for us).

6 Category of sheaves

We will discuss some basic properties of the category of sheaves over X, denoted $\mathbf{Sh}(X)$. This is important as we wish to calculate cohomology of its objects, hence we would require the notion of injective and projective resolutions of sheaves. We covered the homological methods necessary for this section in the Homological Methods, Chapter **??**. Let us first begin with a more categorical definition of sheaves.

Definition 6.0.1. **(Sheaf of sets - categorical defn.)** *Suppose X is a topological space and $\mathbf{O(X)}$ is the posetal category of open sets of X, ordered by inclusion. Then a presheaf*

$$F : \mathbf{O(X)}^{\text{op}} \longrightarrow \mathbf{Sets}$$

is a sheaf if for any open set U and any covering of $U = \bigcup_{i\in I} U_i$, we have that

$$\begin{tikzcd} FU \arrow[dr, dashed, "e"] & & \\ & \prod_{i\in I} FU_i \arrow[r, dashed, shift left, "p"] \arrow[r, dashed, shift right, "q"'] & \prod_{i,j\in I} F(U_i \cap U_j) \end{tikzcd}$$

is an equalizer diagram, where the unique maps e, p & q are given as:

- *e : for a $f \in FU$, e maps it as*

$$e(f) = \{\underbrace{F(U_i \subset U)}_{FU \to F(U_i)}(f)\} \in \prod_i F(U_i)$$

That is, e maps each element f of the FU via the set map under the functor F of the inclusion $U_i \subset U$.

- *p : for a sequence $\{f_i\} \in \prod_{i\in I} FU_i$, p maps it as*

$$p(\{f_i\}) = \{\underbrace{F(U_i \cap U_j \subset U_i)}_{FU_i \to F(U_i \cap U_j)}(f_i)\} \in \prod_{i,j\in I} F(U_i \cap U_j)$$

That is, p maps each component y_i of the sequence $\{y_i\}$ via the set map under the functor F of the inclusion $U_i \cap U_j \subset U_i$.

- *q : for a sequence $\{f_i\} \in \prod_{i\in I} FU_i$, q maps it as*

$$q(\{f_i\}) = \{\underbrace{F(U_i \cap U_j \subset U_j)}_{FU_j \to F(U_i \cap U_j)}(f_j)\} \in \prod_{i,j\in I} F(U_i \cap U_j)$$

That is, q maps each component y_i of the sequence $\{y_i\}$ via the set map under the functor F of the inclusion $U_i \cap U_j \subset U_j$.[9]

6.1 Coverings, bases & sheaves

We now quickly discuss some easy properties of sheaves. In the following, a **Subsheaf** of a sheaf F is defined as a subfunctor of F which also satisfies the sheaf property (is a sheaf itself).

Proposition 6.1.1. *A subfunctor S of a sheaf F is a subsheaf if and only if for any open set U and it's open covering $\bigcup_{i\in I} U_i$ together with an $f \in FU$, we have $f \in SU$ if and only if $f|_{U_i} \in SU_i \ \forall\ i \in I$.*

Proof. ($\mathbf{L} \implies \mathbf{R}$) Suppose S is a subsheaf, then clearly for any $f \in SU \subset FU$, we must have $f|_{U_i} \in SU_i$ for all $i \in I$ and for any such collection of $f|_{U_i}$, by the sheaf property of S, $f \in SU$.
($\mathbf{R} \implies \mathbf{L}$) Since S is a subfunctor of F, therefore $SV \subset FV$ for any open V. With this, because F is a sheaf, we have the following diagram:

$$\begin{tikzcd}
SU \arrow[r] \arrow[d, tail] & \prod_i SU_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & \prod_{i,j} S(U_i \cap U_j) \arrow[d, tail] \\
FU \arrow[r] & \prod_i FU_i \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} F(U_i \cap U_j)
\end{tikzcd}$$

where the bottom row is the equalizer. The condition on the right says that for $f \in FU$, $f \in SU \iff \{f|_{U_i}\} \in \prod_i SU_i$, which means that the left square is a pullback. Now because SU is universal due to it being a pullback, and since the top row infact commutes, therefore SU is universal with top row commuting, hence, it is an equalizer. □

6.1.1 Sheaf itself is local

Define **restriction of a sheaf** F on X restricted to open $U \subset X$ to be the $F|_U(V) = F(V)$ where $V \subset U$, and $F|_U(U) = F(\phi) = \{*\}$ if $V \not\subset U$.

[9] Refraining to write $F(V \subset U) = FU \to FV$ to be equal to the restriction $(-)|_V$ exaggerates the emphasis on the abstract nature of sheaf F, that is, it helps to imagine that FU might not always be a set of specific maps over U, even though in most examples of interest it is the case.

Theorem 6.1.2. *Suppose X is a space with a given open covering $X = \bigcup_{k\in I} W_k$. If there are sheaves for each k,*

$$F_k : \mathbf{O(Wk)}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$$

[10]*such that*

$$F_k|_{W_k\cap W_l} = F_l|_{W_k\cap W_l}$$

[11] *then, $\exists$ a sheaf F on X,*

$$F : \mathbf{O(X)}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$$

unique upto isomorphism such that

$$F|_{W_k} \cong F_k.$$

This theorem hence shows that the restriction functor $U \mapsto \mathrm{Sh}\,(U)$ and $V \subset U \mapsto (\mathrm{Sh}\,(U) \to \mathrm{Sh}\,(V), F|_U \mapsto F|_V)$ on $\mathbf{O(X)}$ is local enough to be *almost* a sheaf. If only for any sheaf F, G on X, we had that $F|_{W_k} = G|_{W_k} \,\forall\, k$ would imply that $F = G$, which is not the case in general however, then we would have said that this restriction functor is also a sheaf.

6.1.2 Sheaf over a basis of X

A **basis** of a space X is a subset of topology $\mathcal{B} \subset \mathcal{O}(X)$ such that for any open $U \in \mathcal{O}(X)$, $\exists \{B_i\} \subseteq \mathcal{B}$ such that $U = \bigcup_i B_i$.
It turns out that the restriction functor $r : \mathrm{Sh}\,(X) \longrightarrow \mathrm{Sh}\,(X_{\mathcal{B}})$ which restricts each sheaf over X to that of open sets of basis $\mathcal{B}$ establishes an equivalence of categories!

Theorem 6.1.3. *Suppose X is a topological space and $\mathcal{B}$ is a basis for X. Then, the restriction functor*

$$\begin{aligned} r : Sh\,(X) &\longrightarrow Sh\,(X_{\mathcal{B}}) \\ F &\longmapsto F|_{\mathcal{B}} \\ \eta : F \Longrightarrow G &\longmapsto \eta|_{\mathcal{B}} : F|_{\mathcal{B}} \Longrightarrow G|_{\mathcal{B}} \end{aligned}$$

establishes an equivalence of categories between $Sh\,(X)$ and $Sh\,(X_{\mathcal{B}})$.

Proof. For any sheaves F, G in $\mathrm{Sh}\,(X)$, we want to show that $\mathrm{Hom}_{\mathrm{Sh}(X)}\,(F, G) \cong \mathrm{Hom}_{\mathrm{Sh}(X_{\mathcal{B}})}\,(rF, rG)$, that is, r is fully faithful. One can see that there r is an injection between the above hom-sets as for any $\epsilon, \eta : F \Rightarrow G$, if $rF = F|_{\mathcal{B}} = G|_{\mathcal{B}} = rG$, then due to the commutation of the two squares below because of naturality, (take $U = \bigcup_i B_i$ to be any open set and it's trivial open covering from basic open sets)

$$\begin{tikzcd} FU \arrow[r, "e_F"] \arrow[d, shift right, "\epsilon U"'] \arrow[d, shift left, "\eta U"] & \prod_i FB_i \arrow[d, shift right, "\prod_i \epsilon B_i"'] \arrow[d, shift left, "\prod_i \eta B_i"] \\ GU \arrow[r, "e_G"] & \prod_i GB_i \end{tikzcd}$$

---

[10]where $\mathbf{O(Wk)}^{\mathrm{op}}$ is the opposite category of all open subsets of open set W_k and inclusion.

[11]This condition implies that for any open subsets $V_k \subset W_k$ and $V_l \subset W_l$, $F(V_k \cap W_k \cap W_l) = F(V_l \cap W_k \cap W_l)$ and for arrows $X_1 \subset X_2$ in $\mathbf{O(Wk)}$ & $Y_1 \subset Y_2$ in $\mathbf{O(Wl)}$, $F(X_1 \cap W_k \cap W_l \subset X_2 \cap W_k \cap W_l) = F(Y_1 \cap W_k \cap W_l \subset Y_2 \cap W_k \cap W_l)$.

one can infer $\epsilon U = \eta U$ (e_F and e_G are equalizers, so are monic).
With the information $\kappa : rF \Rightarrow rG$, one can construct a natural transformation $\gamma : F \Rightarrow G$ by defining FU and GU, for any open U with it's basic cover $U = \bigcup_i B_i$ where $B_i \in \mathcal{B}$, as the equalizer of the parallel arrows $\prod_i F|_{\mathcal{B}} B_i \rightrightarrows \prod_{i,j} F|_{\mathcal{B}} B_i \cap B_j$ and $\prod_i G|_{\mathcal{B}} B_i \rightrightarrows \prod_{i,j} G|_{\mathcal{B}} B_i \cap B_j$, respectively. Then, one defines $\gamma U : FU \to GU$ by noticing that the former forms a cone over the latter, due to arrows $\prod_i \kappa B_i : \prod_i F|_{\mathcal{B}} B_i \to \prod_i G|_{\mathcal{B}} B_i$ and $\prod_{i,j} \kappa(B_i \cap B_j) : \prod_{i,j} F|_{\mathcal{B}} B_i \cap B_j \Rightarrow \prod_{i,j} G|_{\mathcal{B}} B_i \cap B_j$, so that there exists a unique arrow $FU \to GU$, which we just define as γU.
With this, we see that r is fully faithful. Finally, with the above definitions, $rF \cong F|_{\mathcal{B}}$ where $F \in \mathrm{Sh}(X)$ is the sheaf obtained by the above process from $F|_{\mathcal{B}} \in \mathrm{Sh}(X_{\mathcal{B}})$ because both of them are equalizers of the same diagram for any open set $U = \bigcup_i B_i$ and it's basic covering (note that any covering of U can be decomposed into basic covering). □

6.2 Sieves as general covers

This is related to generalization of sheaves to topos theory. As we saw in Definition **??**, a subfunctor of $\mathbf{Yon}(C) = \mathrm{Hom}(-, C)$ is a sieve, therefore this notion would allow us to generalize the notion of *covering* of a space, as we will see later. But for now, the *shadow* of that more general notion can still be felt in the usual category $\mathbf{O(X)}$ of open sets of X.

Definition 6.2.1. (**Principal Sieve**) *Suppose X is a topological space and U is open. Then the sieve S,* generated *from U, that is,*

$$S = \{V \ : \ \textit{open } V \subset U\}$$

is said to be a principal sieve, denoted $S = \langle U \rangle$, generated by a single open set.

With Definition 6.2.1, we can now define a new notion of *covering* of an open set, purely in terms of arrows onto it!

Definition 6.2.2. (**Covering Sieve**) *Suppose X is a topological space and U is open in it. A sieve S on U is said to cover U if*

$$U = \bigcup_{W \in S} W.$$

That is, when U is union of all open sets in the sieve S.

Remark 6.2.3. It can be seen quite easily that a subfunctor S of $\mathbf{Yon}(U)$ is a principal sieve over U if and only if S is a subsheaf. L $\Longrightarrow$ R by Proposition **??** and R $\Longrightarrow$ L by noting that the union of all sets in S would generate it. Remember that you can take covers of only those open sets which are members of S because S is a subsheaf.

The above definition in effect can be replaced with in the definition of sheaves!

Proposition 6.2.4. *A presheaf $P : \mathbf{O(X)}^{\mathrm{op}} \longrightarrow$ **Sets** on a topological space X is a sheaf if and only if for any open U and a covering sieve S over U, we have that the inclusion nat. trans. $i_S : S \Longrightarrow \boldsymbol{Yon}(U)$ induces an isomorphism:*

$$\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(S, P) \cong \mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(\boldsymbol{Yon}(U), P).$$

Proof. We can re-derive the sheaf condition in terms of the covering sieve as follows. For an open $U = \bigcup_i U_i$, if$\{f_i\} \in \prod_i PU_i$ is such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then because S is a covering sieve of U, therefore this condition is equivalent to a sequence $\{f_V\} \in PV$ for all $V \in S$ such that $f_V|_{V'} = f|_{V'}$ whenever $V' \subset V$. It can also be seen that every natural transformation η between S and P can be mapped to an element of $\prod_{V \in S} PV$ by forming the collection $\{\eta_V(*)\}$. Similarly, for any $\{f_V\} \in \prod_{V \in S} PV$ we can construct a nat. trans. $\{f_V : SV = \{*\} \to PV\}$. Now, with this, we can obtain the result by a basic diagram chase around the left square of the following

$$\begin{tikzcd}
\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(S,P) \arrow[r, "d"] & \prod_i PU_i \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} P(U_i \cap U_j) \\
\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(\mathbf{Yon}(U),P) \arrow[u, "\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(i_S,P)"] \arrow[r, equal] & PU \arrow[u, "e"'] &
\end{tikzcd}$$

where d is the equalizer of the parallel arrows on the right (the fact that this set is the equalizer is established in the prev. paragraph) $\square$

6.3 Sh (X) has all small limits

We now see that $\mathrm{Sh}(X)$ has all small limits and the inclusion of $\mathrm{Sh}(X)$ to $\widehat{\mathbf{O(X)}}$ preserves these limits.

Proposition 6.3.1. *For any topological space X, the category $Sh(X)$ has all small limits and the inclusion functor*

$$i : Sh(X) \rightarrowtail \widehat{\mathbf{O(X)}}$$

preserves all those limits.

Proof. To show that $\mathrm{Sh}(X)$ has all small limits, we can first notice that the singleton functor is a sheaf, which is the terminal object in $\mathrm{Sh}(X)$. Now, to see equalizers, take any parallel arrows in $\mathrm{Sh}(X)$ as $F \rightrightarrows G$. Since $\widehat{\mathbf{O(X)}}$ has all small limits, therefore, we can take the equalizer of this in it, in turn of taking equalizer in $\mathrm{Sh}(X)$. With this, there exists E, the equalizer of $F \rightrightarrows G$ in $\widehat{\mathbf{O(X)}}$. Now because covariant hom-functors preserves limits, therefore for any open U, the $\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(\mathbf{Yon}(U), E)$ and $\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(S, E)$ acts as equalizers in the diagram below:

$$\begin{tikzcd}
\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(\mathbf{Yon}(U),E) \arrow[r, "e\circ-"] \arrow[d, "-\circ i_s"', "f_E"] & \mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(\mathbf{Yon}(U),F) \arrow[r, shift left, "f\circ-"] \arrow[r, shift right, "g\circ-"'] \arrow[d, "-\circ i_s"', "f_F"] & \mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(\mathbf{Yon}(U),G) \arrow[d, "-\circ i_s"', "f_G"] \\
\mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(S,E) \arrow[r, "e\circ-"'] & \mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(S,F) \arrow[r, shift left, "f\circ-"] \arrow[r, shift right, "g\circ-"'] & \mathrm{Hom}_{\widehat{\mathbf{O(X)}}}(S,G)
\end{tikzcd}$$

Using Proposition 6.2.4, f_F and f_G are isomorphisms. A simple diagram chase on the left square then shows f_E is also an isomorphism. Binary products exists by the same process. $\square$

The above proposition hence allows us to infer what it means to be a subobject of a sheaf in $\mathrm{Sh}(X)$.

Corollary 6.3.2. *For any topological space X, any subobject of a sheaf F in $Sh(X)$ is isomorphic to a subsheaf of F.*

Proof. Suppose $H \Rightarrow F$ is a monic, so a subobject of F. Since $\mathrm{Sh}(X)$ has all limits (Proposition 6.3.1), so the kernel pair of this arrow would exist in $\mathrm{Sh}(X)$ and it's inclusion in $\widehat{\mathbf{O(X)}}$ would preserve it. By point-wise construction of presheaves in $\widehat{\mathbf{O(X)}}$, we can see that H would be isomorphic to some some subfunctor of F, which would be a sheaf too because it is isomorphic to H, a sheaf. □

6.3.1 Topology of X $\cong$ Subobjects of $\mathbf{Yon}(X)$ in $\mathbf{Sh}(X)$

Finally, we observe that the topology of X is actually isomorphic to subobjects of $\mathbf{Yon}(X)$[12] in $\mathrm{Sh}(X)$!

Proposition 6.3.3. *For any topological space X, there exists an isomorphism of the following posets*

$$\mathcal{O}(X) \cong Sub_{Sh(X)}(\boldsymbol{Yon}(X))$$

which is moreover order preserving.[13]

6.4 Direct and inverse limits in $\mathbf{Sh}(X)$

Since Grothendieck-abelian categories have all colimits, therefore it also has direct limits. We now show that the direct limits in $\mathbf{Sh}(X)$ are obtained by sheafifying the corresponding direct limit in $\mathbf{PSh}(X)$.

Lemma 6.4.1. [14] *Let X be a topological space and $\{\mathcal{F}_i\}$ be a direct system of sheaves over X. Then, the direct limit $\varinjlim_i \mathcal{F}_i$ in $\mathbf{Sh}(X)$ is formed by sheafification of the presheaf $U \mapsto \varinjlim_i \mathcal{F}_i(U)$.*

Proof. Let F denote the presheaf obtained by $U \mapsto \varinjlim_i \mathcal{F}_i(U)$ and further denote $\mathcal{F} = F^{++}$, the sheafification of F. Note that we have $\mathcal{F}_i \xrightarrow{j_i} F \to \mathcal{F}$. We wish to show that $\mathcal{F}$ satisfies the universal property of direct limits in $\mathbf{Sh}(X)$. Indeed, take any other sheaf $\mathcal{G}$ for which there are maps $f_i : \mathcal{F}_i \to \mathcal{G}$ which further satisfies that for any $j \geq i$ in the direct set indexing the system, we have that the following triangle commutes:

$$\begin{tikzcd} & \mathcal{G} & \\ \mathcal{F}_i \arrow[ur, "f_i"] \arrow[rr, "\varphi_{ij}"'] & & \mathcal{F}_j \arrow[ul, "f_j"'] \end{tikzcd} .$$

[12] Remember that $\mathbf{Yon}(X)$ is the terminal object in $\mathrm{Sh}(X)$.

[13] Remember Proposition **??**. Therefore this isomorphism could be extended as:

$$\mathcal{O}(X) \cong \mathrm{Sub}_{\mathrm{Sh}(X)}(\mathbf{Yon}(X)) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbf{Yon}(X), \Omega)$$

when Ω exists. This is the first sign of how sheaves might be related to topoi.

[14] Exercise II.1.10 of Hartshorne.

We wish to show that there exists a unique map $\tilde{f} : \mathcal{F} \to \mathcal{G}$ such that for all i, the following commutes:

$$\begin{tikzcd} \mathcal{G} & \mathcal{F} \arrow[l, dashed, "\tilde{f}"'] \\ \mathcal{F}_i \arrow[u, "f_i"] \arrow[r, "j_i"'] & F \arrow[u] \end{tikzcd}.$$

But this is straightforward, as by the universal property of direct limits in $\mathbf{PSh}(X)$, we first have a map $f : F \to \mathcal{G}$ which makes the bottom left triangle in the above commute. Then, by the universal property of sheafification (Theorem 2.0.1), we get a corresponding $\tilde{f} : \mathcal{F} \to \mathcal{G}$ which makes the top right triangle in the above commute. Consequently, we have obtained $\tilde{f}$ which makes the square commute. $\square$

7 Classical Čech cohomology

Sheaf cohomology becomes an important tool to any attempt at understanding any sophisticated geometric situation in topology. It is a tool which measures the obstructions faced in extending a local construction (which are usually not too difficult to make) to a global one (which are most of the time very difficult to make). To get a feel of why such questions and tools developed to solve them are important, one may look no further than basic analysis; say in case of $\mathbb{R}^n$, we wish to extend a local isometry from an open set of $\mathbb{R}^n$ to $\mathbb{R}^m$, into a global one between $\mathbb{R}^n$ and $\mathbb{R}^m$. Clearly the former is much, much easier than the latter. In the same vein, we wish to understand obstructions faced in making local-to-global leaps in the context of schemes, which covers almost all range of algebro-geometric situations.

Construction 7.0.1 (*Čech cochain complex and Čech cohomology of an abelian presheaf.*)**.** Let X be a topological space and F be an abelian presheaf over X. We will construct and discuss the Čech cohomology groups $\check{H}^q(X;F)$. After giving the basic constructions, we will specialize to the case of schemes in Chapter **??**, to prove the Serre's theorem on invariance of affine refinements of cohomology of coherent sheaves.

We first construct the Čech cochain complex of F w.r.t. to an open cover $\mathcal{U}$. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a fixed open cover of X. We can then define for each $i = 0, 1, 2, \dots$, a group called the group of *i-cochains of F w.r.t. $\mathcal{U}$*:

$$C^i(\mathcal{U}, F) := \prod_{(\alpha_0,\dots,\alpha_i)\in I^{i+1}} F(U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_i}).$$

where the product runs over all increasing $i+1$-tuples with entries in I[15]. A typical element $s \in C^i(\mathcal{U}, F)$ is called an i-cochain, whose part corresponding to $(\beta_0, \dots, \beta_i) \in I^{i+1}$ is denoted by $s(\beta_0, \dots, \beta_i) \in F(U_{\beta_0} \cap \dots \cap U_{\beta_i})$. For example, the set of all 0-cochains is $\prod_{\alpha_0 \in I} F(U_{\alpha_0})$, which is equivalent to choosing a section for each element of the cover. Similarly, choosing an element from $C^1(\mathcal{U}, F)$ can be thought of as choosing a section for each intersection of two open sets from $\mathcal{U}$. Similarly one can interpret the higher cochains.

Next, we give the sequence of groups $\{C^i(\mathcal{U}, F)\}_{i\in\mathbb{N}\cup 0}$ the structure of a cochain complex. Indeed, one defines the required *differential* in quite an obvious manner, if one knows the construction of singular homology. Define a map

$$\begin{aligned} d : C^i(\mathcal{U}, F) &\longrightarrow C^{i+1}(\mathcal{U}, F) \\ s = (s(\alpha_0, \dots, \alpha_i)) &\longmapsto ds \end{aligned}$$

where the components of ds are given as follows for $\beta_0, \dots, \beta_{i+1} \in I$:

$$\begin{aligned} (ds)(\beta_0, \dots, \beta_{i+1}) &:= \sum_{j=0}^{i+1} (-1)^j \rho_j (s(\beta_0, \dots, \beta_{j-1}, \beta_{j+1}, \dots, \beta_{i+1})) \\ &= \sum_{j=0}^{i+1} (-1)^j \rho_j (s(\widehat{\beta_j})) \end{aligned}$$

[15]we choose increasing tuples only to make sure we don't repeat an open set in the product.

where ρ_j is the following restriction map of the presheaf F:

$$\rho_j : F(U_{\beta_0} \cap \cdots \cap U_{\beta_{j-1}} \cap U_{\beta_{j+1}} \cap U_{\beta_{i+1}}) \longrightarrow F(U_{\beta_0} \cap \cdots \cap U_{\beta_{j-1}} \cap U_{\beta_j} \cap U_{\beta_{j+1}} \cap U_{\beta_{i+1}}),$$

that is, the one where the open set U_{β_j} is dropped from intersection.

This differential can be understood in the simple case of $i = 0$ as follows. Take $s = (s(\alpha_0)) \in C^0(\mathcal{U}, F)$. Then $ds \in C^1(\mathcal{U}, F)$ and it corresponds to a choice of a section in the intersection on each pair of open sets in $\mathcal{U}$. For $\beta_0, \beta_1 \in I$, this choice is given by

$$(ds)(\beta_0, \beta_1) = \rho_0(s(\beta_1)) - \rho_1(s(\beta_0)).$$

This is interpreted as "how much far away $s(\beta_1) \in F(U_{\beta_1})$ and $s(\beta_0) \in F(U_{\beta_0})$ are in the intersection $U_{\beta_1} \cap U_{\beta_0}$". If $d(s) = 0$, then $s \in C^0(\mathcal{U}, F)$ corresponds to a matching family.

Similarly, for a $s \in C^1(\mathcal{U}, F)$, we can think of it as a choice of a section on each intersecting pair of open sets of $\mathcal{U}$. Then, the differential $ds \in C^2(\mathcal{U}, F)$ for any $(\beta_0, \beta_1, \beta_2) \in I^3$ has the component

$$(ds)(\beta_0, \beta_1, \beta_2) = \rho_0(s(\beta_1, \beta_2)) - \rho_1(s(\beta_0, \beta_2)) + \rho_2(s(\beta_0, \beta_1)).$$

If this quantity is non zero, then it measures "how much the three elements $s(\beta_1, \beta_2) \in F(U_{\beta_1} \cap U_{\beta_2})$, $s(\beta_0, \beta_2) \in F(U_{\beta_0} \cap U_{\beta_2})$ and $s(\beta_0, \beta_1) \in F(U_{\beta_0} \cap U_{\beta_1})$ differs in the combined intersection $U_{\beta_0} \cap U_{\beta_1} \cap U_{\beta_2}$". Indeed, suppose the three agree on $F(U_{\beta_0} \cap U_{\beta_1} \cap U_{\beta_2})$. Then, we have $\rho_0(s(\beta_1, \beta_2)) = \rho_1(s(\beta_0, \beta_2)) = \rho_2(s(\beta_0, \beta_1))$. Consequently, $ds(\beta_0, \beta_1, \beta_2) = \rho_2(s(\beta_0, \beta_1))$.

Now it is quite obvious that in order to measure the failure of an element of $C^i(\mathcal{U}, F)$ to "match up in one level above" will be measured by the homology of the cochain complex. Indeed that is what we do now.

For any $s \in C^i(\mathcal{U}, F)$, it is observed by doing the summation and using the fact that the restriction maps ρ are group homomorphisms that

$$d^2 = 0.$$

Hence, we have a cochain complex, called the **Čech cochain complex w.r.t. $\mathcal{U}$**:

$$C^0(\mathcal{U}, F) \xrightarrow{d} C^1(\mathcal{U}, F) \xrightarrow{d} C^2(\mathcal{U}, F) \xrightarrow{d} \cdots$$

The cohomology of this complex is denoted by

$$H^q(\mathcal{U}; F) := \frac{\operatorname{Ker}(d)}{\operatorname{Im}(d)} =: \frac{Z^q\mathcal{U}, \mathcal{F}}{B^q(\mathcal{U}, \mathcal{F})}$$

for $C^{q+1}(\mathcal{U}, F) \leftarrow C^q(\mathcal{U}, F) \leftarrow C^{q-1}(\mathcal{U}, F)$. The subgroup $B^q(\mathcal{U}, \mathcal{F}) = \operatorname{Im}(d) =\subseteq C^q(\mathcal{U}, F)$ is called the *group of q-coboundaries*, whereas the group $Z^q(\mathcal{U}, \mathcal{F}) = \operatorname{Ker}(d) \subseteq C^q(\mathcal{U}, F)$ is called the *group of q-cocycles.*

To define the general Čech cohomology groups, we need to take limit of cohomology groups with respect to finer and finer open covers. To this end, we first define the following. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in J}$ be two open covers. Then, $\mathcal{V}$ is said to be *finer* than $\mathcal{U}$ if for all $j \in J$, there is an $i \in I$ such that $V_j \subseteq U_i$. We therefore obtain a function $\sigma : J \to I$ such that $V_j \subseteq U_{\sigma(j)}$.

For two open covers $\mathcal{U}, \mathcal{V}$ where $\mathcal{V}$ is finer than $\mathcal{U}$ as above, we first get a map of cochain complexes given by

$$\begin{aligned} r_{\mathcal{U},\mathcal{V}} : C^q(\mathcal{U}, F) &\longrightarrow C^q(\mathcal{V}, F) \\ s &\longmapsto r_{\mathcal{U},\mathcal{V}}(s) \end{aligned}$$

where for any $(\beta_0, \dots, \beta_q) \in J^{q+1}$, we define

$$r_{\mathcal{U},\mathcal{V}}(s)(\beta_0, \dots, \beta_q) = \rho\left(s(\sigma\beta_0, \dots, \sigma\beta_q)\right)$$

for $\rho : F(U_{\sigma\beta_0} \cap \dots \cap U_{\sigma\beta_q}) \longrightarrow F(V_{\beta_0} \cap \dots \cap V_{\beta_q})$ is the restriction map of F. As restriction homomorphisms commute with themselves, therefore we have that the following square commutes

$$\begin{tikzcd} C^q(\mathcal{U}, F) \arrow[r, "d"] \arrow[d, "r_{\mathcal{U},\mathcal{V}}"'] & C^{q+1}(\mathcal{U}, F) \arrow[d, "r_{\mathcal{U},\mathcal{V}}"] \\ C^q(\mathcal{V}, F) \arrow[r, "d"'] & C^{q+1}(\mathcal{V}, F) \end{tikzcd},$$

showing that $r_{\mathcal{U},\mathcal{V}} : C^\bullet(\mathcal{U}, F) \to C^\bullet(\mathcal{V}, F)$ is a map of cochain complexes. Consequently, we get a map at the level of cohomology also denoted by

$$r_{\mathcal{U},\mathcal{V}} : H^q(\mathcal{U}, F) \longrightarrow H^q(\mathcal{V}, F).$$

We call the above the *refinement homomorphism.*

We now wish to show that if $\mathcal{V}$ is a refinement of $\mathcal{U}$ via $\sigma : J \to I$, then the refinement homomorphism $r_{\mathcal{U},\mathcal{V}}$ on cohomology doesn't depend on σ; there might be many such σ making $\mathcal{V}$ finer than $\mathcal{U}$, but all give same refinement homomorphism on cohomology.

Lemma 7.0.2. *The refinement homomorphism $r_{\mathcal{U},\mathcal{V}}$ is independent of σ.*

Proof. Let $r, r' : C^q(\mathcal{U}, F) \to C^q(\mathcal{V}, F)$ be the refinement homomorphisms on cochain level for $\sigma, \tau : J \to I$ respectively. Pick any q-cocycle $s \in C^q(\mathcal{U}, F)$. We wish to show that $r(s) - r'(s)$ is a q-coboundary w.r.t. $\mathcal{V}$. The following $t \in C^{q-1}(\mathcal{V}, F)$

$$t(\alpha_0, \dots, \alpha_{q-1}) := \sum_{j=0}^{q-1} (-1)^j \rho\left(s\left(\sigma\alpha_0, \dots, \sigma\alpha_j, \tau\alpha_j, \tau\alpha_{j+1}, \dots, \tau\alpha_{i-1}\right)\right)$$

where $\rho : F(U_{\sigma\alpha_0} \cap \dots \cap U_{\sigma\alpha_j} \cap U_{\tau\alpha_j} \cap \dots \cap U_{\tau\alpha_{i-1}}) \longrightarrow F(V_{\alpha_0} \cap \dots \cap V_{\alpha_j} \cap \dots \cap V_{\alpha_{i-1}})$ is such that

$$r(s) - r'(s) = dt$$

in $C^q(\mathcal{V}, F)$. This can be checked by expanding dt and using the fact that $ds = 0$. This calculation is omitted for being too cumbersome to write. $\square$

This finally allows us to define Čech cohomology of a presheaf over a topological space as follows. Let $\mathcal{O}$ be the poset of all open covers of X ordered by refinement. The **Čech cohomology groups of presheaf** F are then defined to be

$$\check{H}^q(X, F) := \varinjlim_{\mathcal{U} \in \mathcal{O}} H^q(\mathcal{U}, F).$$

Diagrammatically, we have for any two open covers $\mathcal{U}$ and $\mathcal{V}$ where $\mathcal{V}$ is a refinement of $\mathcal{U}$ the following

$$\begin{tikzcd} & \check{H}^q(X,F) \\ H^q(\mathcal{U},F) \arrow[ur] \arrow[r, "r_{\mathcal{U},\mathcal{V}}"'] & H^q(\mathcal{V},F) \arrow[u] \end{tikzcd}.$$

This completes the construction of Čech cohomology groups.

Let us first see something that we hinted during the construction.

Lemma 7.0.3. *Let X be a space and $\mathcal{F}$ be a sheaf over X. Then, for any open cover $\mathcal{U}$ of X, we have*

$$H^0(\mathcal{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F}).$$

Consequently, we have $\check{H}^0(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F})$.

Proof. We first have $H^0(X,F) = \mathrm{Ker}\,(d)$ where $d : C^0(\mathcal{U},F) \to C^1(\mathcal{U},F)$. But any $s \in \mathrm{Ker}\,(d)$ is equivalent to the data of a matching family over $\mathcal{U}$. As $\mathcal{F}$ is a sheaf, therefore this gives rise to a unique element in $\Gamma(X,\mathcal{F})$. Conversely, by restriction, we get an element of $\mathrm{Ker}\,(d)$ via a global section. □

Let us first see an example computation of $\check{H}^1(X,F)$.

Example 7.0.4. Let $X = S^1$ and $F = \mathcal{K}$ be the constant sheaf of a field K. Further, let $\mathcal{U}$ be the open cover obtained by dividing S^1 into n-open intervals $U_1,\dots,U_n$ where $U_i \cap U_{i+1}$ and $U_i \cap U_{i-1}$ are non-empty and $U_i \cap U_j$ is empty for all $j \neq i, i+1, i-1$. We wish to calculate $H^1(\mathcal{U},\mathcal{K})$. To this end, we first see that

$$C^0(\mathcal{U},\mathcal{K}) = \prod_{i=1}^{n} \mathcal{K}(U_i) = K^{\oplus n}$$

and

$$C^1(\mathcal{U},\mathcal{K}) = \prod_{i=1}^{n} \mathcal{K}(U_i \cap U_{i+1}) = K^{\oplus n}.$$

For $q \geq 2$, we clearly have $C^q(\mathcal{U},\mathcal{K}) = 0$ as there are no higher intersections. The differential $d : C^0(\mathcal{U},\mathcal{K}) \to C^1(\mathcal{U},\mathcal{K})$ maps as

$$d(x_1,\dots x_n) = (x_2 - x_1, x_3 - x_2, \dots, x_1 - x_n).$$

Consequently,

$$H^0(\mathcal{U},\mathcal{K}) = \mathrm{Ker}\,(d) = \{(x_1,\dots,x_n) \in C^0(\mathcal{U},\mathcal{K}) \mid x_1 = x_2 = \dots = x_n\} \cong K$$

and

$$H^1(\mathcal{U},\mathcal{K}) = \frac{C^1(\mathcal{U},\mathcal{K})}{\mathrm{Im}\,(d)} \cong K$$

as $C^1(\mathcal{U},\mathcal{K})$ is an n-dimensional K-vector space and $\mathrm{Im}\,(d)$ is of dimension $n-1$ because its defined by one equation deeming the sum of all entries to be 0.

Construction 7.0.5 (Map in cohomology)**.** Any map of abelian sheaves over X yields a map in the cohomology as well. Indeed, let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then we get a map

$$\begin{aligned} \varphi^q : C^q(\mathcal{U}, \mathcal{F}) &\longrightarrow C^q(\mathcal{U}, \mathcal{G}) \\ s = (s(\alpha_0, \dots, \alpha_q)) &\longmapsto \varphi^q(s) = \left(\varphi_{\alpha_0 \dots \alpha_q}(s(\alpha_0, \dots, \alpha_q))\right) \end{aligned}$$

where $\varphi_{\alpha_0 \dots \alpha_q} = \varphi_{U_{\alpha_0} \cap \dots \cap U_{\alpha_q}}$.

It then follows quite immediately from the fact that each $\varphi_{\alpha_0 \dots \alpha_q}$ is a group homomorphism that $d\varphi^q = \varphi^{q+1} d$. It follows that we get a map of chain complexes

$$\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{G}).$$

Hence, we get a map in cohomology

$$\varphi^q : H^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(\mathcal{U}, \mathcal{G}).$$

Finally, this gives by universal property of direct limits a unique map

$$\varphi^q : \check{H}^q(X, \mathcal{F}) \longrightarrow \check{H}^q(X, \mathcal{G})$$

such that for every open cover $\mathcal{U}$, the following diagram commutes:

$$\begin{tikzcd} \check{H}^q(X, \mathcal{F}) \arrow[r, dashed, "\varphi^q"] & \check{H}^q(X, \mathcal{G}) \\ H^q(\mathcal{U}, \mathcal{F}) \arrow[u] \arrow[r, "\varphi^q"'] & H^q(\mathcal{U}, \mathcal{G}) \arrow[u] \end{tikzcd}$$

where vertical maps are the maps into direct limits.

The main tool for calculations with cohomology theories is the cohomology long exact sequence. We put below, without proof, the main theorem of Čech cohomology which gives a condition for an exact sequence of sheaves to induce this long exact sequence in cohomology. Recall X is paracompact if it is Hausdorff and every open cover has a locally finite refinement. Such spaces are always normal. We first give an explicit description of the first connecting homomorphism.

Construction 7.0.6 (Connecting homomorphism)**.** Let X be a topological space and

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

be an exact sequence of sheaves on X. We define the connecting homomorphism

$$\check{H}^0(X, \mathcal{H}) \xrightarrow{\delta} \check{H}^1(X, \mathcal{F})$$

as follows. First, pick any $h \in H^0(X, \mathcal{H}) = \Gamma(X, \mathcal{H})$. As ψ is surjective therefore there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X and $g_i \in \mathcal{G}(U_i)$ such that $\psi_{U_i}(g_i) = h|_{U_i}$. Using (g_i) and (U_i) we construct a 1-cocycle for $\mathcal{F}$ as follows. Observe that for each $i, j \in I$, we have $\psi_{U_i \cap U_j}(g_i - g_j) = 0$ in $\mathcal{H}(U_i \cap U_j)$. Thus, $g_i - g_j \in \operatorname{Ker}\left(\psi_{U_i \cap U_j}\right)$. By exactness guaranteed by Lemma 3.0.8, it follows that there exists $f_{\alpha_0\alpha_1} \in \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1})$ such that $\varphi_{U_{\alpha_0} \cap U_{\alpha_0}}(f_{\alpha_0\alpha_1}) = g_{\alpha_0} - g_{\alpha_1}$, for each $\alpha_0, \alpha_1 \in I$. We claim that the element

$$f := (f_{\alpha_0\alpha_1})_{\alpha_0,\alpha_1} \in \prod_{(\alpha_0,\alpha_1)\in I^2} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) = C^1(\mathcal{U}, \mathcal{F})$$

is a 1-cocycle. Indeed, we need only check that $df = 0$ in $C^2(\mathcal{U}, \mathcal{F})$. Pick any $(\alpha_0, \alpha_1\alpha_2) \in I^3$. We wish to show that $df(\alpha_0, \alpha_1\alpha_2) = 0$. Indeed,

$$\begin{aligned} df(\alpha_0, \alpha_1\alpha_2) &= \sum_{j=0}^{2} (-1)^j \rho_j \left(f_{\alpha_0\hat{\alpha}_j\alpha_2}\right) \\ &= f_{\alpha_1\alpha_2} - f_{\alpha_0\alpha_2} + f_{\alpha_0\alpha_1} \end{aligned}$$

in $\mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2})$. We claim the above is zero. Indeed, By Lemma 3.0.8 on $V := U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}$ we get that φ_V is injective. But since

$$\begin{aligned} \varphi_V(f_{\alpha_1\alpha_2} - f_{\alpha_0\alpha_2} + f_{\alpha_0\alpha_1}) &= \varphi_V(f_{\alpha_1\alpha_2}) - \varphi_V(f_{\alpha_0\alpha_2}) + \varphi_V(f_{\alpha_0\alpha_1}) \\ &= g_{\alpha_1} - g_{\alpha_2} - (g_{\alpha_0} - g_{\alpha_2}) + g_{\alpha_0} - g_{\alpha_1} \\ &= 0, \end{aligned}$$

hence it follows that $df(\alpha_0, \alpha_1\alpha_2) = 0$, as required. Hence $f \in C^1(\mathcal{U}, \mathcal{F})$ is a 1-cocycle. Thus we get an element $[f] \in H^1(\mathcal{U}, \mathcal{F})$. This defines a group homomorphism $\check{H}^0(X, \mathcal{H}) \to H^1(\mathcal{U}, \mathcal{F})$. Further by passing to direct limit, we get an element $[f] \in \check{H}^1(X, \mathcal{F})$. We thus define

$$\delta(f) := [f] \in \check{H}^1(X, \mathcal{F}).$$

This defines the required group homomorphism δ.

Theorem 7.0.7. *Let X be a paracompact space and the following be an exact sequence of sheaves over X*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

Then, there is a long exact sequence in cohomology

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(X, \mathcal{F}_1) \arrow[r] & \check{H}^0(X, \mathcal{F}_2) \arrow[r] & \check{H}^0(X, \mathcal{F}_3) \arrow[dll] \\ & \check{H}^1(X, \mathcal{F}_1) \arrow[r] & \check{H}^1(X, \mathcal{F}_2) \arrow[r] & \check{H}^1(X, \mathcal{F}_3) \arrow[dll, dashed, no head] \\ & & {} & \end{tikzcd}.$$

8 Derived functor cohomology

We will here define the cohomology of abelian sheaves over a topological space as right derived functors of the left exact global-sections functor (see Section **??** for preliminaries on derived functors).

Let X be a topological space. In Section 6, we showed that the category of abelian sheaves $\mathbf{Sh}(X)$ has enough injectives. We now use it to define cohomology of $\mathcal{F} \in \mathbf{Sh}(X)$.

Definition 8.0.1. (**Sheaf cohomology functors**) Let X be a topological space and $\mathbf{Sh}(X)$ be the category of abelian sheaves over X. The i^{th}-cohomology functor $H^i(X,-) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$ is defined to be the i^{th}-right derived functor of the global sections functor $\Gamma(-,X) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$. In other words, $H^i(X,\mathcal{F})$ for $\mathcal{F} \in \mathbf{Sh}(X)$ is defined by choosing an injective resolution $0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^\bullet$ in $\mathbf{Sh}(X)$ and then

$$H^i(X,\mathcal{F}) := h^i(\Gamma(X,\mathcal{I}^\bullet)).$$

As sheaf cohomology functors are in particular derived functors, so they satisfy results from Section **??**. The main point in particular being that sheaf cohomology induces a long exact sequence in cohomology from a short exact sequence of sheaves. This will be our primary source of computations.

8.1 Flasque sheaves & cohomology of $\mathcal{O}_X$-modules

We would like to see the following theorem.

Theorem 8.1.1. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the right derived functors of $\Gamma(-,X)$: $\mathbf{Mod}(\mathcal{O}_X) \to \mathbf{AbGrp}$ is equal to the restriction of the cohomology functors $H^i(X,-) : \mathbf{Sh}(X) \to \mathbf{AbGrp}$.*

Remember that $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives (Theorem **??**) but, apriori, the above two functors might be different because an injective object in $\mathbf{Mod}(\mathcal{O}_X)$ may not be injective in $\mathbf{Sh}(X)$. Consequently, the above result is important because its relevance in rectifying the cohomology of $\mathcal{O}_X$-modules (which are of the only utmost interest in algebraic geometry) to that of the usual sheaf cohomology functors. Hence, we may completely work inside the module category $\mathbf{Mod}(\mathcal{O}_X)$. Clearly to prove such a result, we need a bridge between injective modules in $\mathbf{Mod}(\mathcal{O}_X)$ and either injective or acyclic objects in $\mathbf{Sh}(X)$. Indeed, we will see that this bridge is provided by the realization that injective modules in $\mathbf{Mod}(\mathcal{O}_X)$ are acyclic because they are *flasque*.

Definition 8.1.2 (Flasque sheaves). A sheaf $\mathcal{F}$ on X is said to be flasque if all restriction maps of $\mathcal{F}$ are surjective.

The following is a simple, yet important class of examples of flasque sheaves.

Example 8.1.3. Let X be an irreducible topological space and $\mathcal{A}$ be the constant sheaf over X for an abelian group A. We claim that $\mathcal{A}$ is flasque. Indeed, first recall that any open subspace $U \subseteq X$ is irreducible, therefore connected. Conseuqently, all restrictions are $\rho : \mathcal{A}(V) \to \mathcal{A}(U)$ are identity maps $\mathrm{id} : A \to A$ (see Remark 1.0.3). In-fact this shows that on an irreducible space, any constant sheaf $\mathcal{A}$ of abelian group A has section over any open set U as $\mathcal{A}(U) = A$ and all restrictions are identities.

An important property of flasque sheaves is that they have no obstruction to lifting of sections, a hint to their triviality in cohomology. However, the proof of this is quite non-constructive and thus a bit enlightening.

Theorem 8.1.4. *Let X be a space. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaves and $\mathcal{F}_1$ is flasque, then we have an exact sequence of sections over any open $U \subseteq X$*

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0.$$

Proof. By left-exactness of global sections functor, we need only show the surjectivity of $\Gamma(\mathcal{F}_2, X) \to \Gamma(\mathcal{F}_3, X)$. To this end, pick any $s \in \Gamma(\mathcal{F}_3, X)$. We wish to lift this to an element of $\Gamma(\mathcal{F}_2, X)$. Consider the following poset

$$\mathcal{P} = \{(U, t) \mid U \subseteq X \text{ open } \& \ t \in \mathcal{F}_2(U) \text{ is a lift of } s|_U\}$$

where $(U, t) \leq (U', t')$ iff $U' \supseteq U$ and $t'|_U = t$. We reduce to showing that $\mathcal{P}$ has a maximal element and it is of the form (X, t). This will conclude the proof.

To show the existence of a maximal element, we will use Zorn's lemma. Pick any toset of $\mathcal{P}$ denoted $\mathcal{T}$. We wish to show that it is upper bounded. Indeed, let $V = \bigcup_{(U,t)\in\mathcal{T}} U$ and $\tilde{t} \in \mathcal{F}_2(V)$ be the section obtained by gluing $t \in \mathcal{F}_2(U)$ for each $(U, t) \in \mathcal{T}$ (they form a matching family because $\mathcal{T}$ is totally ordered). We thus have $(V, \tilde{t})$ which we wish to show is in $\mathcal{P}$. Indeed, as $\tilde{t}$ is obtained by lifts of restrictions of s, therefore $\tilde{t}$ is a lift of $s|_V$ by locality of sheaf $\mathcal{F}_3$. This shows that $\mathcal{P}$ has a maximal element, denote it by $(V, \tilde{t})$.

We finally wish to show that $V = X$. Indeed, if not, then $V \subsetneq X$. Pick any point $x \in X \setminus V$. Since we have a surjective map on stalks $\mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$, hence the germ $(X, s)_x \in \mathcal{F}_{3,x}$ can be lifted to $(U, a)_x$ for some open $U \ni x$ and $a \in \mathcal{F}_2(U)$. We now have two cases. If $U \cap V = \emptyset$, then $(V \cup U, \tilde{t} \amalg a)$ is a lift of $s|_{V\cup U}$, contradicting the maximality of $(V, \tilde{t})$. On the other hand, suppose we have $U \cap V \neq \emptyset$. Let $W = U \cap V$. Since $W \subseteq V$, therefore we have $t_W \in \mathcal{F}_2(W)$ a lift of $s|_W$. Moreover, by restriction, we have $a \in \mathcal{F}_2(W)$ also a lift of $s|_W$. It follows that $a - t_W \in \mathcal{F}_1(W)$. As $\mathcal{F}_1$ is flasque, therefore there exists $b \in \Gamma(\mathcal{F}_1, X)$ which extends $a - t_W$. Consequently, we have $a - b = t_W \in \mathcal{F}_2(W)$. Observe that $a - b \in \mathcal{F}_2(U)$ is also a lift of $s|_U$ because $b = 0$ in $\Gamma(\mathcal{F}_3, X)$ by the left-exactness of global sections functor. It follows that $(U, a - b)$ and $(V, \tilde{t})$ is a matching family, which glues to $(U \cup V, c)$ where c is a lift of $s|_{U\cup V}$ as well, contradicting the maximality of $(V, \tilde{t})$. □

Corollary 8.1.5. *Let X be a space. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of sheaves where $\mathcal{F}_2$ is flasque, then $\mathcal{F}_3$ is flasque.*

Proof. This is immediate from Theorem 8.1.4 and the following diagram where $U \supseteq V$ an inclusion of open subsets of X:

$$\begin{tikzcd}
\mathcal{F}_2(U) \arrow[r] \arrow[d, two heads, "\rho"'] & \mathcal{F}_3(U) \arrow[r] \arrow[d, "\rho"] & 0 \\
\mathcal{F}_2(V) \arrow[r] & \mathcal{F}_3(V) \arrow[r] & 0
\end{tikzcd}.$$

□

Lemma 8.1.6. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{O}_U = i_!\mathcal{O}_{X|U}$ to be the extension by zeros of $\mathcal{O}_{X|U}$ for any open set $i : U \hookrightarrow X$. Then,*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{F}) \cong \mathcal{F}(U).$$

Proof. Indeed, we have the following isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{O}_{U|U}, \mathcal{F}_{|U}) \cong \mathrm{Hom}_{\mathcal{O}_{X|U}}(\mathcal{O}_{X|U}, \mathcal{F}_{|U}) \cong \mathcal{F}(U).$$

The first isomorphism follows from the universal property of sheafification. The second isomorphism follows from the observation that $\mathcal{O}_{U|U} = \mathcal{O}_{X|U}$ as is clear from Definition 3.0.9 and the fact that sheafification of a sheaf is that sheaf back. The last isomorphism follows from Lemma **??**, 2. □

Proposition 8.1.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is flasque.*

Proof. Let $i : U \hookrightarrow X$ be an open set. Denote $\mathcal{O}_U = i_! \mathcal{O}_{X|U}$ (see Definition 3.0.9). We know from Lemma 8.1.6 that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \cong \mathcal{I}(U)$ for any open $U \subseteq X$. Now, let $U \subseteq V$ be an inclusion of open sets. To this, we get $\rho : \mathcal{I}(V) \to \mathcal{I}(U)$ the restriction map. Restricting to open set V, we get the following injective map by Corollary 3.0.11

$$0 \to \mathcal{O}_U \to \mathcal{O}_V.$$

Using injectivity of $\mathcal{I}$, we obtain a surjection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \to 0.$$

Consequently, we have

$$\mathcal{I}(V) \to \mathcal{I}(U) \to 0$$

where the map is the restriction map of sheaf $\mathcal{I}$. Indeed, this follows from the explicit isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \cong \mathcal{I}(X)$ constructed in the proof of Lemma **??**, 2. □

Finally, we see that flasque sheaves have trivial cohomology.

Proposition 8.1.8. *Let X be a space and $\mathcal{F}$ be a flasque sheaf over X. Then*

$$H^i(X, \mathcal{F}) = 0$$

for all $i \geq 1$. That is, flasque sheaves are acyclic for the global sections functor.

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}$ be an injective map where $\mathcal{I}$ is an injective sheaf. Consequently, we have an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$$

where $\mathcal{G} = \mathcal{I}/\mathcal{F}$. It follows from Proposition 8.1.7 that $\mathcal{I}$ is flasque. By Corollary 8.1.5 it follows that $\mathcal{G}$ is flasque. By Theorem **??**, we have a long exact sequence in cohomology

$$\begin{tikzcd}
H^i(X,\mathcal{F}) \arrow[r] & H^i(X,\mathcal{I}) \arrow[r] & H^i(X,\mathcal{G}) \arrow[dll, "\delta_i"'] \\
H^{i+1}(X,\mathcal{F}) \arrow[r] & H^{i+1}(X,\mathcal{I}) \arrow[r] & H^{i+1}(X,\mathcal{G}) \arrow[dll, dotted, "\delta_{i+1}"'] \\
{} & &
\end{tikzcd}.$$

Since $\mathcal{I}$ is injective, therefore by Remark **??**, we have $H^i(X, \mathcal{I}) = 0$ for all $i \geq 1$. It follows from exactness of the above diagram that δ_i are isomorphisms for each $i \geq 1$, that is,

$$H^i(X, \mathcal{G}) \cong H^{i+1}(X, \mathcal{F}).$$

But since $\mathcal{G}$ is also flasque, therefore by repeating the above process, we deduce that $H^{i+1}(X,\mathcal{F}) \cong H^1(X,\mathcal{H})$ where $\mathcal{H}$ is some other flasque sheaf. It thus suffices to show that $H^1(X,\mathcal{F}) = 0$. This follows immediately as we have an exact sequence

$$0 \to \Gamma(\mathcal{F}, X) \to \Gamma(\mathcal{I}, X) \to \Gamma(\mathcal{G}, X) \to H^1(X,\mathcal{F}) \to 0$$

where by Theorem 8.1.4, the map $\Gamma(\mathcal{I}, X) \to \Gamma(\mathcal{G}, X)$ is surjective and since $\Gamma(\mathcal{G}, X) \to H^1(X,\mathcal{F})$ is surjective by exactness, it follows that the map $\Gamma(\mathcal{G}, X) \to H^1(X,\mathcal{F})$ is the zero map and $H^1(X,\mathcal{F}) = 0$, as required. □

An immediate corollary is the proof of Theorem 8.1.1.

Proof of Theorem 8.1.1. Pick any $\mathcal{F} \in \mathbf{Mod}(\mathcal{O}_X)$ and pick an injective resolution of $\mathcal{F}$ in $\mathbf{Mod}(\mathcal{O}_X)$

$$0 \to \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^\bullet.$$

By Proposition 8.1.7, it follows that each $\mathcal{I}^i$ is flasque. By Proposition 8.1.8, it follows that the above is an acyclic resolution for the sheaf $\mathcal{F}$ in $\mathbf{Sh}(X)$. Denote by $\bar{\Gamma} : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{AbGrp}$ the restriction of the global sections functor. We wish to show that $R^i\bar{\Gamma}(\mathcal{F}) \cong H^i(X,\mathcal{F})$. By Proposition **??**, we have the following isomorphism

$$R^i\bar{\Gamma}(\mathcal{F}) \cong h^i(\bar{\Gamma}(\mathcal{I}^\bullet)) = h^i(\Gamma(\mathcal{I}^\bullet)) \cong H^i(X,\mathcal{F}),$$

as needed. □

An important property of flasque sheaves over noetherian spaces is that it is closed under direct limits.

Proposition 8.1.9. *Let X be a noetherian space and $\{\mathcal{F}_\alpha\}$ be a directed system of flasque sheaves. Then $\varinjlim \mathcal{F}_\alpha$ is a flasque sheaf as well.*

8.1.1 Examples

We now present some computations.

Example 8.1.10. [16] Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k and $\mathbb{Z}$ be the constant sheaf over X. Let $P, Q \in X$ be two distinct closed points and let $U = X \setminus C$ where $C = \{P, Q\}$ be an open set. Denote $\mathbb{Z}_U$ to be the extension by zero sheaf of $\mathbb{Z}_{|U}$ over X. We claim that

$$H^1(X, \mathbb{Z}_U) \neq 0.$$

We will use the extension by zero short exact sequence of Corollary 3.0.11. Denote $i : C \hookrightarrow X$ to be the inclusion. Then, we have

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to i_*\mathbb{Z}_{|C} \to 0.$$

By Theorem **??** and Example 8.1.3, it follows that the following sequence is exact

$$0 \to \Gamma(\mathbb{Z}_U, X) \to \Gamma(\mathbb{Z}, X) \to \Gamma(i_*\mathbb{Z}_{|C}, X) \to H^1(X, \mathbb{Z}_U) \to 0.$$

[16] Exercise III.2.1, a) of Hartshorne.

Now suppose that $H^1(X, \mathbb{Z}_U) = 0$. It follows that the map $\Gamma(\mathbb{Z}, X) \to \Gamma(i_*\mathbb{Z}_{|C}, X)$ is surjective. Since X is irreducible and hence connected, we yield $\Gamma(\mathbb{Z}, X) = \mathbb{Z}$. Consequently, we have a surjective map $\mathbb{Z} \to \Gamma(i_*\mathbb{Z}_{|C}, X)$. It follows that $\Gamma(i_*\mathbb{Z}_{|C}, X) = \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. We claim that this is not possible by showing that $\Gamma(i_*\mathbb{Z}_{|C}, X)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which will yield a contradiction.

We first observe that $\Gamma(i_*\mathbb{Z}_{|C}, X) = \Gamma(\mathbb{Z}_{|C}, C)$. Recall that $\mathbb{Z}_{|C} = i^{-1}\mathbb{Z}$. Note that $(\mathbb{Z}_{|C})_P = \mathbb{Z}_p = \mathbb{Z} = (\mathbb{Z}_{|C})_Q$ by Lemma 5.0.3. Hence, by Definition 5.0.2 and Remark 2.0.4, we deduce that $(i^+\mathbb{Z})_P = (i^+\mathbb{Z})(\{P\}) = \mathbb{Z}_P = \mathbb{Z} = (i^+\mathbb{Z})_Q$ and

$$\Gamma(\mathbb{Z}_{|C}, C) = \left\{ \begin{matrix} (s,t) \in \mathbb{Z} \oplus \mathbb{Z} \mid \exists \text{ opens } U_P \ni P, U_Q \ni \\ Q \text{ in } C \ \& \ s' \in i^+\mathbb{Z}(U_P) \ \& \ t' \in \\ i^+\mathbb{Z}(U_Q) \text{ s.t. } s = s'_P,\ t = t'_Q,\ s = t'_P \text{ if } P \in \\ U_Q \ \&\ t = s'_Q \text{ if } Q \in U_P. \end{matrix} \right\}$$

With this, we observe that for each $(s,t) \in \mathbb{Z} \oplus \mathbb{Z}$, if we keep $U_P = \{P\}$ and $U_Q = \{Q\}$ (which is possible since $P \neq Q$ are closed points in X), we obtain $i^+\mathbb{Z}(U_P) = \mathbb{Z} = i^+\mathbb{Z}(U_Q)$. Then, we may take $s' = s$ and $t' = t$ to obtain that $\Gamma(\mathbb{Z}_{|C}, C) \cong \mathbb{Z} \oplus \mathbb{Z}$. This completes the proof.

Moreover, one can see that the only properties of $\mathbb{A}^1_k$ that we needed was that it is irreducible and $P, Q \in \mathbb{A}^1_k$ are distinct closed points. Consequently, the above result holds true for X an arbitrary irreducible space and $U = X \setminus \{P, Q\}$ where P, Q are two distinct closed points.

Example 8.1.11. Consider the notations of Example 8.1.10. As an exercise in working with sheaves and sheafification, one can also show that

$$\Gamma(\mathbb{Z}_U, X) = 0.$$