Problems on Vector Bundles & Characteristic Classes

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Contents

1 Smooth manifolds

Question 1. Show that the following two are equivalent:

1. X is a smooth manifold as in Milnor-Stasheff.
2. X is a smooth manifold as is defined contemporarily.

Proof. (1. $\Rightarrow$ 2.) Let $X \subseteq \mathbb{R}^A$ be a smooth n-manifold as in Milnor-Stasheff. We wish to produce an atlas of X such that its transition maps are smooth. By definition, we have local parameterizations $(U_\alpha, h_\alpha)_\alpha$ where $U_\alpha \subseteq \mathbb{R}^n$ and $h_\alpha : U_\alpha \to X$ is an open embedding such that $\cup_\alpha h_\alpha(U_\alpha) = X$. Denote $V_\alpha = h_\alpha(U_\alpha)$. We claim that the collection (U_α, h_α) forms an atlas of X in the contemporary sense.

Indeed, we need only show that for any two α, β, the transition map

$$h_\beta^{-1} \circ h_\alpha : h_\alpha^{-1}(V_\alpha \cap V_\beta) \to h_\beta^{-1}(V_\alpha \cap V_\beta)$$

is smooth. Indeed, this is what Lemma 1.1 of Milnor-Stasheff says, which we have done in class.

(2. $\Rightarrow$ 1.) Consider $A = \mathcal{C}^\infty(X; \mathbb{R})$ to be the $\mathbb{R}$-algebra of all smooth maps on the contemporary smooth n-manifold X. Let the charts of X be (U_α, h_α) where $U_\alpha \subseteq \mathbb{R}^n$ open and $h_\alpha : U_\alpha \to X$ is an open embedding with smooth transitions. Denote $V_\alpha = h_\alpha(U_\alpha)$. Consider the function

$$\begin{aligned} \varphi : X &\longrightarrow \mathbb{R}^A \\ x &\longmapsto (f(x))_{f \in A}. \end{aligned}$$

Our first claim is that f is an injective continuous map. We first show continuity. If $V \subseteq \mathbb{R}^A$ is a basic open set, then $V = \prod_{\alpha\in A} U_\alpha$ where for all but finitely many α is U_α proper, say for $\alpha_1, \ldots, \alpha_k$. Thus

$$\begin{aligned}\varphi^{-1}(V) &= \{x \in X \mid f_\alpha(x) \in U_\alpha\} \\ &= \{x \in X \mid f_{\alpha_i}(x) \in U_{\alpha_i}, i = 1, \ldots, k\} \\ &= \left\{x \in X \mid x \in f_{\alpha_i}^{-1}(U_{\alpha_i}) \forall i\right\} \\ &= \bigcap_{i=1}^{k} f_{\alpha_i}^{-1}(U_{\alpha_i})\end{aligned}$$

and the latter is open in X as $f_{\alpha_i} \in A$ are smooth. Next, we show injectivity of φ. If $\varphi(x) = \varphi(y)$, then for all $f \in A$, $f(x) = f(y)$. This follows from Proposition 2.25 of Lee.

Next, we show that the chart $\{(U_\alpha, h_\alpha)\}_\alpha$ of X gives a local parameterization of X in the sense of Milnor-Stasheff. To this end, we first have to show that the composite $\varphi \circ h_\alpha : U_\alpha \to \mathbb{R}^A$ is a smooth map. Indeed, it suffices to show that each for each projection $\pi_f : \mathbb{R}^A \to \mathbb{R}$ for $f \in A$, the composition $\pi_f \circ \varphi \circ h_\alpha : U_\alpha \to \mathbb{R}$ is a smooth map. As $\pi_f \circ \varphi \circ h_\alpha(u) = f(h_\alpha(u)) = f \circ h_\alpha(u)$. As f is smooth, hence so is $f \circ h_\alpha$. This shows that $\varphi \circ h_\alpha$ is smooth.

Finally, we wish to show that the derivative $D(\varphi \circ h_\alpha) : \mathbb{R}^n \to \mathbb{R}^A$ is of rank n. We'll show that there is an $n \times n$ submatrix of $A \times n$ matrix $D(\varphi \circ h_\alpha)$ which is full rank. Indeed, consider a chart $h : U \to M$ with $V = h(U)$ and consider the projection map $p_i : V \to \mathbb{R}$ given by $\pi_i \circ h^{-1}$. There exists a smooth map $\psi : M \to \mathbb{R}$ such that $\mathrm{Supp}\,(\psi) \subseteq V$. We may thus define the map

$$\tilde{\pi}_i(x) = \begin{cases} 0 & \text{if } x \notin V \\ \psi p_i & \text{if } x \in V. \end{cases}$$

This is a smooth map. Moreover, $\mathrm{Supp}\,(\tilde{\pi}_i) \subseteq V$. It is immediate to see that these n maps $\tilde{\pi}_1, \ldots, \tilde{\pi}_n$ are such that $D(\tilde{\pi}_i \circ h)$ is linearly independent set for all points in U. □

Question 2. Let $M \subseteq \mathbb{R}^A$ be a smooth n-manifold and for $x \in M$, let $h : U \to M \subseteq \mathbb{R}^A$ be a chart with $h(u) = x$. Then show that any linear combination of the vectors $\vec{v}_i = \frac{\partial h}{\partial u_i}(u) \in \mathbb{R}^A$ in T_xM is again a tangent vector. This shows that T_xM is a vector space.

Proof. By translations, we may assume that $u = 0$ so that $h(0) = x$. Let $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Consider the path $\eta : (-\epsilon, \epsilon) \to U$ given by $t \mapsto \sum_{i=1}^n tc_ie_i$. Let $\gamma = h \circ \eta : (-\epsilon, \epsilon) \to M$. We get

$$\begin{aligned}\frac{d\gamma}{dt}(0) = \frac{dh \circ \eta}{dt}(0) &= \sum_{i=1}^{n} \frac{\partial h}{\partial u_i}(0) \cdot \frac{d\eta_i}{dt} \\ &= \sum_{i=1}^{n} c_i \frac{\partial h}{\partial u_i}(0) \\ &= \sum_{i=1}^{n} c_i \vec{v}_i,\end{aligned}$$

as required. □

Question 3. Let $M_1 \subseteq \mathbb{R}^A$ and $M_2 \subseteq \mathbb{R}^B$ be two manifolds of dimensions n and m respectively. Show the following:

1. $M_1 \times M_2$ has the structure of a smooth $n+m$-manifold.
2. $T(M_1 \times M_2)$ is diffeomorphic to $TM_1 \times TM_2$.

Proof. 1. Consider $(U_\alpha, g_\alpha)_\alpha$ be an atlas for M_1 and $(V_\beta, h_\beta)_\beta$ be an atlas for M_2 where $U_\alpha \subseteq \mathbb{R}^n$ and $V_\beta \subseteq \mathbb{R}^m$. We claim that $(U_\alpha \times V_\beta, g_\alpha \times h_\beta)_{\alpha,\beta}$ forms an atlas for $M_1 \times M_2$. Indeed, pick any point $x \times y \in M_1 \times M_2$. Then for some α and β, we'll have $x \times y \in g_\alpha(U_\alpha) \times h_\beta(V_\beta)$. Denote

$$\begin{aligned} k_{\alpha\beta} = g_\alpha \times h_\beta : U_\alpha \times V_\beta &\longrightarrow \mathbb{R}^{A \amalg B} \\ (u,v) &\longmapsto (g_\alpha(u), h_\beta(v)) \end{aligned}$$

We need to show that $k_{\alpha,\beta}$ is smooth. To this end, it suffices to show that $\pi_j \circ k_{\alpha\beta} : U_\alpha \times V_\beta \to \mathbb{R}$ is smooth for any projection $\pi_j : \mathbb{R}^{A \amalg B} \to \mathbb{R}$. If $j \in A$, then note that $\pi_j \circ k_{\alpha\beta} = \pi_j \circ g_\alpha$, where the RHS is smooth as (U_α, g_α) is a smooth chart for M_1. Similarly, if $j \in B$. Hence, $k_{\alpha\beta}$ are smooth maps, as required.

Next, we show that $k_{\alpha\beta}$ is an open embedding. To this end, we need only observe that product of two open embeddings is an open embedding. Finally, we have to show that $D(k_{\alpha\beta})$ is a collection of $n+m$-linearly independent vectors in $\mathbb{R}^{A \amalg B}$. Observe that

$$D(k_{\alpha\beta}) = \begin{bmatrix} \frac{\partial g_\alpha}{\partial u} & 0 \\ 0 & \frac{\partial h_\beta}{\partial v} \end{bmatrix} = \begin{bmatrix} D(g_\alpha) & 0 \\ 0 & D(h_\beta) \end{bmatrix}.$$

As $D(g_\alpha)$ is of column rank n and $D(h_\beta)$ is of column rank m, hence $D(k_{\alpha\beta})$ is of column rank $n+m$, as required. This completes the proof of item 1.

2. It is first easy to see that $T_{(m_1,m_2)} M_1 \times M_2 = T_{m_1} M_1 \times T_{m_2} M_2$ for any $m_1 \in M_1, m_2 \in M_2$. This is essentially because local charts of $M_1 \times M_2$ are product of those for M_1 and M_2.

Define the map

$$\begin{aligned} \varphi : T(M_1 \times M_2) &\longrightarrow TM_1 \times TM_2 \\ (x_1, x_2, \vec{v}_1, \vec{v}_2) &\longmapsto ((x_1, \vec{v}_1), (x_2, \vec{v}_2)) . \end{aligned}$$

We wish to show that this is a diffeomorphism. First, observe that φ is a homeomorphism as φ is the restriction of the permutation homeomorphism

$$\begin{aligned} \tilde{\varphi} : M_1 \times M_2 \times \mathbb{R}^A \times \mathbb{R}^B &\longrightarrow M_1 \times \mathbb{R}^A \times M_2 \times \mathbb{R}^B \\ (m_1, m_2, \vec{v}_1, \vec{v}_2) &\longmapsto (m_1, \vec{v}_1, m_2, \vec{v}_2). \end{aligned}$$

Hence, we need only show that φ is a smooth map with a smooth inverse. Indeed, pick any chart $k = g \times h : U \times V \to M_1 \times M_2$ of $M_1 \times M_2$ where (U, g) and (V, h) are open charts for M_1 and M_2 respectively. Recall that we then have a chart

$$\begin{aligned} k \times \partial : U \times V \times \mathbb{R}^n \times \mathbb{R}^m &\longrightarrow T(M_1 \times M_2) \\ (u, v, \vec{a}, \vec{b}) &\longmapsto \left(g(u), h(v), \vec{a} \cdot \frac{\partial g}{\partial u}, \vec{b} \cdot \frac{\partial h}{\partial v} \right). \end{aligned}$$

We need only show that the map

$$\begin{aligned}\varphi\circ(k\times\partial): U\times V\times\mathbb{R}^n\times\mathbb{R}^m &\longrightarrow M_1\times\mathbb{R}^A\times M_2\times\mathbb{R}^B\subseteq\mathbb{R}^A\times\mathbb{R}^A\times\mathbb{R}^B\times\mathbb{R}^B\\ (u,v,\vec{a},\vec{b}) &\longmapsto \left(g(u),\vec{a}\cdot\frac{\partial g}{\partial u},h(v),\vec{b}\cdot\frac{\partial h}{\partial v}\right).\end{aligned}$$

is smooth. But this is immediate as g,h are smooth charts and taking inner product is a linear operation. One can similarly show that the inverse map

$$\begin{aligned}\varphi^{-1}: TM_1\times TM_2 &\longrightarrow T(M_1\times M_2)\\ ((x_1,\vec{v}_1),(x_2,\vec{v}_2)) &\longmapsto (x_1,x_2,\vec{v}_1,\vec{v}_2)\end{aligned}$$

is also smooth. This completes the proof. □

Question 4. Let $\mathbb{P}^n$ be the set of all 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$ and consider the quotient $q:\mathbb{R}^{n+1}-0\to\mathbb{P}^n$. Define $F=\{f:\mathbb{P}^n\to\mathbb{R}\mid f\circ q:\mathbb{R}^{n+1}-0\to\mathbb{R}\text{ is smooth}\}$. Show that

1. F is a smoothness structure on $\mathbb{P}^n$.
2. Let $M=\{A\in M_{n+1}(\mathbb{R})\mid A\text{ is symmetric},\operatorname{Tr}(A)=1\ \&\ A\cdot A=A\}$. Show that $\mathbb{P}^n$ is diffeomorphic to M.
3. Show that $\mathbb{P}^n$ is compact and $V\subseteq\mathbb{P}^n$ is open if and only if $q^{-1}(V)\subseteq\mathbb{R}^{n+1}-0$ is open.

Proof. 1. We first show that F separates points of $\mathbb{P}^n$. Assuming to the contrary, we get that there exists $[x],[y]\in\mathbb{P}^n$ two distinct points such that for all $f\in F$, $f([x])=f([y])$. Indeed, consider f_{ij} given by

$$\begin{aligned}f_i:\mathbb{P}^n &\longrightarrow\mathbb{R}\\ [z] &\longmapsto \frac{z_iz_j}{\sum_{k=0}^n z_k^2}.\end{aligned}$$

By our assumption, we get

$$\frac{x_ix_j}{\sum_{k=0}^n x_k^2}=\frac{y_iy_j}{\sum_{k=0}^n y_k^2}.$$

from which we deduce that for each

$$\frac{x_i}{y_i}=\frac{x_j}{y_j}\sqrt{\frac{\sum_{k=0}^n x_k^2}{\sum_{k=0}^n y_k^2}}.$$

The square root is a constant, say $\alpha>0$. As $[x],[y]\in\mathbb{P}^n$, we may take $x,y\in S^{n+1}$ as $S^{n+1}\to\mathbb{P}^n$ is a quotient map. Thus, we get $\alpha=1$, that is

$$\frac{x_i}{y_i}=\frac{x_j}{y_j}$$

for all $i,j=0,\ldots,n$. This shows that $[x]=[y]$ in $\mathbb{P}^n$, a contradiction. Hence F separates points.

Next we wish to show that the image of the map

$$\begin{aligned}\varphi:\mathbb{P}^n &\longrightarrow\mathbb{R}^F\\ [x] &\longmapsto (f([x]))_{f\in F}\end{aligned}$$

is a smooth manifold. Indeed, let $M = \varphi(\mathbb{P}^n)$ and consider the subsets of $\mathbb{P}^n$ given by U_i, $i = 0, \dots, n$ where $U_i = \{[x_0 : \cdots : x_n] \mid x_i \neq 0\}$. Let $V_i = \varphi(U_i) \subseteq M \subseteq \mathbb{R}^F$. We claim that the maps

$$\varphi \circ h_i : \mathbb{R}^n \xrightarrow{h_i} U_i \xrightarrow{\varphi} M$$

where h_i is given by $(x_1, \dots, x_n) \mapsto [x_1 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n]$ is smooth. Indeed, $\varphi \circ h_i$ composed with projection on $f \in F$ is the composite $f \circ h_i$ which is smooth as it is the restriction of $f \circ q$ to the open subspace of $\mathbb{R}^{n+1} - 0$ where $x_i \neq 0$.

Finally, to show that $\varphi \circ h_i$ is a local parameterization for M, we need to show that $D(\varphi \circ h_i)$ is of rank n. Again, as in Question 1, it suffices to find n-functions $f_j \in F$ so that the corresponding $n \times n$ submatrix of $D(\varphi \circ h_i)$ is of full rank. One can check that this is done by the following n-functions

$$\begin{aligned} f_j : \mathbb{P}^n &\longrightarrow \mathbb{R} \\ [x] &\longmapsto \frac{x_j^2}{\sum_{k=0}^n x_k^2} \end{aligned}$$

for each $i = 0, \dots, n$ and $j \neq i$. This shows that $\mathbb{P}^n$ has F as a smoothness structure.

2. Consider the map

$$\begin{aligned} \varphi : \mathbb{P}^n &\longrightarrow M \subseteq \mathbb{R}^{(n+1)^2} \\ [x] &\longmapsto A_x \end{aligned}$$

where $A_x = (f_{ij}([x]))_{0 \leq i,j \leq n}$ and

$$\begin{aligned} f_{ij} : \mathbb{P}^n &\longrightarrow \mathbb{R} \\ [x] &\longmapsto \frac{x_i x_j}{\sum_{k=0}^n x_k^2}. \end{aligned}$$

It is immediate to see that indeed $A_x \in M$. We need to show that φ is a diffeomorphism. Clearly, φ is smooth as for the charts (U_i, h_i) of $\mathbb{P}^n$ as in item 1, the composition

$$\begin{aligned} \varphi \circ h_i : U_i &\longrightarrow M \\ (x_1, \dots, x_n) &\longmapsto A_x \end{aligned}$$

is smooth as each entry of A_x is a rational function in $x_1, \dots, x_n$ with denominator never vanishing as $0 \notin U_i$. This shows that φ is smooth.

We construct a smooth inverse of φ as follows:

$$\begin{aligned} \psi : M &\longrightarrow \mathbb{P}^n \\ A = (a_{ij}) &\longmapsto [l] \end{aligned}$$

where l is the column space of A, i.e. the linear space spanned by $n+1$-columns $A_0, \dots, A_n$ of A. Indeed, as A is a symmetric idempotent matrix of trace 1, therefore A is a projection matrix onto a 1-dimensional subspace of $\mathbb{R}^{n+1}$, spanned by the columns. Hence ψ is well-defined. Moreover, it is smooth as ψ on each coordinate is a linear combination of entries of A. This shows that ψ is smooth. Then ψ is an inverse of φ. This shows that M and $\mathbb{P}^n$ are diffeomorphic.

3. The map $q : \mathbb{R}^{n+1} - 0 \to \mathbb{P}^n$ is a quotient map. As we have the following commutative diagram

$$\begin{array}{ccc} & & \mathbb{R}^{n+1} - 0 \\ & \nearrow & \downarrow q \\ S^n & \xrightarrow[p]{} & \mathbb{P}^n \end{array},$$

therefore we have a quotient map $p : S^n \to \mathbb{P}^n$. Thus, $\mathbb{P}^n$ is compact. By definition of quotient maps, $V \subseteq \mathbb{P}^n$ is open if and only if $q^{-1}(V)$ is open. $\square$

Question 5. Let M be a smooth n-manifold and $R = \mathcal{C}^\infty(M;\mathbb{R})$.

1. Show that R is an $\mathbb{R}$-algebra.
2. Every point $x \in M$ determines $\mathrm{ev}_x : R \to \mathbb{R}$ an $\mathbb{R}$-algebra homomorphism. That is, we have a function
$$\mathrm{ev} : M \longrightarrow \mathrm{Hom}\,(R,\mathbb{R})$$
$$x \longmapsto \mathrm{ev}_x$$
3. If M is compact, then there is a bijection of sets
$$\mathrm{mspec}(R) \cong M.$$
4. If M is second-countable, then the map
$$\mathrm{ev} : M \to \mathrm{Hom}\,(R,\mathbb{R})$$
is a bijection.
5. For any $x \in M$, consider the $\mathbb{R}$-algebra map $\mathrm{ev}_x : R \to \mathbb{R}$, $f \mapsto f(x)$. Hence for each $x \in M$, we get that $\mathbb{R}$ is an R-module via the map ev_x and we denote $\mathbb{R}$ with this R-module structure as $\mathbb{R}_x$. Then show that any $\mathbb{R}$-linear map
$$X : R \longrightarrow \mathbb{R}$$
satisfying $X(fg) = X(f)\cdot g(x) + f(x)\cdot X(g)$ for some fixed $x \in M$ is uniquely determined by a choice of a vector $\vec{v} \in T_xM$. That is, if $\mathrm{Der}_{\mathbb{R}}(R,\mathbb{R}_x)$ denotes the set of all $\mathbb{R}$-linear maps $d : R \to \mathbb{R}_x$ satisfying $d(fg) = d(f)\cdot g(x) + f(x)\cdot d(g)$ for some $x \in M$, then we have a bijection
$$\mathrm{Der}_{\mathbb{R}}(R,\mathbb{R}_x) \cong T_xM.$$

Proof. 1. This is immediate by pointwise addition and multiplication.

2. Define for any $x \in M$ the following $\mathbb{R}$-algebra homomorphism:

$$\mathrm{ev}_x : R \longrightarrow \mathbb{R}$$
$$f \longmapsto f(x)$$

which is the evaluation at x. This is the required homomorphism. Note that $\mathrm{Ker}\,(\mathrm{ev}_x)$ is a maximal ideal since $R/\mathrm{Ker}\,(\mathrm{ev}_x) \cong \mathbb{R}$.

3. Define the map

$$\begin{aligned} \alpha : M &\longrightarrow \mathrm{mspec}(R) \\ x &\longmapsto \mathrm{Ker}\,(\mathrm{ev}_x). \end{aligned}$$

By item 2, $\alpha(x)$ is indeed a maximal ideal of R. Pick any maximal ideal $\mathfrak{m} \in \mathrm{mspec}(R)$. We show that it is kernel of evaluation at some point. If not, then for all $x \in M$, there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. As $f_x : M \to \mathbb{R}$ is continuous, therefore there exists an open $x \in U \subseteq M$ such that $f_x(y) \neq 0$ for all $y \in U_x$. We have thus obtained a cover of M by $\{U_x\}$. By shrinking each U_x if necessary, we may assume that $U_x \subseteq C_x \subseteq V_x$ where C_x is a compact set of M and V_x is open in M. It follows by compactness that there is a finite cover $M = \bigcup_{i=1}^n U_{x_i}$. As M is compact Hausdorff, therefore there exists smooth bump functions on each open U_{x_i}. Thus we have maps $\rho_i : M \to \mathbb{R}$ such that $\rho_i = 1$ on U_{x_i}. Consider then the map $g = \sum_{i=1}^n \rho_i f_{x_i}^2$. This is a global smooth map $g : M \to \mathbb{R}$ such that $g(x) = \sum_{i=1}^n \rho_i f_{x_i}^2(x) \neq 0$ as for any $x \in X$, there are finitely many U_{x_i} containing x on which atleast one of f_{x_i} is non-zero and ρ_i is 1. Hence g is invertible. As $f_{x_i}^2 \in \mathfrak{m}$, therefore $g \in \mathfrak{m}$ and hence $\mathfrak{m} = R$, a contradiction. Thus α is surjective.

We next show injectivity of α. If $\mathfrak{m}_x = \mathfrak{m}_y$ and $x \neq y$, then by Hausdorff property, we may separate x and y by opens U and V. Consider the singleton $\{x\} \subseteq U$ which is compact. By Proposition 2.29 of Lee, we deduce that there exists $f : M \to \mathbb{R}$ smooth such that $f(x) \neq 0$ and $f = 0$ on $X \setminus V$. Thus $f(x) \neq 0$ and $f(y) = 0$, as required. This establishes that α is an isomorphism.

4. It is injective as if $f \mapsto f(z)$ is same for x and y, then $\mathfrak{m}_x = \mathfrak{m}_y$ by item 3. By injectivity of α we deduce that $x = y$. For surjectivity, pick $\psi : R \to \mathbb{R}$ an $\mathbb{R}$-algebra homomorphism. Then, we claim that there exists $x \in M$ such that $\psi(f) = f(x)$. We give a simple proof if M is compact. Indeed, by item 3, we have that $\mathrm{Ker}\,(\psi) = \mathfrak{m}_x$ for some point $x \in M$. We claim that $\psi = \mathrm{ev}_x$. Indeed, if not then there exists $g_x \in R$ such that $\psi(g_x) \neq g_x(x)$. Consider the map $f = g_x - \psi(g_x)$ in R where we assume $\psi(g_x)$ as a constant function. Applying ψ to it, we get

$$\psi(f) = \psi(g_x) - \psi(\psi(g_x)) = \psi(g_x) - \psi(g_x) = 0.$$

Thus $f \in \mathfrak{m}_x$ and hence $g_x(x) = \psi(g_x)$, a contradiction. This completes the proof of item 4 for the case when M is compact.

Now consider M to be only second countable. If $\psi \neq \mathrm{ev}_x$ for all x, then for all $x \in M$, there exists $g_x \in R$ such that $\psi(g_x) \neq g_x(x)$. We thus get maps $f_x = g_x - \psi(g_x) \in R$ which are non-zero at x. It follows that there exists opens U_x containing x such that f_x on U_x is non-vanishing, so we may assume wlog that $f_x > 0$. Note that for each f_x, we have

$$\psi(f_x) = 0.$$

Thus, each $f_x \in \mathrm{Ker}\,(\psi)$. We next deduce by second countability, in particular, by Lindelöf property that there exists a countable subcover U_{x_n} of M. By refinement, we may assume that $\{U_{x_n}\}_n$ is a locally finite cover. This allows us to construct a non-zero global smooth map given by

$$g = \sum_n f_{x_n}$$

where $g(x) = \sum_n f_{x_n}(x)$ is finite as $\{U_{x_n}\}$ is a locally finite cover, so there exists $U \ni x$ such that $U \cap U_{x_n} \neq \emptyset$ only for finitely many n. Consequently, $\mathrm{Ker}\,(\psi)$ contain g which is non-zero and hence a unit, a contradiction to non-triviality of ψ.

5. Consider the map

$$\begin{aligned} \varphi : T_xM &\longrightarrow \mathrm{Der}_{\mathbb{R}}(R, \mathbb{R}_x) \\ \vec{v} &\longmapsto X_{x,\vec{v}} \end{aligned}$$

where $X_{x,\vec{v}}(f) = Df_x(\vec{v}) \in \mathbb{R}$, the directional derivative of f at x along $\vec{v}$. This is injective as if $X_{x,\vec{v}} = X_{x,\vec{w}}$ for $\vec{v}, \vec{w} \in T_xM$, then $Df_x(\vec{v}) = Df_x(\vec{w})$ for all $f \in R$. As $Df_x(\vec{v}) = \nabla f_x \cdot \vec{v}$ where $\nabla f_x \in \mathbb{R}^n$ is the gradient vector of f at x, therefore we get that $\nabla f_x(\vec{v} - \vec{w}) = 0$ for all $f \in R$. We may let f to be the global projection maps on M obtained by using partitions of unity and the n-projection maps on the local charts. This yields that $\vec{v} - \vec{w} = 0$, as required.

For surjectivity, consider any derivation $d : R \to \mathbb{R}_x$. We wish to show that $d = X_{x,\vec{v}}$ for some $\vec{v} \in T_xM$. Begin by fixing a coordinate chart $h : U \to M$ such that $h(0) = x$ where $U \subseteq \mathbb{R}^n$ is an open neighborhood of 0. Consider the basis vectors $\vec{v}_i \in T_xM$ given by

$$\vec{v}_i = \frac{\partial h}{\partial u_i}(0).$$

We do the Taylor expansion of $f \circ h : U \to \mathbb{R}$, so that for some open $U' \subseteq U$ around 0, we can write

$$\begin{aligned} f \circ h(p) &= f \circ h(0) + D(f \circ h)_0(p) + p^T H(f \circ h)_0 p \\ &= f(x) + \sum_{i=1}^{n} \frac{\partial f \circ h}{\partial x_i}(0) \cdot p_i + \sum_{i,j=1}^{n} \frac{\partial^2 f \circ h}{\partial x_i x_j}(0) p_i p_j. \end{aligned}$$

where $H(f \circ h)_0$ is the Hessian of $f \circ h$ at $0 \in U'$. Let $\phi : V' = h(U') \to U'$ be the inverse of h on U'. Thus for any $y \in V'$, we may get the following by replacing p by $\phi(y)$:

$$f(y) = f(x) + \sum_{i=1}^{n} \frac{\partial f \circ h}{\partial x_i}(0) \cdot \phi_i(y) + \sum_{i,j=1}^{n} \frac{\partial^2 f \circ h}{\partial x_i x_j}(0) \phi_i(y) \phi_j(y).$$

Applying d on above equation, we get

$$d(f) = d(f(x)) + \sum_{i=1}^{n} \frac{\partial f \circ h}{\partial x_i}(0) \cdot d(\phi_i) + \sum_{i,j=1}^{n} \frac{\partial^2 f \circ h}{\partial x_i x_j}(0) d(\phi_i \phi_j)$$

Now note that derivation applied at a constant is 0, so $d(f(x)) = 0$. Further, $d(\phi_i \phi_j) = \phi_i(x) d(\phi_j) + \phi_j(x) d(\phi_i) = 0$ as $\phi_i(x) = 0 = \phi_j(x)$. Hence, we get

$$d(f) = \sum_{i=1}^{n} \frac{\partial f \circ h}{\partial x_i}(0) \cdot d(\phi_i).$$

Now, observe that

$$\begin{aligned} d(\phi_i) \cdot X_{x,\vec{v}_i}(f) &= d(\phi_i) \cdot Df_x(\vec{v}_i) \\ &= \frac{\partial f \circ h}{\partial x_i}(0) \cdot d(\phi_i). \end{aligned}$$

Hence, we get that

$$d(f)=\sum_{i=1}^{n} d(\phi_i)X_{x,\vec{v_i}} = X_{x,\sum_{i=1}^{n} d(\phi_i)\vec{v}_i}(f),$$

as required. □

2 Vector bundles

Question 6. Let M be a smooth n-manifold. Show that the tangent manifold TM with the projection map $\pi : TM \to M$ is a vector bundle of rank n.

Proof. Fix a point $x \in M$. We wish to find an open set $U \ni x$ in M such that $\pi : \pi^{-1}(U) \to U$ is a trivial bundle. Indeed, consider a chart (U, h) around x, so that $h : U \to \mathbb{R}^n$ is an open embedding. We claim that $\pi^{-1}(U)$ is the tangent manifold of $U \subseteq M$. Indeed, $\pi^{-1}(U) = \{(x, \vec{v}) \in U \times \mathbb{R}^n \mid \vec{v} \in T_xU\}$ and since $T_xU = T_xM$, therefore $\pi^{-1}(U) = TU$. It hence suffices to show that $TU \cong U \times \mathbb{R}^n$, that is, U is parallelizable. We claim that $TU \cong TV$ where $h : U \to V \subseteq \mathbb{R}^n$ is a homeomorphism. Indeed, as each chart is a diffeomorphism by inverse function theorem, hence the map $dh : TU \to TV$ induced by h is a fiberwise isomorphism. It now suffices to show that TV is trivial.

To complete the proof, it suffices to show that any open subset of $\mathbb{R}^n$ is parallelizable. Indeed for $V \subseteq \mathbb{R}^n$ and $x \in V$, we may consider $T_xV = \mathbb{R}^n$ obtained by shifting the origin to x. Consider the sections $s_i : V \to TV$ mapping $x \mapsto (x, \vec{e}_i)$ where $\vec{e}_i$ is the i^{th} standard vector in $T_xV = \mathbb{R}^n$. This is continuous since it is continuous as a map $V \to V \times \mathbb{R}^n$. Note that this may not be continuous if V was not a manifold with one chart, as then different charts would give different coordinates for the same tangent vector. As the collection $s_1, \dots, s_n$ is everywhere independent collection of global sections of $TV \to V$ hence $TV \to V$ is trivial. □

Question 7. Show the following about n-spheres.

1. If S^n admits a non-vanishing vector field, then the identity map $\text{id} : S^n \to S^n$ is homotopic to the antipodal map $a : S^n \to S^n$.
2. If n is even, then antipodal $a : S^n \to S^n$ is homotopic to the reflection $r : S^n \to S^n$ given by $(x_1, x_2 \dots, x_{n+1}) \mapsto (-x_1, x_2, \dots, x_{n+1})$.
3. If $n \geq 2$ is even, then S^n is not parallelizable.

Proof. 1. Let $s : S^n \to TS^n$ be a non-vanishing vector field, so that $s : x \mapsto (x, \tilde{s}(x))$ where $\tilde{s} : S^n \to \mathbb{R}^n$ is a continuous map. We construct the following homotopy:

$$\begin{aligned} H : S^n \times I &\longrightarrow S^n \\ (x,t) &\longmapsto x\cos(t\pi) + \frac{\tilde{s}(x)}{\|\tilde{s}(x)\|}\sin(t\pi) \end{aligned}$$

Indeed, as $\langle x, \tilde{s}(x)\rangle = 0$, therefore H is a well-defined homotopy from id to a.

2. Done in class.

3. Suppose $n \geq 2$ is even. Assume that S^n is parallelizable. Then $TS^n = S^n \times \mathbb{R}^n$. Consequently, S^n admits a non-vanishing vector field, say $f : S^n \to TS^n$, $x \mapsto (x, \vec{v})$ where $\vec{v}$ is fixed in $\mathbb{R}^n$. By item 1, we get $\text{id} \sim a$ and by item 2, we further get $a \sim r$. Thus $\text{id} \sim r$. But $\deg \text{id} = 1$, $\deg r = -1$, a contradiction to the homotopy invariance of degree map. □

Question 8. Show that any vector bundle $p : E \to B$ where B is a paracompact space has an Euclidean metric.

Proof. Cover B by local trivializations $\{U_\alpha\}$ such that for each α, we have isomorphisms of families:

$$\begin{array}{ccc} U_\alpha \times \mathbb{R}^n & \xrightarrow[\cong]{h_\alpha} & p^{-1}(U_\alpha) \\ {\scriptstyle \pi_1}\downarrow & \swarrow {\scriptstyle p} & \\ U_\alpha & & \end{array}.$$

Define on each $p^{-1}(U_\alpha)$ the following Euclidean metric:

$$\mu_\alpha : p^{-1}(U_\alpha) \xrightarrow{h_\alpha^{-1}} U_\alpha \times \mathbb{R}^n \xrightarrow{\sum_i x_i^2} \mathbb{R}$$

which maps as $e \mapsto (p(e), k_\alpha(e)) \mapsto \sum_{i=1}^n k_{\alpha_i}(e)^2$ where $k_\alpha(e) = (k_{\alpha_i}(e))$. We will patch these μ_α up by using partitions of unity. First, by paracompactness of B, we may assume that $\{U_\alpha\}$ is a locally finite cover. Consequently, $p^{-1}(U_\alpha)$ is a locally finite cover of E. By partitions of unity, we get maps $\rho_\alpha : B \to \mathbb{R}$ with $\sum_\alpha \rho_\alpha = 1$ and $\text{Supp}\,(\rho_\alpha) \subseteq U_\alpha$. Denote $\sigma_\alpha = \rho_\alpha \circ p : E \to \mathbb{R}$ and observe that $\sum_\alpha \sigma_\alpha = 1$ and $\text{Supp}\,(\sigma_\alpha) \subseteq p^{-1}(U_\alpha)$. We will now patch up μ_α.

Define $\mu = \sum_\alpha \sigma_\alpha \cdot \mu_\alpha$ which is a map $E \to \mathbb{R}$. This is well-defined by construction. We need only show that for each $b \in B$, the map on fibers $E_b \to \mathbb{R}$ is a positive definite quadratic form. Indeed, by local finiteness of $\{U_\alpha\}$, we get that each $b \in B$ is contained in say $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{m_b}}$ and hence $E_b \subseteq p^{-1}(U_{\alpha_1}) \cap \cdots \cap p^{-1}(U_{\alpha_{m_b}})$. Consequently on fiber E_b, the map μ is

$$\mu_b = \sum_{j=1}^{m_b} \sigma_{\alpha_i} \cdot \mu_{\alpha_i},$$

where each σ_{α_i} is a constant function on E_b as for any $e \in E_b$, $\sigma_{\alpha_i}(e) = \rho_{\alpha_i} \circ p(e) = \rho_{\alpha_i}(b) \in \mathbb{R}_{\geq 0}$. Hence μ_b is a positive definite quadratic form, as required. □

Question 9 (*Alexandroff line*)**.** Show that the Alexandroff line doesn't admit a Riemannian metric.

Proof. Alexandroff line L is a 1-dimensional smooth connected manifold. Recall that every Riemannian manifold has a metric space structure. But since L is not paracompact and every metric space is paracompact, therefore L cannot admit a Riemannian structure. □

Question 10 (*Isometry theorem*)**.** Let $p : E \to B$ be a vector bundle and μ, μ' be two Euclidean metrics on E. Denote $E = (E, \mu)$ and $E' = (E, \mu')$. Show that there exists an isomorphism $f : E \to E'$ of vector bundles such that for all $b \in B$, the linear map $f_b : (E_b, \mu_b) \to (E_b, \mu'_b)$ is a linear isometric isomorphism.

Proof. Fix $b \in B$. Observe that for any $\vec{v} \in E_b$, we have $\mu_b(\vec{v}) = \vec{v}^T A_b \vec{v}$ and $\mu'_b(\vec{v}) = \vec{v}^T A'_b \vec{v}$ where A_b, A'_b are positive definite symmetric matrices corresponding to the positive definite quadratic forms $\mu_b, \mu'_b : E_b \to \mathbb{R}$, respectively. Recall that every positive definite symmetric matrix M has a unique square root, that is, a positive definite symmetric matrix $\sqrt{M}$ such that $(\sqrt{M})^2 = M$. Since a positive definite matrix is always invertible as it has all positive eigenvalues, therefore if we write

$$A_b = \sqrt{A_b} \cdot \sqrt{A_b}$$
$$A'_b = \sqrt{A'_b} \cdot \sqrt{A'_b},$$

then for $B_b = (\sqrt{A'_b})^{-1} \cdot \sqrt{A_b}$ we get

$$B_b^T \cdot A'_b \cdot B_b = A_b.$$

We thus define a map

$$\begin{aligned} f_b : E_b &\longrightarrow E'_b \\ \vec{v} &\longmapsto B_b \vec{v}. \end{aligned}$$

Observe that $\mu'_b(f_b(\vec{v})) = (B_b\vec{v})^T A'_b (B_b \vec{v}) = \vec{v}^T B_b^T A'_b B_b \vec{v} = \vec{v}^T A_b \vec{v} = \mu_b(\vec{v})$, hence f_b is a linear isometric isomorphism. Thus we get a function $f : E \to E'$, which is isomorphism on fibers. To see the continuity of f, we need only show that the mapping $b \mapsto B_b$ is continuous as b varies in B. As the map $b \mapsto B_b$ is the product of $b \mapsto (\sqrt{A'_b})^{-1}$ and $b \mapsto \sqrt{A_b}$, and since the mapping $b \mapsto A_b$, $b \mapsto A'_b$ are continuous by continuity of μ and μ', therefore it is sufficient to show that for the mapping $M \mapsto \sqrt{M}$ for positive definite symmetric matrices M is continuous. This is immediate from power series expansion of $\sqrt{M}$. □

3 Constructions on vector bundles

Question 11. Let M, N be two smooth manifolds of dimension m and n. Let $g : M \to N$ be a smooth map which is a submersion. Construct a subbundle $\text{Ker}\,(g)$ of TM whose fibers are $\text{Ker}\,\big(g_x : T_xM \to T_{g(x)}N\big)$. If M is Riemannian, show that

$$TM \cong \text{Ker}\,(g) \oplus g^*(TN).$$

Proof. As a submersion is of locally constant rank, therefore by Theorem 3.0.1, we have the kernel bundle $\text{Ker}\,(g)$. Now suppose M is Riemannian. Let U be a common trivialization of both $\text{Ker}\,(g) \oplus g^*TN$ and TM. To show the splitting, it is sufficient to show that there is a map $h_x : T_{g(x)}N \to T_xM$ such that $x \mapsto h_x$ is continuous and h_x is a splitting of the following s.e.s:

$$0 \longrightarrow \text{Ker}\,(g_x) \longrightarrow T_xM \underset{h_x}{\overset{g_x}{\rightleftarrows}} T_{g(x)}N \longrightarrow 0\ .$$

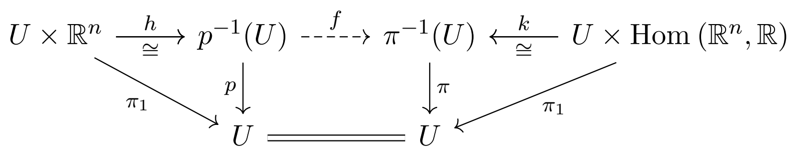
Indeed, let $\mu : TM \to \mathbb{R}$ be the given Euclidean metric, thus each T_xM is an inner product space. Recall from linear algebra that $\text{Ker}\,(g_x) \oplus (\text{Ker}\,(g_x))^\perp = T_xM$. Thus we have $g_x : (\text{Ker}\,(g_x))^\perp \to T_{g(x)}N$ is an isomorphism. Let $h_x : T_{g(x)}N \to (\text{Ker}\,(g_x))^\perp$ be its inverse. As taking inverse of linear isomorphisms is a continuous map, therefore $x \mapsto h_x$ is continuous, as required. □

Question 12. Let $\xi = (E, p, B)$ and $\eta = (E', q, B)$ be two bundles of rank n and m respectively. Let $f : B \to \mathcal{H}om(\xi, \eta)$ be a global section such that the map $b \mapsto \dim \operatorname{Im}(f(b) : E_b \to E'_b)$ is a locally constant function. Then construct $\operatorname{Ker}(f)$ and $\operatorname{CoKer}(f)$ two vector bundles over B whose fibers are $\operatorname{Ker}(f(b))$ and $\operatorname{CoKer}(f(b))$ respectively.

Proof. This is done in Theorem 3.0.1 as any $f : B \to \mathcal{H}om(\xi, \eta)$ is equivalent to the data of a vector bundle map $f : \xi \to \eta$. □

Question 13. If a vector bundle $\xi = (E, p, B)$ admits a Euclidean metric, then it is isomorphic to the dual $\mathcal{H}om(\xi, \epsilon^1)$.

Proof. Let $\mu : E \to \mathbb{R}$ be the Euclidean metric and ξ be of rank n. Let $U \subseteq B$ be a common local trivialization of both E and $\mathcal{H}om(\xi, \epsilon^1)$. We then have the following diagram:



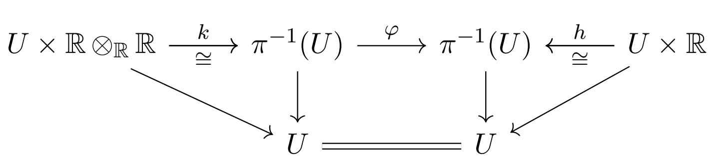
where we define f by defining the map $k^{-1} \circ f \circ h : U \times \mathbb{R}^n \to U \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ as follows:

$$k^{-1} \circ f \circ h : (b, \vec{v}) \mapsto (b, \langle \vec{v}, - \rangle_b)$$

where $\langle -, - \rangle_b$ is the inner product on E_b defined by the positive definite quadratic form $\mu_b : E_b \to \mathbb{R}$. As $\mu : E \to \mathbb{R}$ is continuous, therefore the $\langle -, - \rangle_b : E_b \times E_b \to \mathbb{R}$, is continuous in $b \in B$ and thus the above map $k^{-1} \circ f \circ h$ is continuos. This defines a global continuous map $f : E \to \mathcal{H}om(\xi, \epsilon^1)$. To show that f is an isomorphism, it is sufficient to show that $f_b : E_b \to \operatorname{Hom}(E_b, \mathbb{R})$ is a linear isomorphism for each b. Indeed, this is clear as E_b is an inner-product space, therefore the map $f_b : e \mapsto \langle e, - \rangle_b$ is an isomorphism by Riesz-representation theorem, as required. □

Question 14. Construct the Picard group $\operatorname{Pic}(B)$ of a space B and show that those elements of $\operatorname{Pic}(B)$ of order ≤ 2 are equivalent to Euclidean line bundles on B.

Proof. Let $\operatorname{Pic}(B)$ denote the set of isomorphism classes of line bundles over B. For two line bundles $\xi = (L_1, \pi_1, B)$ and $\eta = (L_2, \pi_2, B)$, we define $\xi \otimes \eta$ to be the tensor product bundle $(L_1 \otimes L_2, \pi, B)$. As rank of $L_1 \otimes L_2$ is equal to product of ranks of L_1 and L_2, therefore $\pi : L_1 \otimes L_2 \to B$ is also a line bundle. As $\xi \otimes \eta \cong \eta \otimes \xi$, therefore we have a well-defined commutative product on $\operatorname{Pic}(B)$. To show group structure, it suffices to show that $\xi \otimes \epsilon^1 \cong \xi$ and the existence of inverses. Indeed, define $\varphi : \xi \otimes \epsilon^1 \to \xi$ which on fiber at $b \in B$ is $L_{1,b} \otimes_{\mathbb{R}} \mathbb{R} \to L_{1,b}$ mapping as $e \otimes \lambda \mapsto \lambda e$. Clearly φ is an isomorphism on fibers. To see continuity, consider the following diagram for a common trivializing open $U \subseteq B$ for both the bundles:



As the horizontal map can be checked is equal to $(b, \lambda \otimes \gamma) \mapsto (b, \lambda\gamma)$, which is continuous, therefore φ is continuous. Hence, φ is an isomorphism, as required.

Next suppose $\xi = (L, \pi, B)$ is a line bundle and let $\check{\xi} = \mathcal{H}om(\xi, \epsilon^1)$ be the dual bundle. Note that $\check{\xi}$ is also a line bundle. We claim that $\xi \otimes \mathcal{H}om(\xi, \epsilon^1) \cong \epsilon^1$. Indeed, define the map $\varphi : \xi \otimes \mathcal{H}om(\xi, \epsilon^1) \to \epsilon^1$ which on fiber at $b \in B$ is defined as $L_{1,b} \otimes \mathrm{Hom}\,(L_{1,b}, \mathbb{R}) \to \mathbb{R}$, $e \otimes \varphi \mapsto \varphi(e)$. Clearly, this is an isomorphism on fibers. To show continutiy, take $U \ni b$ a trivializing neighborhood of b for both the bundles. By drawing the similar diagram as above, we conclude that φ is continuous and isomorphism on fibers, hence an isomorphism.

Now pick $\xi = (L, \pi, B)$ to be a Euclidean line bundle on B which is not trivial. We then claim that $\xi \otimes \xi \cong \epsilon^1$. Indeed, first observe that for any inner product space V, there is an isomorphism $V \otimes_{\mathbb{R}} V \to \mathbb{R}$ given by $v \otimes w \mapsto \langle v, w \rangle$. We use this to define an isomorphism $\xi \otimes \xi \cong \epsilon^1$. Define $\varphi : \xi \otimes \xi \to \epsilon^1$ on fiber at $b \in B$ by $L_{1,b} \otimes L_{1,b} \to \mathbb{R}$, $e \otimes f \mapsto \langle e, f \rangle_b$ where $\langle -, - \rangle_b$ is the inner product on fibers given by the Euclidean structure. Clearly φ is a fiber isomorphism. We need only show that it is continuous. Indeed, drawing the same diagram as above one immediately sees this. Hence, ξ is an order 2 element of $\mathrm{Pic}(B)$.

Conversely, pick an order 2 element $\xi \in \mathrm{Pic}(B)$. Then, there is an isomorphism $\varphi : \xi \otimes \xi \to \epsilon^1$. It follows that we have an isomorphism $\varphi_b : L_b \otimes L_b \to \mathbb{R}$ which varies continuously on b. As $L_b \otimes L_b \cong \mathbb{R}$, therefore we have $\varphi_b : \mathbb{R} \to \mathbb{R}$. It is sufficient to show that φ_b is positive definite. To this end, we need to show that $\varphi_b(\lambda^2) > 0$ for all $\lambda \neq 0$ in $\mathbb{R}$. As $\varphi_b(\lambda^2) = \lambda^2 \varphi_b(1)$, therefore we need only show that $\varphi_b(1) > 0$. Observe that the map $b \mapsto \varphi_b(1)$ is continuous and since each φ_b is an isomorphism, therefore by intermediate value theorem, either $\varphi_b(1) > 0$ or < 0 for each $b \in B$. If $\varphi_b(1) < 0$, we may replace φ by $-\varphi$. Hence we have $\varphi_b(1) > 0$ for all $b \in B$, as required. □

Theorem 3.0.1 (Theorem 8.2 of Husemoller). *Let (E', π', B') and (E, π, B) be vector bundles of ranks n and m respectively and $(f, g) : (E', \pi', B') \to (E, \pi, B)$ be a map of vector bundles where $f : E' \to E$ is of locally constant rank. Then, there exists bundles K_g over B' and C_g over B such that fiber of K_g and C_g at $x \in B$ is $\mathrm{Ker}\left(g_x : E'_x \to E_{g(x)}\right)$ and $\mathrm{CoKer}(g_x : E'_x \to E_{g(x)})$, respectively.*

Proof. Define $K_g = \amalg_{x \in B'} \mathrm{Ker}\left(f_x : E'_x \to E_{g(x)}\right)$ and $C_g = \amalg_{x \in B} \mathrm{CoKer}(f_x : E'_x \to E_{g(x)})$. Give K_g the subspace topology of E'. It is thus sufficient to show that $\pi' : K_g \to B'$ is locally trivial. To this end, pick any point $b \in B'$. There exists trivializing neighborhood $U \ni b$ and $V \ni g(b)$ such that $g^{-1}(V) = U$ such that $f : \pi'^{-1}(U) \to \pi^{-1}(V)$ is of constant rank k. We thus have the following diagram:

$$\begin{array}{ccccccc}
U \times \mathbb{R}^n & \xrightarrow[\cong]{h} & \pi'^{-1}(U) & \xrightarrow{f} & \pi^{-1}(V) & \xleftarrow[\cong]{k} & V \times \mathbb{R}^m \\
& \searrow^{\pi_1} & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & \swarrow_{\pi_1} & \\
& & U & \xrightarrow[g]{} & V & &
\end{array}.$$

Let the horizontal composite be $u : U \times \mathbb{R}^n \to V \times \mathbb{R}^m$ and denote for each $b \in U$ the corresponding linear map as $u_b : \mathbb{R}^n \to \mathbb{R}^m$.

We now have the following split exact sequences for each $x \in U$

$$0 \to \mathrm{Ker}\,(u_x) \to \mathbb{R}^n \xrightarrow{u_x} \mathrm{Im}(u_x) \to 0$$

and

$$0 \to \mathrm{Im}(u_x) \to \mathbb{R}^m \to \mathrm{CoKer}(u_x) \to 0.$$

Thu for a fixed $b_0 \in U$, we may write

$$\mathbb{R}^n = V_1 \oplus V_2$$
$$\mathbb{R}^m = W_1 \oplus W_2$$

where $V_1 \cong \mathrm{Im}(u_{b_0})$, $V_2 = \mathrm{Ker}\,(u_{b_0})$, $W_1 = \mathrm{Im}(u_{b_0})$ and $W_2 \cong \mathrm{CoKer}(u_{b_0})$. Now construct the following linear map for each $x \in U$:

$$V = \mathbb{R}^n \oplus W_2 = V_1 \oplus V_2 \oplus W_2 \xrightarrow{w_x} W_1 \oplus W_2 \oplus V_2 = \mathbb{R}^m \oplus V_2 = W,$$

where w_x on V_1 is u_x, on V_2 is $u_x \oplus \mathrm{id}_{V_2}$ and on W_2 is id_{W_2}. Note that w_{b_0} is a linear isomorphism. Since u_x is continuous in x and isomorphisms form an open subset of linear maps, hence we may assume by shrinking U appropriately that w_x is isomorphism for all $x \in U$. Let $v_x : W \to V$ be the inverse of w_x. Note that $x \mapsto v_x$ is also continuous.

Using w_x, we show that K_g is locally trivial. Indeed, a vector $(\vec{v}_1, \vec{v}_2) \in \mathbb{R}^n = V_1 \oplus V_2$ is in $\mathrm{Ker}\left(g_x : E'_x \to E_{g(x)}\right)$ if and only if $w_x(\vec{v}_1, \vec{v}_2, 0) = (0, \vec{v}_2, 0)$. Thus, $(\vec{v}_1, \vec{v}_2) \in \mathrm{Ker}\,(f_x)$ if and only if $v_x(0, \vec{v}_2, 0) = (\vec{v}_1, \vec{v}_2)$. Hence the map

$$\begin{aligned} U \times V_2 &\longrightarrow U \times \mathbb{R}^n \xrightarrow{h} \pi'^{-1}(U) \\ (x, \vec{v}_2) &\longmapsto (x, v_x(0, \vec{v}_2, 0)) \longmapsto h(x, v_x(0, \vec{v}_2, 0)) \end{aligned}$$

maps $U \times V_2$ isomorphically onto $\pi'^{-1}(U) \cap K_g$, thus giving a local trivialization of K_g, as required.

Finally, we show that C_g is locally trivial. Observe that $\mathrm{Im}(u_x) \cap W_2 = 0$. Indeed, if not then for some $(\vec{v}_1, \vec{v}_2) \in \mathbb{R}^n = V_1 \oplus V_2$, we have $u_x(\vec{v}_1, \vec{v}_2) \in W_2 \subseteq \mathbb{R}^m$. Then, $u_x(\vec{v}_1, \vec{v}_2, y) = 0$ and by injectivity of u_x, we conclude that $\vec{v}_i = 0$. Hence, we may define

$$\begin{aligned} V \times W_2 &\longrightarrow V \times \mathbb{R}^{m-k} \\ (x, \vec{w}_2) &\longmapsto (x, \vec{w}_2 + \mathrm{Im}(u_x)). \end{aligned}$$

This gives the required local trivialization for C_g. □

4 Stiefel-Whitney classes & Grassmannian

Question 15. Show that for two vector bundles $\xi = (E_1, \pi_1, B_1)$ and $\eta = (E_2, \pi_2, B_2)$, we have

$$w_k(\xi \times \eta) = \sum_{i=0}^{k} w_i(\xi) \times w_{k-i}(\eta)$$

where $\times$ denotes the cohomology cross product.

Proof. Recall that $\xi \times \eta = p_1^* \xi \oplus p_2^* \eta$ where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are projections.

Consequently,

$$\begin{aligned} w_k(\xi \times \eta) = w_k(p_1^* \xi \oplus p_2^* \eta) &= \sum_{i=0}^{k} w_i(p_1^* \xi) \smile w_{k-i}(p_2^* \eta) \\ &= \sum_{i=0}^{k} p_1^*(w_i(\xi)) \smile p_2^*(w_{k-i}(\eta)) \\ &= \sum_{i=0}^{k} w_i(\xi) \times w_{k-i}(\eta). \end{aligned}$$

This completes the proof. □

Question 16. Show that for $n+1 = 2^r m$ where $m \geq 3$ is odd, there are no 2^r everywhere independent vector fields on $\mathbb{P}^n$.

Proof. Let $\{X_i\}_{i=1,\ldots,2^r}$ be 2^r-everywhere independent vector fields on $\mathbb{P}^n$. These define independent sections $X_i : \mathbb{P}^n \to T\mathbb{P}^n$. Hence, this defines a trivial subbundle $E \subseteq T\mathbb{P}^n$ of rank 2^r. As $\mathbb{P}^n$ is compact, therefore $T\mathbb{P}^n$ has an Euclidean metric. Consequently, there exists $E^\perp \subseteq T\mathbb{P}^n$ of rank $n - 2^r = 2^r m - 1 - 2^r = 2^r(m-1) - 1$ such that

$$E \oplus E^\perp = T\mathbb{P}^n.$$

By Whitney product formula, we have

$$w(T\mathbb{P}^n) = w(E) \cdot w(E^\perp)$$

where the product is in $H^\Pi(B; \mathbb{Z}_2)$. As E is trivial, therefore $w(E) = 1$. Hence, $w(T\mathbb{P}^n) = w(E^\perp)$. Note $w(T\mathbb{P}^n) = (1+a)^{n+1} = (1+a)^{2^r m} = ((1+a)^{2^r})^m = (1+a^{2^r})^m$. This has largest term given by $m \cdot a^{2^r(m-1)}$. This is non-zero as m is odd, so mod 2 it is non-zero. On the other hand, the largest possible non-zero term of $w(E^\perp)$ is $a^{2^r(m-1)-1}$ by above. This contradicts the conclusion that $w(T\mathbb{P}^n) = w(E^\perp)$, as required. □

Question 17. Show that $\mathbb{P}^n$ admits a field of tangent 1-planes if and only if n is odd. Show that $\mathbb{P}^4$ and $\mathbb{P}^6$ doesn't admit a field of tangent 2-planes.

Proof. Let E be the subbundle of $T\mathbb{P}^n$ of rank 1. We show that n is odd. Note we have a decomposition $T\mathbb{P}^n \cong E \oplus E^\perp$, where $E^\perp$ has rank $n-1$. By product formula, we have

$$(1+a)^{n+1} = w(E) \cdot w(E^\perp).$$

We have two cases. First, if $w(E) = 1$, then $w(E^\perp) = (1+a)^{n+1}$. As the largest possible degree term of $(1+a)^{n+1}$ is $(n+1)a^n$ and for $w(E^\perp)$ the largest possible degree term is a^{n-1}, thus we must have $n+1 = 0 \mod 2$, that is, n is odd. In the second case, $w(E) = 1+a$. Then, $w(E^\perp) = (1+a)^n$, whose largest non-zero term is a^n. But the largest non-zero term of $w(E^\perp)$ must be a^{n-1}, a contradiction. Conversely, if n is odd, then $\mathbb{P}^n$ admits a non-vanishing vector field as S^n has a non-vanishing vector field for n odd. This shows the first part.

Suppose $E \subseteq T\mathbb{P}^4$ is a rank 2 subbundle of $T\mathbb{P}^4$. Then, we have $E \oplus E^\perp \cong T\mathbb{P}^4$, where $E^\perp$ is also rank 2. By product formula,

$$(1+a)^5 = 1 + a + a^4 = w(E) \cdot w(E^\perp).$$

As $w(E), w(E^\perp) = 1, 1+a, 1+a^2, 1+a+a^2$, one then easily sees that none of their product ever gives $1 + a + a^4$, as required.

Similarly, if $T\mathbb{P}^6 \cong E \oplus E^\perp$ where E and $E^\perp$ are of rank 2 and 4 respectively, then

$$(1+a)^7 = w(E) \cdot w(E^\perp).$$

Now, $w(E) = 1, 1+a, 1+a^2, 1+a+a^2$. One then again checks similar to the $\mathbb{P}^4$ case that in all cases for $w(E)$ we get a contradiction. □

Question 18. If an n-manifold M can be immersed into $\mathbb{R}^{n+1}$, then show that $w_k(M) = w_1(M)^{\smile k}$. If $\mathbb{P}^n$ can be immersed into $\mathbb{R}^{n+1}$, then $n = 2^r - 1$ or $n = 2^r - 2$.

Proof. Let $NM \to M$ be the normal bundle of rank 1, hence either $w(NM) = 1$or $1 + b$, for $b = w_1(NM) \in H^1(M; \mathbb{Z}_2)$. As $TM \oplus NM \cong \epsilon^{n+1}$, therefore if $w(NM) = 1$, then $w(TM) = 1$ and hence $w_1(TM) = 0$ and hence $w_i(TM) = w_1(TM)^{\smile k}$ vacuously. On the other hand, if $w(NM) = 1 + b$, then

$$w(TM) = \overline{w(NM)} = \overline{1+b} = 1 + b + b^2 + \cdots + b^n.$$

From above expression, we deduce that $w_1(NM) = w_1(TM) = b$. Hence for any $k \geq 1$, we have $w_k(TM) = w_1(TM)^k$, as required.

For the second statement, suppose $\mathbb{P}^n$ immerses into $\mathbb{R}^{n+1}$. Thus we have a splitting

$$T\mathbb{P}^{n+1} \oplus L \cong \epsilon^{n+1},$$

where L is a line bundle. Thus we have

$$w(T\mathbb{P}^n) \cdot w(L) = 1.$$

Now either $w(L) = 1$ or $1 + a$. Hence, $w(T\mathbb{P}^n) = 1$ or $(1+a)^{-1}$. In the former, by the theorem that says $w(T\mathbb{P}^n) = 1$ if and only if $n + 1 = 2^r$, we deduce that $n = 2^r - 1$. In the latter, we have $(1+a)^{n+1} = w(T\mathbb{P}^n) = (1+a)^{-1}$. Consequently, $(1+a)^{n+2} = 1$. If $n+2$ is not a power of 2, then we have $n + 2 = 2^r m$ where $m > 1$ is odd. Expanding $(1+a)^{n+2}$, we get

$$1 = (1+a)^{n+2} = (1+a)^{2^r m} = (1 + a^{2^r})^m = 1 + m a^{2^r} + \ldots.$$

This yields that m is even, a contradiction, as required. □

Question 19 (*Unoriented cobordism group*)**.** Let $\mathcal{M}_n$ denote the collection of all n-dimensional closed manifolds. Denote

$$\Omega_n^O = \mathcal{M}_n \,/\sim$$

where $M \sim N$ if and only if there exists W an $n+1$-dimensional compact manifold such that $\partial W = M \amalg N$. The set Ω_n^O is called the unoriented cobordism group.

1. Show that Ω_n^O is an abelian group under disjoint union.
2. Show that Ω_n^O is a finite-dimensional $\mathbb{Z}_2$-vector space.
3. Show that Ω_4^O has atleast four distinct elements.

Proof. 1. Define

$$[M] + [N] := [M \amalg N].$$

We first show that this is well-defined. If $[M] = [M']$ and $[N] = [N']$, then there exists A, B compact $n+1$-dimensional manifolds such that $\partial A = M \amalg M'$ and $\partial B = N \amalg N'$. Consequently, $A \amalg B$ is a compact $n+1$-dimensional manifold with boundary $\partial(A \amalg B) = \partial A \amalg \partial B = M \amalg M' \amalg N \amalg N'$. Hence, $[M \amalg N] = [M' \amalg N']$, as required. Associativity and commutativity is immediate. Moreover, identity of Ω_n^O is given by the empty manifold $\emptyset$, which is considered to be a manfiold of every dimension. Finally the additive inverse of $[M]$ is given by $[M]$ itself since $M \amalg M$ is the boundary of $M \times I$.

2. Since for any $[M] \in \Omega_n^O$, we have $[M] + [M] = 0$, hence we have a natural $\mathbb{Z}_2$-vector space structure on Ω_n^O. To show finite dimensionality, it is sufficient to show that Ω_n^O is finite. Indeed, by Thom-Pontryagin theory, two closed n-manifolds $M, N \in \mathcal{M}_n$ give $[M] = [N]$ if and only if all of their Stiefel-Whitney numbers are same. As there are only finitely many possibilities for Stiefel-Whitney numbers for a given closed n-manfiold, therefore there can atmost be finitely many cobordism classes with different Stiefel-Whitney numbers. Hence there are only finitely many cobordism classes, as required.

3. We will show that $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^4$ are not cobordant. It will then follow that Ω_4^O has atleast three elements. Since Ω_n^O is a $\mathbb{Z}_2$-vector space, therefore it must then atleast have four elements, as required. To this end, it suffices to show that there exists a Stiefel-Whitney monomial which evaluates to different numbers for $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^4$. We first calculate the Stiefel-Whitney classes for both these spaces.

For $\mathbb{P}^4$, we have

$$w(\mathbb{P}^4) = (1+a)^5 = 1 + a + a^4$$

where $a \in H^1(\mathbb{P}^4\ Z_2)$ is the non-zero term. On the other hand, the tangent bundle on $\mathbb{P}^2 \times \mathbb{P}^2$ is given by the Cartesian product $T\mathbb{P}^2 \times T\mathbb{P}^2$. If $p, q : \mathbb{P}^2 \times \mathbb{P}^2 \rightrightarrows \mathbb{P}^2$ are the coordinate projections, then we can write this Cartesian product as

$$T\mathbb{P}^2 \times T\mathbb{P}^2 = p^*(T\mathbb{P}^2) \oplus q^*(T\mathbb{P}^2).$$

Consequently, by naturality of Stiefel-Whitney classes, we must have

$$w(T\mathbb{P}^2 \times T\mathbb{P}^2) = p^*(w(T\mathbb{P}^2)) \cdot q^*(w(T\mathbb{P}^2)).$$

As $p^* = q^*$, we further have

$$w(T\mathbb{P}^2 \times T\mathbb{P}^2) = p^*(w(T\mathbb{P}^2) \cdot w(T\mathbb{P}^2)).$$

Let $b \in H^1(\mathbb{P}^2\ \mathbb{Z})$ be the non-zero element. Then $w(T\mathbb{P}^2) = (1+b)^3 = 1 + b + b^2$. Consequently, we have

$$w(T\mathbb{P}^2 \times T\mathbb{P}^2) = p^*((1 + b + b^2)^2) = p^*(1 + b^2) = 1 + (p^*b)^2.$$

Now, consider the Stiefel-Whitney monomial w_4. We claim that $w_4[\mathbb{P}^4] \neq 0$. Indeed, as $H^4(\mathbb{P}^4;\mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2}\left(H_4(\mathbb{P}^4;\mathbb{Z}_2),\mathbb{Z}_2\right)$ by the evaluation map and since $w_4(\mathbb{P}^4) \neq 0$, hence $w_4[\mathbb{P}^4] \neq 0$. On the other hand, $w_4(\mathbb{P}^2 \times \mathbb{P}^2) = 0$, as calculated above. Thus, $w_4[\mathbb{P}^2 \times \mathbb{P}^2] = 0$. This shows that $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^4$ have different Stiefel-Whitney number corresponding to the top monomial w_4, hence they are not cobordant, as required. □

Question 20 (*Smooth structure on* $\mathrm{Gr}_k(n)$)**.** Show that $\mathrm{Gr}_k(n)$ is a smooth manifold of dimension $k(n-k)$.

Proof. We'll show that $\mathrm{Gr}_k(n)$ is a closed subamnifold of $\mathbb{P}(\wedge^k\mathbb{R}^n)$. Consider the function

$$\begin{aligned} P : \mathrm{Gr}_k(n) &\longrightarrow \mathbb{P}\wedge^k V \\ \Lambda &\longmapsto [v_1 \wedge \cdots \wedge v_k] \end{aligned}$$

where Λ has basis $\{v_1,\dots,v_k\}$. This is well-defined as if $\{w_1,\dots,w_k\}$ forms another basis of Λ, then $w_1 \wedge \cdots \wedge w_k = d \cdot (v_1 \wedge \cdots \wedge v_k)$ where d is the determinant of the change of basis matrix, and thus they determine same point in $\mathbb{P}\wedge^k V$.

We next wish to write P in projective coordinates of $\mathbb{P}\wedge^k V$. To this end, fix a basis $\{e_1,\dots,e_n\}$ of V. Writing each v_i in this basis, we deduce that the k-plane Λ is the row space of the $k \times n$ matrix A_Λ whose rows are v_i. We can then write

$$v_1 \wedge \cdots \wedge v_k = \sum_{I \in \mathrm{Inc}(k,n)} p_I e_I$$

where $I = (i_1,\dots,i_k)$ is an increasing sequence of elements from $\{1,\dots,n\}$, $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ forms the basis of $\wedge^k V$ and $p_I = \det A_\Lambda[I]$, the $k \times k$-minor of A_Λ determined by columns with index I. In projective coordinates (of which there are ${}^nC_k$ many), the map P is merely

$$P : \Lambda \mapsto [p_I]_{I \in \mathrm{Inc}(k,n)}$$

where $p_I = \det A_\Lambda[I]$ is a polynomial in the entries of a general $k \times n$ matrix.

We first wish to show that this function is injective. Indeed, if $P(\Lambda) = P(\Lambda')$, then $v_1 \wedge \cdots \wedge v_k = d \cdot w_1 \wedge \dots w_k$ for $d \in \mathbb{R}^\times$ where $\{v_1,\dots,v_k\}$ is a basis of Λ and $\{w_1,\dots,w_k\}$ is a basis of Λ'. If $[p_I]_I$ and $[q_I]_I$ are projective coordinates of $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ respectively, then $p_I = d \cdot q_I$. It follows that every $k \times k$ minor of A_Λ is a common multiple of the same minor of $A_{\Lambda'}$. Consequently, A_Λ and $A_{\Lambda'}$ have same row space, as required.

The map P embeds $\mathrm{Gr}_k(\mathbb{R}^n)$ as a subspace of $\mathbb{P}(\wedge^k V)$. We next claim that $\mathrm{Gr}_k(n)$ is in-fact a closed subspace. We need only show that the image of P is closed. To this end, we first claim that

$$\mathrm{Im}\, P = \left\{ [\eta] \in \mathbb{P}(\wedge^k V) \mid \dim \mathrm{Im}\, V \xrightarrow{\wedge\eta} \wedge^{k+1} V \leq n-k \right\}.$$

Indeed, image of P consists of classes of all those $\eta \in \wedge^k V$ where $\eta = v_1 \wedge \cdots \wedge v_k$ for $v_i \in V$, i.e. η is a pure tensor. The vector η is of this form if and only if $\dim \mathrm{Ker}\left(V \xrightarrow{\wedge\eta} \wedge^k V\right) \geq k$ and hence the desired claim follows.

As $\wedge\eta$ is a linear map, therefore $\dim \mathrm{Im}\, V \xrightarrow{\wedge\eta} \wedge^{k+1} V \leq n-k$ if and only if all $n-k+1$ minors of $\wedge\eta$ are 0. This is a closed condition, as required. □

Question 21 (Tangent bundle of $\mathrm{Gr}_k(n)$)**.** Let $\mathcal{V}^k \to \mathrm{Gr}_k(n)$ be the universal k-plane bundle on the Grassmannian of k-planes in $\mathbb{R}^n$.

1. Show that the tangent bundle of $\mathrm{Gr}_k(n)$ is isomorphic to
$$\mathcal{H}om(\mathcal{V}^k, \mathcal{Q})$$
where $\mathcal{Q}$ is the orthogonal complement of $\mathcal{V}^k$ in ϵ^n. In other words, $\mathcal{Q} = \epsilon^n/\mathcal{V}^k$ is the universal quotient $n-k$-plane bundle over $\mathrm{Gr}_k(n)$.
2. Let $M \subseteq \mathbb{R}^n$ be a smooth manifold of dimension k with normal bundle ν. If $g : M \to \mathrm{Gr}_k(n)$ is the Gauss map of M, then show that g determines a unique global section of the bundle $\mathcal{H}om(TM \otimes TM, \nu)$.

Proof. 1. We denote $G = \mathrm{Gr}(k, V)$ and $p : TG \to G$ and $q : \mathcal{H}om(\mathcal{V}, \mathcal{Q}) \to G$ be the two given rank $k(n-k)$ bundles. Let $\Gamma \subseteq V$ be an $n-k$ plane of V and consider the open affine patch U_Γ of all k-planes linearly disjoint to Γ. For a fixed $\Omega \in U_\Gamma$, we have $U_\Gamma = \mathrm{Hom}\,(\Omega, \Gamma)$. Then, $TG|_{U_\Gamma} = U_\Gamma \times \mathrm{Hom}\,(\Omega, \Gamma)$ since TG is trivial over any affine chart of G. Our first claim is that fibers of $\mathcal{H}om(\mathcal{V}, \mathcal{Q})$ at $\Omega \in U_\Gamma$ is isomorphic to $(TG)_\Omega$. Indeed, as $(TG)_\Omega = \mathrm{Hom}\,(\Omega, \Gamma)$, therefore we need only show that $\mathcal{V}_\Omega = \Omega$ and $\mathcal{Q}_\Omega = \Gamma$. To this end, by construction $\mathcal{V}_\Omega = \Omega$ and $\mathcal{Q}_\Omega = V/\Omega = \Gamma$ since $V = \Omega \oplus \Gamma$. Consequently we have isomorphism

$$\varphi_\Omega : (TG)_\Omega \longrightarrow \mathcal{H}om(\mathcal{V}, \mathcal{Q})_\Omega$$

for each $\Omega \in G$. We claim that these define a bundle isomorphism. To this end, we need only show that transition maps $U_\Gamma \cap U_{\Gamma'} \to \mathrm{GL}_k(\mathbb{R})$ that both the bundle induces are isomorphic for any two affine open patches $U_\Gamma, U_{\Gamma'}$ of G. To this end, we first observe the transition maps for TG. Recall that transitions for tangent bundle comes from the derivative of transition maps of the base manifold. As the transition of the G from U_Γ to U_Γ' is given by (denote $U = U_\Gamma \cap U_{\Gamma'}$)

$$\psi : U_\Gamma = \mathrm{Hom}\,(\Omega, \Gamma) \longrightarrow U_{\Gamma'} = \mathrm{Hom}\,\left(\Omega, \Gamma'\right)$$

which is obtained by the composite linear isomorphisms $\Gamma \xrightarrow{\alpha} V/\Omega \xrightarrow{\beta^{-1}} \Gamma'$, the transition map of TG is the differential of ψ:

$$d\psi : U \times \mathrm{Hom}\,(\Omega, \Gamma) \to U \times \mathrm{Hom}\,\left(\Omega, \Gamma'\right)$$

which is again same as ψ on second factor as ψ is linear. We next wish to show that $\mathcal{H}om(\mathcal{V}, \mathcal{Q})$ has the same transitions. Indeed, by theory of continuous functor it is immediate that the transition of $\mathcal{H}om(\mathcal{V}, \mathcal{Q})$ on U is same as $d\psi$.

2. Given the map $g : M \to \mathrm{Gr}_k(n)$ which maps $x \mapsto T_xM$, we get the map on tangent bundles

$$dg : TM \to \mathcal{H}om(\mathcal{V}^k, \mathcal{Q})$$

which takes $(x, \vec{v}) \mapsto dg(x, \vec{v})$ where $dg(x, \vec{v})$ is a bundle map given by (note $\mathcal{V}^k_{T_xM} = T_xM$ and $\mathcal{Q}_{T_xM} = V/T_xM = \nu_x$)

$$dg(x, \vec{v}) : T_xM \longrightarrow \nu_x.$$

It follows that we have a bundle map $dg : TM \to \mathcal{H}om(TM,\ nu)$. Consequently, g gives a global section dg of $\mathcal{H}om(TM, \mathcal{H}om(TM, \nu)) \cong \mathcal{H}om(TM \otimes TM, \nu)$ and this section is unique w.r.t. the property that it is the differential of the map $g : M \to \mathrm{Gr}_k(n)$, as required. □

Question 22 (Universal property of Grassmannians)**.** Let B be a paracompact space. Then there is a one-to-one correspondence between maps $B \to \mathrm{Gr}_k$ to Grassmannian of k-planes in $\mathbb{R}^\infty$ and collection of k-plane bundles over B.

Proof. Let $\mathrm{Bun}_k(B)$ be the collection of all k-plane bundles over B and $\mathcal{V}^k$ be the universal k-plane bundle over Gr_k. Consider the map

$$\begin{aligned} \mathrm{Hom}\,(B, \mathrm{Gr}_k) &\longrightarrow \mathrm{Bun}_k(B) \\ \varphi &\longmapsto \varphi^*(\mathcal{V}^k). \end{aligned}$$

We claim that this map is a bijection. We first show injectivity. If we have bundle equality $\varphi^*(\mathcal{V}^k) = \psi^*(\mathcal{V}^k)$, then we at once have $\varphi = \psi$. The difficult part now is to show that the above map is surjective.

Let $\xi = (E, p, B)$ be a k-plane bundle over B. We wish to construct $\varphi : B \to \mathrm{Gr}_k$ such that $\varphi^*\mathcal{V}^k = \xi$. By Lemma 5.9 of Milnor-Stasheff, it follows by paracompactness that there exists a countable cover $\{U_i\}$ of B such that restriction of ξ to U_i is trivial. By partitions of unity, there exists maps $\rho_i : B \to \mathbb{R}$ and $W_i \subseteq \overline{W_i} \subseteq V_i \subseteq \overline{V_i} \subseteq U_i$ such that $\rho_i = 1$ on $\overline{W_i}$, $\mathrm{Supp}(\rho_i) = \overline{V_i}$ and V_i covers B. Let h_i be the composition of local trivialization with projection:

$$h_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^k \to \mathbb{R}^k.$$

Note that $h_{i,b} : E_b \to \mathbb{R}^k$ is a linear isomorphism. We then extend h_i to whole of E by using ρ_i as follows; define

$$\begin{aligned} \hat{h}_i : E &\longrightarrow \mathbb{R}^k \\ e &\longmapsto \begin{cases} \rho_i(p(e))h_i(e) & \text{if } p(e) \in U_i \\ 0 & \text{else.} \end{cases} \end{aligned}$$

At this point, we have a countable family of maps $\{\hat{h}_i\}$. Using the we construct the following map

$$\begin{aligned} \hat{f} : E &\longrightarrow \mathbb{R}^k \times \mathbb{R}^k \times \cdots = \mathbb{R}^\infty \\ e &\longmapsto (\hat{h}_1(e), \hat{h}_2(e), \dots). \end{aligned}$$

This is continuous as it is coordinatewise so. Moreover, for $b \in B$, the restriction

$$\hat{f} : E_b \longrightarrow \mathbb{R}^\infty$$

is linear and injective. Indeed, linearity is immediate and if for some $e \in E_b$, we have $\hat{f}(e) = 0$, then since for some i, $\rho_i(p(e)) \neq 0$, hence $h_i(e) = 0$. By injectivity of h_i on E_b, it follows that $e = 0$, as required.

We finally construct the pullback square

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & \mathcal{V}^k \\ {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow \\ B & \xrightarrow[\ \varphi\ ]{} & \mathrm{Gr}_k \end{array}$$

by first defining f as follows:

$$\begin{aligned} f : E &\longrightarrow \mathcal{V}^k \\ e &\longmapsto \left(\hat{f}(E_{p(e)}), \hat{f}(e)\right) \end{aligned}$$

which then induces the map $\varphi : b \mapsto \hat{f}(E_b)$. Observe further that f is an isomorphism on fibers as $f_b : E_b \to \hat{f}(E_b)$ is the map $\hat{f} : E_b \to \mathbb{R}^\infty$, which is proved to be injective. To complete the proof, we need only show that f is continuous.

To see continuity of f, it is sufficient to show that f composed with local trivializations of ξ are continuous. Indeed, let $k_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ be the local trivialization of ξ as stated in the beginning. Then, we claim that the map $f \circ k_i^{-1}$ is continuous. Indeed, we have

$$f \circ k_i^{-1} : (b, \vec{v}) \longmapsto \left(\hat{f}(E_b), \hat{f}(k_i^{-1}(b, \vec{v}))\right).$$

This is continuous as both factors are so by continuity of $\hat{f}$ and k_i^{-1}. This completes the proof. □

5 Cohomology ring of Grassmannian

Question 23. Show that the inclusion $i : \mathrm{Gr}_k(n) \hookrightarrow \mathrm{Gr}_k(\infty)$ induces an isomorphism

$$i^* : H^p(\mathrm{Gr}_k(\infty); R) \longrightarrow H^p(\mathrm{Gr}_k(n); R)$$

for every $p < n - k$ and for any ring R.

Proof. Let $X = \mathrm{Gr}_k(\infty)$ and $A = \mathrm{Gr}_k(n)$. From the long exact sequence of pairs, we get the following exact sequence

$$\cdots \to H^p(X, A; R) \to H^p(X; R) \to H^p(A; R) \to H^{p+1}(X, A; R) \to \cdots.$$

We claim that $H^{p+1}(X, A; R) = 0 = H^p(X, A; R)$. Indeed, pick $0 \leq q \leq n - k$. We will show that the relative cellular chain group $C_q(X, A; R)$ is 0. This is sufficient as then $H^q(X, A; R) = 0$ and thus the above exact sequence i^* will be an isomorphism.

We have

$$C_q(X, A; R) = \frac{C_q(X; R)}{C_q(A; R)} = \frac{R^d}{R^e} = R^{d-e}$$

where d and e are the number of q-cells in X and A respectively. As

$$\begin{aligned} d &= \#\{n - k \geq a_1 \geq \cdots \geq a_k \geq 1 \mid a_1 + \cdots + a_k = q\} \\ e &= \#\{\infty > b_1 \geq \cdots \geq b_k \geq 1 \mid b_1 + \cdots + b_k = q\} \end{aligned}$$

and $q \leq n - k$, therefore $b_1 \leq n - k$ always. Hence $d = e$ and thus $C_q(X, A; R) = 0$, as required. □

Question 24. Let $f : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_{k+1}(\mathbb{R} \oplus \mathbb{R}^n)$ be defined by $X \mapsto \mathbb{R} \oplus X$.

1. Show that f is an embedding.

2. Show that there is a fiber square:

$$\begin{array}{ccc} \epsilon^1 \oplus \mathcal{V}^k & \xrightarrow{\quad g \quad} & \mathcal{V}^{k+1} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathrm{Gr}_k(\mathbb{R}^n) & \xrightarrow[f]{} & \mathrm{Gr}_{k+1}(\mathbb{R} \oplus \mathbb{R}^n) \end{array} .$$

3. Let $e(\vec{\sigma})$ be an r-cell of $\mathrm{Gr}_k(\mathbb{R}^n)$ determined by the Schubert symbol $\vec{\sigma} = (\sigma_1, \dots, \sigma_k)$ with the partition of r being $(i_1, \dots, i_s)$. Show that $f(e(\sigma))$ is also an r-cell of $\mathrm{Gr}_{k+1}(\mathbb{R} \oplus \mathbb{R}^n)$ with the partition of r being same $(i_1, \dots, i_s)$.

Proof. 1. We have to show that f is injective, smooth, homeomorphic to its image and an immersion. Injectivity is immediate since if $\mathbb{R} \oplus X = \mathbb{R} \oplus Y$ in $\mathbb{R} \oplus \mathbb{R}^n$ for $X, Y \subseteq \mathbb{R}^n$, then $X = Y$. For smoothness, we use Plücker coordinates. Observe that we have a map $\wedge^k(\mathbb{R}^n) \to \wedge^{k+1}(\mathbb{R} \oplus \mathbb{R}^n)$ which maps on the basis vector as $e_{i_1} \wedge \dots \wedge e_{i_k} \mapsto e_{n+1} \wedge e_{i_1} \wedge \dots \wedge e_{i_k}$. This defines a map $\mathbb{P}(\wedge^k \mathbb{R}^n) \to \mathbb{P}(\wedge^{k+1}(\mathbb{R} \oplus \mathbb{R}^n))$. Clearly, this is a smooth map as it is so coordinatewise. This further restricts to the closed subspace $\mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_{k+1}(\mathbb{R} \oplus \mathbb{R}^n)$ and the map becomes $[v_1 \wedge \dots \wedge v_k] \mapsto [e_{n+1} \wedge v_1 \wedge \dots \wedge v_k]$. This shows smoothness of f. The map f is homeomorphic to its image as f is injective and $\mathrm{Gr}_k(\mathbb{R}^n)$ is already compact. We need only show f is an immersion. Indeed the map on tangent spaces induced by f is

$$df : \mathcal{H}om(\mathcal{V}^k, \mathcal{Q}^{n-k}) \longrightarrow \mathcal{H}om(\mathcal{V}^{k+1}, \mathcal{Q}^{n-k})$$

which defined on $\Lambda \in \mathrm{Gr}_k(\mathbb{R}^n)$ maps

$$\begin{aligned} df_\Lambda : \mathrm{Hom}\,(\Lambda, V/\Lambda) &\longrightarrow \mathrm{Hom}\,(\mathbb{R} \oplus \Lambda, V/\Lambda) \\ \varphi &\longmapsto 0 \oplus \varphi. \end{aligned}$$

This is clearly an injective map, as required. This completes the proof that f is an embedding.

2. Define the map $g : \epsilon^1 \oplus \mathcal{V}^k \longrightarrow \mathcal{V}^{k+1}$ on fiber at $\Lambda \in \mathrm{Gr}_k(\mathbb{R}^n)$ as follows; define

$$g_\Lambda : \epsilon^1_\Lambda \oplus \mathcal{V}^k_\Lambda = \mathbb{R} \oplus \Lambda \longrightarrow \mathcal{V}^{k+1}_{\mathbb{R} \oplus \Lambda} = \mathbb{R} \oplus \Lambda$$

to be identity. Then clearly this defines a continuous map $g : \epsilon^1 \oplus \mathcal{V}^k \to \mathcal{V}^{k+1}$ which is furthermore isomorphism on fibers. As g makes the square commute, therefore g provides the required fiber square.

3. Let $\vec{\sigma} : 1 \leq \sigma_1 < \sigma_2 < \dots < \sigma_k \leq n$ be a Schubert symbol for $\mathrm{Gr}_k(\mathbb{R}^n)$ and let $e(\sigma)$ be the open cell of dimension $r = \sigma_1 - 1 + \sigma_2 - 2 + \dots + \sigma_k - k$ that it determines. We claim that

$$f(e(\vec{\sigma})) = e(\vec{\tau})$$

where $\vec{\tau} = (1, \sigma_1 + 1, \sigma_2 + 1, \dots, \sigma_k + 1)$ in $\mathrm{Gr}_{k+1}(\mathbb{R} \oplus \mathbb{R}^n)$. To this end, observe that if

$$0 \subset \mathbb{R} \subset \mathbb{R}^2 \subset \dots \subset \mathbb{R}^n$$

is the complete flag for $\mathbb{R}^n$, then

$$0 \subset \mathbb{R} \oplus \mathbb{R} \subset \mathbb{R} \oplus \mathbb{R}^2 \subset \dots \mathbb{R} \oplus \mathbb{R}^n$$

is a complete flag for $\mathbb{R} \oplus \mathbb{R}^n$. We now have

$$\begin{aligned} f(e(\vec{\sigma})) &= \{\mathbb{R} \oplus \Lambda \mid \Lambda \in e(\vec{\sigma})\} \\ &= \left\{\mathbb{R} \oplus \Lambda \mid \dim \Lambda \cap \mathbb{R}^{\sigma_i} = i \;\&\; \dim \Lambda \cap \mathbb{R}^{\sigma_i - 1} = i - 1\right\} \\ &= \left\{\mathbb{R} \oplus \Lambda \mid \dim(\mathbb{R} \oplus \Lambda) \cap (\mathbb{R} \oplus \mathbb{R}^{\sigma_i}) = i + 1 \;\&\; \dim(\mathbb{R} \oplus \Lambda) \cap (\mathbb{R} \oplus \mathbb{R}^{\sigma_i - 1}) = i\right\} \\ &= e(\vec{\tau}), \end{aligned}$$

as required. Now, the dimension of $e(\vec{\tau})$ is

$$\dim e(\vec{\tau}) = \sum_{i=1}^{k+1} \tau_i - i = 0 + \sigma_1 - 1 + \sigma_2 - 2 + \cdots + \sigma_k - k = r = \dim e(\vec{\sigma})$$

and from this its also clear that the partition of r that $\vec{\tau}$ gives rise to is same as that of $\vec{\sigma}$, as required. □

Question 25. Let M be an n-dimensional manifold. Show that the number of distinct Stiefel-Whitney numbers for M is $p(n)$, i.e. the number of unordered positive partitions of integer n.

Proof. Our first claim is that a Stiefel-Whitney number is determined by the corresponding Stiefel-Whitney monomial, that is, the monomial $w_1^{r_1} \dots w_n^{r_n}$ determines the number $w_1^{r_1} \dots w_n^{r_n}[M] \in \mathbb{Z}_2$ completely. Indeed, for M, the SW-number corresponding to $w_1^{r_1} \dots w_n^{r_n}$ is given by

$$\langle w_1^{r_1} \dots w_n^{r_n}(TM), \mu_M \rangle \in \mathbb{Z}_2.$$

But since the Kronecker pairing

$$H^n(M; \mathbb{Z}_2) \times H_n(M; \mathbb{Z}_2) \xrightarrow{\langle -,- \rangle} \mathbb{Z}_2$$

is non-degenerate, therefore we have an isomorphism

$$H^n(M; \mathbb{Z}_2) \cong \operatorname{Hom}_{\mathbb{Z}_2}(H_n(M; \mathbb{Z}_2), \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Thus SW-number corresponding to $w_1^{r_1} \dots w_n^{r_n}$ is 1 if and only if $w_1^{r_1} \dots w_n^{r_n}(TM) \in H^n(M; \mathbb{Z}_2)$ is 1. Similarly for 0. Hence we need only count the number of SW-monomials. Indeed, the no. of SW-monomials is same as the size of the set

$$A = \{(r_1, \dots, r_n) \mid r_i \geq 0 \;\&\; r_1 + 2r_2 + \cdots + nr_n = n\}.$$

We claim that there is a bijection from B to A where

$$B = \{(i_1, \dots, i_s) \mid i_j \geq 1 \;\&\; i_1 + \cdots + i_s = n\}.$$

Indeed, define

$$\begin{aligned} \varphi : B &\longrightarrow A \\ I = (i_1, \dots, i_s) &\longmapsto (r_1, \dots, r_n) \end{aligned}$$

where $r_j = \#$ of j in I. Then clearly, $\sum_{j=1}^n j r_j = n$. Converse is also immediate. Hence φ is a bijection and thus $\#A = \#B = p(n)$, as required. □

Question 26. Let ξ and η be two bundles of rank n and m. Show that there is a polynomial p in $n+m$ variables such that

1. we have

$$w(\xi \otimes \eta) = p(w_1(\xi), \ldots, w_n(\xi), w_1(\eta), \ldots, w_m(\eta)),$$

2. if $\sigma_1, \ldots, \sigma_n$ are elementary symmetric functions in variables $t_1, \ldots, t_n$ and $\sigma'_1, \ldots, \sigma'_m$ are in $t'_1, \ldots, t'_m$, then

$$p(\sigma_1, \ldots, \sigma_n, \sigma'_1, \ldots, \sigma'_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + t_i + t'_j).$$

Proof. This is an application of splitting principle which says that for any bundle $\xi = (E, p, B)$, there is a space X and a map $f : X \to B$ such that $f^* : H^*(B; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ is injective and $f^*\xi$ is a sum of line bundles. Thus, any relation we may obtain amongst SW classes while assuming ξ and η are direct sum of line bundles is true in general. We omit the proof of splitting principle as it is well-known.

Assuming the above result, we may complete the proof as follows. We may assume $\xi = \oplus_{i=1}^n L_i$ and $\eta = \oplus_{j=1}^m L'_j$. Then

$$\begin{aligned} \xi \otimes \eta &= \bigoplus_{i=1}^{n} L_i \otimes \bigoplus_{j=1}^{m} L'_j \\ &= \bigoplus_{1 \le i \le n} \bigoplus_{1 \le j \le m} L_i \otimes L'_j. \end{aligned}$$

Thus by Whitney formula

$$\begin{aligned} w(\xi \otimes \eta) &= \prod_{1 \le i \le n} \prod_{1 \le j \le m} w(L_i \otimes L'_j) \\ &= \prod_{1 \le i \le n} \prod_{1 \le j \le m} (1 + a_i + a'_i) \end{aligned}$$

where $w(L_i \otimes L'_j) = 1 + a_i + a'_j$, $w_1(L_i) = a_i$ and $w_1(L'_j) = a'_j$. Since

$$\begin{aligned} w(\xi) &= \prod_{i=1}^{n} (1 + a_i) \\ w(\eta) &= \prod_{j=1}^{m} (1 + a'_j) \end{aligned}$$

therefore $w_p(\xi)$ and $w_q(\eta)$ are elementary symmetric polynomials in a_i and a'_j respectively. This completes the proof. $\square$